TOPOLOGY II

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Abstract. These are lecture notes for the course "Topology II" held at LMU München during the summer term 2024.

Contents

Acknowledgements. These are lecture notes for the course "Topology II" held at LMU München during the summer term 2024. Special thanks go to the attendants of this course, in particular for many questions that helped to weed out mistakes and inconsistencies in these notes. All remaining mistakes are my own.

Since everything in these notes is part of a well-developed theory, the treatment is based on existing literature on the subject. Everything up to the discussion of Poincaré duality is rather close to [Dol80]. Section 4 additionally draws from [Hat02] and [May99]. Section 5 is a mashup of material from [Hat02], [May99], [FP08] and [tD08]. In particular, the proof of the cellular approximation theorem comes from [FP08], while the proof of the homotopy excision theorem comes from [tD08] (who attributes it to Dieter Puppe).

0. Reminder on singular homology

Let us begin with a quick reminder about the basic definitions and statements concerning singular homology, since this is the place from which we will continue this term.

Given a topological space X and an abelian group M, one associates to these data the *singular chain complex* $C^{\mathrm{sing}}(X;M)$: in degree n, this chain complex is given by the abelian group

$$C_n^{\mathrm{sing}}(X;M) := \mathbb{Z}\left[\mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, X)\right] \otimes M,$$

where $\mathrm{Hom}_{\mathrm{Top}}$ denotes the set of continuous maps and

$$\Delta^n_{\mathrm{Top}} := \left\{ \sum_{i=0}^n \lambda_i e_i \mid \lambda_i \in [0,1],\ \sum_{i=0}^n \lambda_i = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

is the standard n-simplex. Its differential in degree n is

$$d_n := \sum_{i=0}^n \delta_i^* \otimes \mathrm{id}_M \colon C_n^{\mathrm{sing}}(X;M) \to C_{n-1}^{\mathrm{sing}}(X;M),$$

where δ_i^* is the homomorphism induced by restriction along the inclusion

$$\delta_i \colon \Delta^{n-1}_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$$

of the i-th face of Δ^n_{Top}.

Recall that a *pair of topological spaces* (X,A) is by definition a topological space X together with a subspace $A \subseteq X$. The singular chain complex $C^{\mathrm{sing}}(X,A;M)$ of such a pair is by definition

$$C^{\mathrm{sing}}(X,A;M) := C^{\mathrm{sing}}(X;M)/C^{\mathrm{sing}}(A;M).$$

Note that $C^{\mathrm{sing}}(X,\varnothing;M) \cong C^{\mathrm{sing}}(X;M)$.

The *n-th singular homology group of the pair (X,A) with coefficients in M* is

$$H_n(X,A;M) := H_n(C^{\mathrm{sing}}(X,A;M)) = \ker d_n / \operatorname{img} d_{n+1},$$

the n-th homology of the corresponding singular chain complex. For each $n \geq 0$, the association $(X,A) \mapsto H_n(X,A;M)$ refines to a functor from the category of pairs of topological spaces and continuous maps of pairs to the category of abelian groups. These functors have the following properties:

(1) $H_n(\varnothing;M) = 0$;
(2) $H_0(*;M) \cong M$ and $H_n(*;M) = 0$ for $n \geq 1$;
(3) the canonical map
$$\bigoplus_{i\in I} H_n(X_i,A_i;M) \to H_n(\coprod_{i\in I}(X_i,A_i);M)$$
is an isomorphism for every collection of pairs $(X_i,A_i)_{i\in I}$, all $n \geq 0$ and all M;
(4) if $f,g \colon (X,A) \to (Y,B)$ are homotopic maps of pairs, then
$$H_n(f) = H_n(g) \colon H_n(X,A;M) \to H_n(Y,B;M);$$
(5) for every pair of topological spaces (X,A), there exist natural maps
$$\partial_n \colon H_n(X,A;M) \to H_{n-1}(A;M)$$
such that the sequence
$$H_n(A;M) \to H_n(X;M) \to H_n(X,A;M) \xrightarrow{\partial_n} H_{n-1}(A;M) \to H_{n-1}(X;M)$$
is exact for all n;

(6) for every pair (X, A) and every subspace $U \subseteq A$ with $\overline{U} \subseteq \operatorname{int}(A)$ the maps

$$H_n(X \setminus U, A \setminus U; M) \to H_n(X, A; M)$$

induced by the inclusion map are isomorphisms;

(7) for every topological space X and all ordered pairs (A, B) of subspaces $A, B \subseteq X$ with $\operatorname{int}(A) \cup \operatorname{int}(B) = X$ there exist natural maps

$$\partial_n^{A,B} \colon H_n(X; M) \to H_{n-1}(A \cap B; M)$$

such that the sequence

$$\begin{aligned} H_n(A \cap B; M) \xrightarrow{(i_*^A, i_*^B)} H_n(A; M) \oplus H_n(B; M) \xrightarrow{j_*^A - j_*^B} H_n(X; M) \\ \xrightarrow{\partial_n^{A,B}} H_{n-1}(A \cap B; M) \xrightarrow{(i_*^A, i_*^B)} H_{n-1}(A; M) \oplus H_{n-1}(B; M) \end{aligned}$$

is exact;

(8) for every pointed space (X, x), there is a natural isomorphism

$$H_n(\Sigma X, N; M) \cong H_{n-1}(X, x; M),$$

where N denotes the north pole of the suspension ΣX;

(9) for every pointed and path-connected space (X, x), the Hurewicz map

$$\pi_1(X, x) \to H_1(X; M)$$

is an abelianisation.

Given a CW-complex X, one defines the associated *cellular chain complex* $C^{\text{cell}}(X; M)$ by setting

$$C_n^{\text{cell}}(X; M) := H_n(X^{(n)}, X^{(n-1)}; M)$$

and equipping this with the differential

$$d_n^{\text{cell}} \colon C_n^{\text{cell}}(X; M) \xrightarrow{\partial_n} H_{n-1}(X^{(n-1)}; M) \to C_{n-1}^{\text{cell}}(X; M).$$

The *cellular homology* of X is defined to by

$$H_n^{\text{cell}}(X; M) := H_n(C^{\text{cell}}(X; M)).$$

There are isomorphisms

$$H_n^{\text{cell}}(X; M) \cong H_n(X; M)$$

which are natural in cellular maps. Moreover, one can identify

$$C^{\text{cell}}(X; M) \cong C^{\text{cell}}(X; \mathbb{Z}) \otimes M,$$

there are relative versions of these definitions and statements which make sense for a CW-complex X together with a subcomplex $A \subseteq X$.

The differentials in the cellular chain complex admit the following description. Let J_n denote the set of n-cells in the CW-complex X. A choice of characteristic maps for the n-cells induces an isomorphism

$$\gamma_n \colon C_n^{\text{cell}}(X) \xrightarrow{\cong} H_n(X^{(n)}/X^{(n-1)}, *) \xleftarrow{\cong} H_n(\bigvee_{J_n} S^n, *) \xleftarrow{\cong} \bigoplus_{J_n} H_n(S^n, *).$$

Then the attaching maps induce a homomorphism

$$\begin{aligned} \delta_n \colon \bigoplus_{J_n} H_n(S^n, *) \cong \bigoplus_{n-1} H_{n-1}(S^{n-1}) \to H_{n-1}(X^{(n-1)}) \\ \to H_{n-1}(X^{(n-1)}/X^{(n-2)}, *) \cong \bigoplus_{J_{n-1}} H_{n-1}(S^{n-1}, *) \end{aligned}$$

which satisfies

$$d_n^{\text{cell}} = \gamma_{n-1}^{-1} \circ \delta_n \circ \gamma_n.$$

This reduces the calculation of differentials in the cellular chain complex to determining certain mapping degrees of maps between spheres.

1. The Künneth and universal coefficient theorems

It is a natural question whether the computation of homology with coefficients in an abelian group M can be related (or even reduced) to the calculation of integral homology groups. In fact, we will study a more general question: how does the homology of a tensor product of chain complexes relate to the homologies of the individual factors?

Let us begin with a reminder on some constructions on graded modules and chain complexes. If we do not say anything else, a morphism between chain complexes is a degree 0 map between the underlying graded modules which is compatible with the differentials. By equipping graded R-modules with the zero differential, the category of graded R-modules embeds fully faithfully into the category of chain complexes over R.

- The shift $C[k]$ of a chain complex C has the module C_{n-k} in degree n and is equipped with the differential $(-1)^k d^C$.
- A degree k chain map $C \to D$ is by definition a chain map $C[k] \to D$.
- Given chain complexes C and D of left R-modules, we obtain a chain complex $\underline{\mathrm{Hom}}_R(C,D)$ of abelian groups by setting
$$\underline{\mathrm{Hom}}_R(C,D)_n := \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(C_p, D_{p+n})$$
and equipping this graded abelian group with the differential
$$d_n^{\underline{\mathrm{Hom}}_R(C,D)}((\varphi_p)_p) := \left(d^D_{p+n} \circ \varphi_p + (-1)^{n+1} \varphi_{p-1} \circ d^C_p\right)_p .$$
Then $H_n(\underline{\mathrm{Hom}}_R(C,D))$ is isomorphic to the abelian group of chain homotopy classes of degree n chain maps.
- Given a chain complex of right R-modules C and a chain complex of left R-modules D, the tensor product $C \otimes_R D$ is defined to be the chain complex of abelian groups which is given by the graded module
$$(C \otimes_R D)_n := \bigoplus_{p+q=n} C_p \otimes_R D_q$$
together with the differential induced by the maps
$$C_p \otimes_R D_q \xrightarrow{d^C_p \otimes \mathrm{id} + (-1)^p \, \mathrm{id} \otimes d^D_q} (C_{p-1} \otimes_R D_q) \oplus (C_p \otimes D_{q-1}).$$
For every chain complex of abelian groups E and every chain complex D of left R-modules, there is an evaluation chain map
$$\mathrm{ev}\colon \underline{\mathrm{Hom}}(D,E) \otimes_R D \to E, \quad (\varphi_p)_p \otimes x \mapsto \varphi_{|x|}(x)$$
which induces an isomorphism
$$\mathrm{Hom}_R(C, \underline{\mathrm{Hom}}(D,E)) \cong \mathrm{Hom}(C \otimes_R D, E) \tag{1.0.1}$$
by sending a chain map $\varphi\colon C \to \underline{\mathrm{Hom}}(D,E)$ to the chain map
$$C \otimes_R D \xrightarrow{\varphi \otimes \mathrm{id}} \underline{\mathrm{Hom}}(D,E) \otimes_R D \xrightarrow{\mathrm{ev}} E.$$
In other words, ev is a counit which exhibits $\underline{\mathrm{Hom}}(D,-)$ as a right adjoint of $- \otimes_R D$.

  Assume in addition that R is commutative. Then $C \otimes_R D$ is even a chain complex of R-modules, and this construction refines to a symmetric monoidal structure on the category of chain complexes over R whose symmetry isomorphism is given by

$$\tau\colon C \otimes_R D \xrightarrow{\cong} D \otimes_R C, \quad x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.$$

Let us now return to the situation that C is a chain complex of right R-modules and D is a chain complex of left R-modules. Let $Z(C)$ and $Z(D)$ be the graded modules of cycles in C and D, respectively. The canonical map $Z(C) \otimes_R Z(D) \to C \otimes_R D$ factors over $Z(C \otimes_R D)$ since

$$d^{C\otimes D}(x \otimes y) = d^C(x) \otimes y + (-1)^p x \otimes d^D(y) = 0$$

for $x \in Z_p(C)$ and $y \in Z_q(D)$. The induced map $Z(C) \otimes_R B(D) \to H(C \otimes_R D)$ is trivial because

$$d^{C\otimes D}(x \otimes y') = d^C(x) \otimes y' + (-1)^p x \otimes d^D(y') = (-1)^p x \otimes y$$

whenever $y = d^C(y')$ for some $y' \in D_{q+1}$. Since $Z(C) \otimes_R -$ preserves cokernels, we obtain an induced map

$$Z(C) \otimes_R H(D) \to H(C \otimes_R D).$$

By a completely analogous argument, this map induces a map

$$\mu\colon H(C) \otimes_R H(D) \to H(C \otimes_R D). \tag{1.0.2}$$

1.0.3. **Remark.** If R is a commutative ring, the maps μ can be used to refine homology (considered as a functor from chain complexes to the category of graded modules) to a lax symmetric monoidal functor.

If one was overly optimistic, one would try to show that μ is an isomorphism. However, this runs into problems quickly. Even if D is concentrated in degree 0, it is unclear whether either $Z(C) \otimes_R D \to Z(C \otimes_R D)$ or $B(C) \otimes_R D \to B(C \otimes_R D)$ is an isomorphism since tensoring with D need not preserves kernels. If $- \otimes_R D$ did preserve kernels, it would follow that μ is an isomorphism. This suggests that we should be interested in the class of modules M for which $- \otimes_R M$ is exact, and try to understand the failure of this functor to be exact in general.

1.1. **Derived functors.**

Throughout this section, fix a ring R. Recall that an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is

(1) *exact* if it preserves exact sequences;
(2) *left exact* if $0 \to F(A_0) \to F(A_1) \to F(A_2)$ is exact for every exact sequence $0 \to A_0 \to A_1 \to A_2 \to 0$ in $\mathcal{A}$;
(3) *right exact* if $F(A_0) \to F(a_1) \to F(a_2) \to 0$ is exact for every exact sequence $0 \to A_0 \to A_1 \to A_2 \to 0$ in $\mathcal{A}$.

Note that F is left exact if and only if F^{op} is right exact.

1.1.1. **Lemma.** *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is left exact if and only if it preserves kernels.*

Proof. Note that a sequence $0 \to A_0 \xrightarrow{i} A_1 \xrightarrow{f} A_2$ is exact if and only if i is a kernel of f.

If F preserves kernels, this implies that $0 \to F(A_0) \to F(A_1) \to F(A_2)$ is exact for every exact sequence $0 \to A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2 \to 0$.

Conversely, let $0 \to \ker(f) \xrightarrow{i} A_1 \xrightarrow{f} A_2$ be given and assume that F is left exact. Then $0 \to \ker(f) \xrightarrow{i} A_1 \xrightarrow{f} \mathrm{img}(f) \to 0$ is exact, so $0 \to F(\ker(f)) \xrightarrow{F(i)} F(A_1) \to F(\mathrm{img}(f))$ is also exact. Moreover, left exactness implies that F preserves monomorphisms since a morphism $f\colon A \to A'$ is a monomorphism if and only if $0 \to A \xrightarrow{f} A'$ is exact. Hence $F(\mathrm{img}(f)) \to F(A_2)$ is a monomorphism. Therefore, for $g\colon b \to F(A_1)$, the composite $b \to F(\mathrm{img}(f))$ is zero if and only if the composite $b \to F(A_2)$ is zero. This shows that $F(i)$ is a kernel of $F(f)$. □

1.1.2. **Definition.** A left R-module M is *flat* if $- \otimes_R M\colon \mathrm{Mod}\text{-}R \to \mathrm{Ab}$ is an exact functor.

1.1.3. **Lemma.** *Let C be a chain complex of right R-modules and let M be a flat left R-module. Then*

$$\mu\colon H_n(C) \otimes_R M \to H_n(C \otimes_R M)$$

is an isomorphism.

Proof. Exercise. □

Ultimately, we will want to get rid of the flatness assumption on M, but let us first try to get an idea how we can recognise flat modules.

1.1.4. **Example.** Every free right R-module is flat. Since retracts of short exact sequences are short exact, this implies that every projective right R-module is flat.

Example 1.1.4 can be generalised further using the following notion.

1.1.5. **Definition.** A category I is *filtered* if the following holds:

(1) for each pair of objects i_1 and i_2 in I, there exists some $i \in I$ and morphisms $i_1 \to i$ and $i_2 \to i$;
(2) for each pair of parallel morphisms $i \overset{f}{\underset{g}{\rightrightarrows}} j$ there exists a morphism $h\colon j \to k$ such that $hf = hg$.

A *filtered colimit* is a colimit whose indexing diagram (without the cone point) is a filtered category.

1.1.6. **Example.** Recall that a poset P is *directed* if for all $p_1, p_2 \in P$ there exists some $p \in P$ with $p_1 \leq p$ and $p_2 \leq p$. Directed posets are filtered categories.

1.1.7. **Lemma.** *Let I be a small filtered category and let $D\colon I \to \mathrm{Set}$ be a diagram. Set $X := \coprod_{i\in I} D(i)$. Define*

$$(i, x) \sim (j, y) \quad :\Longleftrightarrow \quad \text{There are } f\colon i \to k,\ g\colon j \to k \text{ such that } D(f)(x) = D(g)(y).$$

(1) *$\sim$ is an equivalence relation.*
(2) *The quotient $X/{\sim}$, together with the canonical maps $D(i) \to X/{\sim}$, is a colimit of D.*
(3) *The forgetful functor R-Mod $\to$ Set preserves and detects filtered colimits.*

Proof. Exercise. □

1.1.8. **Lemma.** *Let I be a filtered category and let R be a ring. Then*

$$\operatorname*{colim}_I\colon \mathrm{Fun}(I, R\text{-Mod}) \to R\text{-Mod}$$

is an exact functor.

Proof. By Lemma 1.1.1, we have to show that colim_I preserves kernels and cokernels. Since colimits commute among each other, colim_I always preserves cokernels (regardless of the diagram shape). So we only have to show that colim_I preserves kernels. Consider a diagram $\{0 \to K_i \to M_i \xrightarrow{f_i} N_i\}_{i\in I}$ of exact sequences. We apply Lemma 1.1.7 to model the colimits. An element in $\mathrm{colim}_{i\in I} M_i$ is represented by some element $x \in M_i$. If its image in $\mathrm{colim}_{i\in I} N_i$ is trivial, there exists some $\alpha\colon i \to j$ such that $f_j(\alpha_*(x)) = \alpha_*(f_i(x)) = 0 \in N_j$. Then $\alpha_*(x) \in K_j$ as well, so x lifts to $\mathrm{colim}_{i\in I} K_i$.

Similarly, an element in $\operatorname{colim}_{i\in I}$ is represented by some $x \in K_i$. If its image in $\operatorname{colim}_{i\in I} M_i$ is trivial, there exists $\alpha\colon i \to j$ with $\alpha_*(x) = 0 \in M_j$. This implies $\alpha_*(x) = 0$, so x also represents zero in $\operatorname{colim}_{i\in I} K_i$. □

1.1.9. **Proposition** (Lazard). *Let M be a left R-module. The following are equivalent:*

(1) *M is a filtered colimit of free modules;*
(2) *M is a filtered colimit of projective modules;*
(3) *M is flat.*

Proof. The implication (1) ⇒ (2) is obvious. Since the tensor product preserves colimits in each variable separately, Lemma 1.1.8 shows that directed colimits of flat modules are flat. Hence (2) ⇒ (3) follows from Example 1.1.4.

For the implication (3) ⇒ (1), let us first consider an arbitrary R-module M. Define a category $E(M)$ whose objects are R-linear maps $\alpha\colon R^k \to M$ and whose morphisms correspond to commutative triangles

$$\begin{tikzcd} R^k \arrow[d, "f"'] \arrow[dr, "\alpha"] & \\ R^l \arrow[r, "\beta"'] & M \end{tikzcd}$$

We claim that $M \cong \operatorname{colim}_{\alpha\colon R^k\to M} R^k$, where the colimit is indexed by $E(M)$. To see this, it suffices to show that the induced homomorphism

$$\kappa\colon \operatorname{Hom}_R(M, N) \to \lim_{\alpha} \operatorname{Hom}_R(R^k, N)$$

of abelian groups is an isomorphism for every R-module N. Since every element $m \in M$ defines an R-linear map $\alpha_m\colon R \to M$, the map κ is clearly injective.

Let $(n_\alpha)_\alpha \in \lim_\alpha \operatorname{Hom}_R(R^k, N)$ be arbitrary. We show that the assignment $m \mapsto n_{\alpha_m}$ is an R-linear map. For m and m' in M, the equality $\alpha_{m+m'} = \alpha_m + \alpha_{m'}\colon R \to M$ implies that

$$n_{\alpha_{m+m'}} = n_{\alpha_m+\alpha_{m'}} = n_{(\alpha_m\ \alpha'_m)\circ\Delta} = n_{(\alpha_m\ \alpha'_m)} \circ \Delta.$$

The map $n_{(\alpha_m\ \alpha'_m)}\colon R^2 \to N$ is determined by its restriction along the two obvious summand inclusions. Since $n_{(\alpha_m\ \alpha'_m)} \circ \mathrm{inc}_1 = n_{\alpha_m}$ and $n_{(\alpha_m\ \alpha'_m)} \circ \mathrm{inc}_2 = n_{\alpha_{m'}}$, we conclude that

$$n_{(\alpha_m\ \alpha'_m)} = (n_{\alpha_m}\ n_{\alpha_{m'}}),$$

which implies $n_{\alpha_{m+m'}} = n_{\alpha_m} + n_{\alpha_{m'}}$. The equality $n_{\alpha rm} = rn_{\alpha_m}$ for $r \in R$ and $m \in M$ follows similarly. From the preceding discussion, it also follows that each element $n_\alpha\colon R^k \to N$ is determined by its restrictions to the individual summands, so $(m \mapsto n_{\alpha_m})$ is a preimage of $(n_\alpha)_\alpha$ under κ.

This reduces the proposition to showing that $E(M)$ is filtered if M is flat. Given $\alpha\colon R^k \to M$ and $\beta\colon R^l \to M$, we can form the commutative diagram

$$\begin{tikzcd} R^k \arrow[d, "\mathrm{inc}"'] \arrow[dr, "\alpha"] & \\ R^k \oplus R^l \arrow[r, "(\alpha\ \beta)"] & M \\ R^l \arrow[u, "\mathrm{inc}"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Thus we only have to show that for every two morphisms $R^k \underset{g}{\overset{f}{\rightrightarrows}} R^l$, if there exists some $\alpha\colon R^l \to M$ with $\alpha f = \alpha g$, we can factor $\alpha = \beta \circ h$ for some $h\colon R^l \to R^n$ and $\beta\colon R^n \to M$ such that $hf = hg$. By replacing f with $f-g$, we may assume $g = 0$.

An induction on k reduces the claim to the case $k = 1$. Then f corresponds to a tuple $(r_1, \dots, r_l)$, the map α corresponds to a tuple $(m_1, \dots, m_l)$, and the assumption asserts that $\sum_i r_i m_i = 0 \in M$. Let $I := (r_1, \dots, r_l) \subseteq R$ be the right ideal generated by these elements. Since M is flat, the induced map $I \otimes_R M \to R \otimes_R M \cong M$ is injective. In particular, we have that $\sum_{i=1}^{l} r_i \otimes m_i = 0 \in I \otimes_R M$. Again by flatness, the sequence

$$0 \to K \otimes_R M \to R^l \otimes_R M \xrightarrow{\pi \otimes_R M} I \otimes_R M \to 0$$

is exact, where $\pi(e_i) = r_i$ for the standard basis vectors e_i. Then $\sum_{i=1}^{l} e_i \otimes m_i \in R^l \otimes_R M$ maps to zero under $\pi \otimes_R M$, so this element lies in $K \otimes_R M$. Write $\sum_{i=1}^{l} e_i \otimes m_i = \sum_{j=1}^{n} v_j \otimes m'_j$ for appropriate $v_j \in K$ and $m'_j \in M$. As $K \subseteq R^l$, we can express $v_j = \sum_{i=1}^{l} s_{ji} e_i$. Then

$$\sum_{i=1}^{l} e_i \otimes m_i = \sum_{j=1}^{n} (\sum_{i=1}^{l} s_{ji} e_i) \otimes m'_j = \sum_{i=1}^{l} e_i \otimes (\sum_{j=1}^{n} s_{ji} m'_j).$$

Using $R^l \otimes_R M \cong M^n$, we conclude that $m_i = \sum_{j=1}^{n} s_{ji} m'_j$. The given data define R-linear maps

$$h := (s_{j,i})_{1 \le j \le n, 1 \le i \le l} \colon R^l \to R^n \quad \text{and} \quad \beta := (m'_j)_{1 \le j \le n} \colon R^n \to M.$$

Then $\beta \circ h = f$. Moreover, since each v_j is an element in K, we also have $\sum_{i=1}^{l} s_{ji} r_i = 0$, which implies that $h \circ f = 0$. □

1.1.10. **Example.** It follows from Proposition 1.1.9 that $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module. More generally, if we denote for a set S of primes by $\mathbb{Z}[S^{-1}]$ the subring of $\mathbb{Q}$ containing those rational numbers which can be represented by a fraction whose denominator is only divisible by members of S, then $\mathbb{Z}[S^{-1}]$ is flat as a $\mathbb{Z}$-module.

We now introduce a formalism which tries to express that we have a systematic way of measuring the failure of a functor F_0 to be left exact.

1.1.11. **Definition.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A $\mathcal{B}$-valued *δ-functor* on $\mathcal{A}$ is a sequence of additive functors

$$\{F_n \colon \mathcal{A} \to \mathcal{B}\}_{n \ge 0}$$

together with a choice of maps $\{\partial_n \colon F_n(C) \to F_{n-1}(A)\}_{n \ge 1}$ for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that the following holds:

(1) for each short exact sequence $0 \to A \to B \to C \to 0$ and each $n \ge 0$ the sequence

$$F_n(B) \to F_n(C) \xrightarrow{\partial_n} F_{n-1}(A) \to F_{n-1}(B) \to F_{n-1}(C)$$

is exact, where $F_{-1} := 0$;

(2) for each morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

and each $n \ge 1$ the square

$$\begin{tikzcd} F_n(C) \arrow[r, "\partial_n"] \arrow[d, "F_n(h)"'] & F_{n-1}(A) \arrow[d, "F_{n-1}(f)"] \\ F_n(C') \arrow[r, "\partial_n"] & F_{n-1}(A') \end{tikzcd}$$

commutes.

A *morphism* of δ-functors $\tau\colon \{F_n\}_n \Rightarrow \{G_n\}_n$ is a sequence of natural transformations $\{\tau_n\colon F_n \Rightarrow G_n\}_n$ such that the square

$$\begin{tikzcd} F_n(C) \arrow[r, "\partial_n"] \arrow[d, "\tau_n"'] & F_{n-1}(A) \arrow[d, "\tau_{n-1}"] \\ G_n(C) \arrow[r, "\partial_n"] & G_{n-1}(A) \end{tikzcd}$$

commutes for each short exact sequence $0 \to A \to B \to C \to 0$ and each $n \geq 1$.

A δ-functor $\{F_n\}_n$ is *universal* if for every δ-functor $\{G_n\}_n$ and every natural transformation $\tau_0\colon G_0 \Rightarrow F_0$ there exists a unique morphism of δ-functors $\tau\colon \{G_n\}_n \Rightarrow \{F_n\}_n$ extending τ_0.

1.1.12. **Remark.** If a right exact functor $F_0\colon \mathcal{A} \to \mathcal{B}$ admits an extension to a universal δ-functor, then such an extension is unique up to unique isomorphism.

1.1.13. **Proposition.** *Let $\{F_n\}_n$ be a δ-functor. Suppose that F_n is coeffaceable for $n \geq 1$, ie for every $A \in \mathcal{A}$ there exists an epimorphism $p\colon P \twoheadrightarrow A$ such that $F_n(p) = 0$. Then $\{F_n\}_n$ is universal.*

Proof. Suppose that $\tau\colon \{G_n\}_n \Rightarrow \{F_n\}_n$ is a morphism of δ-functors. Let $A \in \mathcal{A}$ be arbitrary and choose an epimorphism $p\colon P \twoheadrightarrow A$ such that $F_n(p) = 0$. In the commutative square

$$\begin{tikzcd} G_n(A) \arrow[r, "\partial_n"] \arrow[d, "\tau_n"'] & G_{n-1}(\ker(p)) \arrow[d, "\tau_{n-1}"] \\ F_n(A) \arrow[r, "\partial_n"] & F_{n-1}(\ker(p)) \end{tikzcd}$$

the bottom horizontal map is a monomorphism. Hence τ_n is uniquely determined by the equation $\partial_n \circ \tau_n = \tau_{n-1} \circ \partial_n$, and it follows by induction that τ is completely determined by τ_0.

At the same time, this also provides a recipe to extend a given natural transformation $\tau_0\colon G_0 \to F_0$ to a morphism of δ-functors. Given $A \in \mathcal{A}$, choose p as above and note that

$$0 \to F_n(A) \xrightarrow{\partial_n} F_{n-1}(\ker(p)) \xrightarrow{F_{n-1}(i)} F_{n-1}(P)$$

is exact. If $\tau_{n-1}\colon G_{n-1} \Rightarrow F_{n-1}$ is already defined, then $F_{n-1}(i) \circ \tau_{n-1} \circ \partial_n = 0$, so $\tau_{n-1} \circ \partial_n$ lifts uniquely to a morphism $\tau_n\colon G_n(A) \to F_n(A)$.

We have to show that τ_n is independent of the choice of p, and that performing this choice for each A yields a natural transformation $\tau_n\colon G_n \Rightarrow F_n$. Note that compatibility with the boundary maps ∂_n holds by construction.

If $q\colon Q \twoheadrightarrow A$ is another epimorphism with $F_n(q) = 0$, then $p + q\colon P \oplus Q \twoheadrightarrow A$ is also an epimorphism with $F_n(p+q) = 0$. By naturality, τ_n also lifts the composite

$$G_n(A) \xrightarrow{\partial_n} G_{n-1}(\ker(p+q)) \xrightarrow{\tau_{n-1}} F_{n-1}(\ker(p+q)).$$

Since the choice of lift is unique, it follows by symmetry that p and q give rise to the same choice of τ_n.

If $f\colon A \to B$ is a morphism in $\mathcal{A}$, pick epimorphisms $p\colon P \twoheadrightarrow A$ and $q\colon Q \twoheadrightarrow B$ with $F_n(p) = 0$ and $F_n(q) = 0$. Then $fp + q\colon P \oplus Q \to B$ is also an epimorphism with $F_n(fp+q) = 0$. Since the square

$$\begin{tikzcd} P \arrow[r, two heads, "p"] \arrow[d, "\mathrm{inc}"'] & A \arrow[d, "f"] \\ P \oplus Q \arrow[r, two heads, "fp+q"] & B \end{tikzcd}$$

commutes, another diagram chase shows that τ_n is natural. $\square$

1.1.14. **Definition.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. A *left derived functor* of F is a universal δ-functor $\{\mathrm{L}F_n\}_n$ together with a natural isomorphism $F_0 \cong F$.

To prove the existence of some left derived δ-functors, we rely on the following two lemmas.

1.1.15. **Lemma** (Horseshoe lemma)**.** *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence and let $P^1_\bullet \to A_1$ and $P^3_\bullet \to A_3$ be projective resolutions. Then there exists a projective resolution $P^2_\bullet \to A_2$ which fits into a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & P^1_\bullet \arrow[r] \arrow[d] & P^2_\bullet \arrow[r] \arrow[d] & P^3_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & 0
\end{tikzcd}$$

in which both rows are exact. In particular, $P^2_n \cong P^1_n \oplus P^3_n$ for all n.

Proof. We construct the required resolution by induction. Setting $P^i_{-1} := A_i$, assume that the resolution has been constructed up to degree n, so we have a diagram

$$\begin{tikzcd}
 & P^1_{n+1} \arrow[d, "p^1_{n+1}"'] & & P^3_{n+1} \arrow[d, "p^3_{n+1}"] & \\
0 \arrow[r] & P^1_n \arrow[r, "f_n"] \arrow[d, "p^1_n"'] & P^2_n \arrow[r, "g_n"] \arrow[d, "p^2_n"] & P^3_n \arrow[r] \arrow[d, "p^3_n"] & 0 \\
0 \arrow[r] & P^1_{n-1} \arrow[r, "f_{n-1}"] & P^2_{n-1} \arrow[r, "g_{n-1}"] & P^3_{n-1} \arrow[r] & 0
\end{tikzcd}$$

in which the lower row is exact. Since P^3_{n+1} is projective, there exists a lift $q^3_{n+1}\colon P^3_{n+1} \to P^2_n$ of p^3_{n+1} along g_n. Then

$$g_{n-1}p^2_n q^3_{n+1} = p^3_n g_n q^3_{n+1} = p^3_n p^3_{n+1} = 0,$$

so $p^2_n q^3_{n+1}$ lifts along f_{n-1} to a map $e'\colon P^3_{n+1} \to P^1_{n-1}$. It is easy to check that $p^1_{n-1}e = 0$, so e' maps to the image of p^1_n by exactness. Using projectivity of P^3_{n+1} once more, we obtain a lift of e' along p^1_n to a map $e\colon P^3_{n+1} \to P^2_n$. In particular, $p^2_n f_n e = p^2_n q^3_{n+1}$. Define

$$p^2_{n+1} := f_n p^1_{n+1} + (q^3_{n+1} - f_n e)\colon P^2_{n+1} := P^1_{n+1} \oplus P^3_{n+1} \to P^2_n.$$

Then the evident inclusion and projection maps $f_{n+1}\colon P^1_{n+1} \to P^2_{n+1}$ and $g_{n+1}\colon P^2_{n+1} \to P^3_{n+1}$ extend the given chain maps, and

$$p^2_n p^2_{n+1} = p^2_n f_n p^1_{n+1} + (q^3_{n+1} - f_n e) = f_{n-1}p^1_n p^1_{n+1} + p^2_n(q^3_{n+1} - f_n e) = 0.$$

The only thing left to show is exactness in degree n, which follows by inspecting the long exact sequence in homology induced by this short exact sequence of (truncated) chain complexes. □

1.1.16. **Lemma.** *Let $\mathcal{A}$ be an abelian category.*

(1) *An object $i\colon P \to Q$ in* $\mathrm{Fun}([1], \mathcal{A})$ *is projective if and only if Q is projective and i is isomorphic to the inclusion of a direct summand.*
(2) *If $\mathcal{A}$ has enough projectives, the same is true for* $\mathrm{Fun}([1], \mathcal{A})$.

Proof. Suppose that Q is projective and that f is a direct summand inclusion $P \rightarrowtail P \oplus Q'$ and consider a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, tail] & M \arrow[d, "h"] & M' \arrow[l, two heads, "p"'] \arrow[d, "h'"] \\
P \oplus Q' \arrow[r, "hf+g"] & N & N' \arrow[l, two heads, "q"']
\end{tikzcd}$$

Since both P and Q' are projective, f lifts along p to a morphism $f'\colon P \to M'$ and g lifts along q to a morphism $g'\colon Q' \to N'$. Then

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, tail] & M' \arrow[d, "h'"] \\ P \oplus Q' \arrow[r, "h'f'+g'"] & N' \end{tikzcd}$$

provides the required lift.

Conversely, if i is projective, it follows that Q is projective by considering for every epimorphism $q\colon N' \twoheadrightarrow N$ and $g\colon Q \to N$ in $\mathcal{A}$ the lifting problem

$$\begin{tikzcd} P \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & P \arrow[d, "gi"] & P \times_N N' \arrow[l, two heads] \arrow[d] \\ Q \arrow[r, "g"] & N & N' \arrow[l, two heads, "q"'] \end{tikzcd}$$

Moreover, the lifting problem

$$\begin{tikzcd} P \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & P \arrow[d] & P \arrow[l, two heads, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\ Q \arrow[r] & 0 & P \arrow[l, two heads] \end{tikzcd}$$

implies that i admits a retraction.

If $f\colon A \to B$ is a morphism in $\mathcal{A}$, pick epimorphisms $p\colon P \twoheadrightarrow A$ and $q\colon Q \twoheadrightarrow B$. Then $fp + q\colon P \oplus Q \to B$ is also an epimorphism and the commutative square

$$\begin{tikzcd} P \arrow[r, two heads, "p"] \arrow[d, "\mathrm{inc}"'] & A \arrow[d, "f"] \\ P \oplus Q \arrow[r, two heads, "fp+q"] & B \end{tikzcd}$$

represents an epimorphism in $\mathrm{Fun}([1], \mathcal{A})$ from a projective object to f. $\square$

1.1.17. **Theorem.** *If $\mathcal{A}$ has enough projectives (ie for every $A \in \mathcal{A}$ there exists an epimorphism $P \twoheadrightarrow A$ with P projective), every right exact functor $F\colon \mathcal{A} \to \mathcal{B}$ has a left derived functor $\{\mathrm{L}F_n\}_n$. Moreover, the value of $\mathrm{L}F_n$ at $A \in \mathcal{A}$ may be computed by the formula*

$$\mathrm{L}F_n(A) \cong H_n(F(P_\bullet)),$$

where $P_\bullet \to A$ is a projective resolution of A.

Proof. For each $A \in \mathcal{A}$, choose a projective resolution

$$\ldots \to P_2^A \to P_1^A \to P_0^A \twoheadrightarrow A.$$

Define

$$\mathrm{L}F_n(A) := H_n(F(P_\bullet^A)).$$

The fundamental theorem of homological algebra implies that this is independent (up to canonical isomorphism) of the choice of projective resolution.

By the fundamental theorem of homological algebra, we obtain for each morphism $f\colon A \to B$ in $\mathcal{A}$ a chain map $\varphi(f)\colon P_\bullet^A \to P_\bullet^B$, unique up to chain homotopy, such that

$$\begin{tikzcd} P_\bullet^A \arrow[r] \arrow[d, "\varphi(f)"'] & A \arrow[d, "f"] \\ P_\bullet^B \arrow[r] & B \end{tikzcd}$$

commutes. In particular, this extends each $\mathrm{L}F_n$ to a functor $\mathrm{L}F_n \colon \mathcal{A} \to \mathcal{B}$. Since F is right exact, the augmentation $P_0^A \to A$ induces a natural isomorphism

$$\mathrm{L}F_0(A) = H_0(F(P_\bullet^A)) \cong \operatorname{coker}(F(P_1^A) \to F(P_0^A)) \cong F(A).$$

Let now $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. The horseshoe lemma provides a projective resolution $Q_\bullet \to B$ which sits in a short exact sequence $0 \to P_\bullet^A \to Q_\bullet \to P_\bullet^C \to 0$ of projective resolutions which is degreewise split. Hence applying F still yields a short exact sequence of chain complexes, so we can define $\partial_n \colon \mathrm{L}F_n(C) \to \mathrm{L}F_n(A)$ as the associated boundary map in homology.

Naturality of the boundary map follows by applying the horseshoe lemma in the abelian category $\mathrm{Fun}([1], \mathcal{A})$ of arrows in $\mathcal{A}$; note that Lemma 1.1.16 not only shows that $\mathrm{Fun}([1], \mathcal{A})$ has enough projectives, but also that a projective resolution in $\mathrm{Fun}([1], \mathcal{A})$ is pointwise a projective resolution.

This defines a δ-functor $\{\mathrm{L}F_n\}_n$. Now observe that $\mathrm{L}F_n(P) = 0$ for $n \geq 1$ and P projective since $P \xrightarrow{\mathrm{id}} P$ is a projective resolution of P. Since $\mathcal{A}$ has enough projectives, this shows that $\mathrm{L}F_n$ is coeffaceable for $n \geq 1$, and universality follows from Proposition 1.1.13. ☐

Theorem 1.1.17 allows us to make the following definition.

1.1.18. **Definition.** The left derived functor of

$$- \otimes_R M \colon \text{Mod-}R \to \mathrm{Ab}$$

is denoted by

$$\left\{\mathrm{Tor}_n^R(-, M)\right\}_n.$$

1.1.19. **Remark.** Applying universality, we find that each Tor_n^R refines to a functor

$$\mathrm{Tor}_n^R \colon \text{Mod-}R \times R\text{-Mod} \to \mathrm{Ab}.$$

Consider the functors

$$\mathrm{Tor}_n^R(N, -) \colon R\text{-Mod} \to \mathrm{Ab}.$$

Pick a projective resolution $P_\bullet \to N$ and consider a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in R-Mod. Since each P_n is projective, we obtain a short exact sequence of chain complexes

$$0 \to P_\bullet \otimes_R M_1 \to P_\bullet \otimes_R M_2 \to P_\bullet \otimes_R M_3 \to 0,$$

which gives rise to natural boundary maps $\partial_n \colon \mathrm{Tor}_n^R(N, M_3) \to \mathrm{Tor}_{n-1}^R(N, M_1)$. As we will see in Lemma 1.1.20 (4) below, $\mathrm{Tor}_n^R(N, -)$ is coeffaceable for $n \geq 1$ since it vanishes on projective modules. Hence we have built a universal δ-functor $\{\mathrm{Tor}_n^R(N, -)\}_n$ with $\mathrm{Tor}_0^R(N, -) \cong N \otimes_R -$. It follows from Theorem 1.1.17 that we may compute $\mathrm{Tor}_n^R(N, M)$ equally well by choosing a projective resolution of M, tensoring with N, and then taking homology. In particular,

$$\mathrm{Tor}_n^R(N, M) \cong \mathrm{Tor}_n^R(M, N)$$

when R is commutative.

1.1.20. **Lemma.**

(1) *the canonical map*
$$\bigoplus_i \operatorname{Tor}_n^R(M_i, N) \to \operatorname{Tor}_n^R(\bigoplus_i M_i, N)$$
is an isomorphism;

(2) *for every filtered category I, the canonical map*
$$\operatorname*{colim}_{i\in I} \operatorname{Tor}_n^R(M_i, N) \to \operatorname{Tor}_n^R(\operatorname*{colim}_{i\in I} M_i, N)$$
is an isomorphism;

(3) *if every R-module admits a projective resolution of length $\leq k$, then*
$$\operatorname{Tor}_n^R(M, N) = 0$$
for all $n > k$;

(4) *for every flat R-module M, we have*
$$\operatorname{Tor}_n^R(-, M) = 0$$
for all $n \geq 1$;

(5) *if submodules of projective modules over R are projective, then $\operatorname{Tor}_n^R(M, N) = 0$ for all $n \geq 2$;*

(6) *if R is an integral domain and $r \neq 0$, then*
$$\operatorname{Tor}_1^R(R/(r), N) \cong \{y \in N \mid r \cdot y = 0\};$$

(7) $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/l) \cong \mathbb{Z}/\gcd(k,l)$ *for $k, l \neq 0$.*

Proof. The third assertion follows from Theorem 1.1.17.

If M is flat, $F := - \otimes_R M$ is exact. Setting $\mathrm{L}F_n := 0$ for $n \geq 1$ provides a universal δ-functor by virtue of Proposition 1.1.13, showing assertion 4.

For the first assertion, note that $\bigoplus_i (M_i \otimes_R -) \to (\bigoplus_i M_i) \otimes_R N$ is an isomorphism, and that $\bigoplus_i \operatorname{Tor}_n^R(M_i, -)$ is a coeffaceable functor since it vanishes on projectives. Hence $\{\bigoplus_i \operatorname{Tor}_n^R(M_i, -)\}_{n\geq 0}$ and $\{\operatorname{Tor}_n^R(\bigoplus_i M_i, -)\}_{n\geq 0}$ are left derived functors of naturally isomorphic functors, which yields an isomorphism of δ-functors.

For the second assertion, we observe similarly that
$$\operatorname*{colim}_i (M_i \otimes_R -) \to \left(\operatorname*{colim}_i M_i\right) \otimes_R -$$
is an isomorphism, and that $\operatorname{colim}_i \operatorname{Tor}_n^R(M_i, -)$ is coeffaceable since it vanishes on projectives.

For the fifth assertion, note that every module has a projective resolution of length 1.

If R has no zero divisors and $r \neq 0$, $R \xrightarrow{r\cdot -} R \to R/(r)$ is a projective resolution of $R/(r)$, so
$$\operatorname{Tor}_1^R(R/(r), N) \cong H_1(N \xrightarrow{r\cdot -} N) = \{n \in N \mid r \cdot n = 0\}. \qquad \square$$

For the last assertion, let g denote the greatest common divisor of k and l, set $T := \{x \in \mathbb{Z}/l \mid kx = 0\}$ and consider the homomorphism
$$\mathbb{Z} \to T, \quad 1 \mapsto \frac{l}{g}.$$
Then $g \cdot \frac{l}{g} = l = 0 \in \mathbb{Z}/l$, so we obtain an induced homomorphism $f\colon \mathbb{Z}/g \to T$. We claim that f is an isomorphism. Since $\frac{k}{g}$ and $\frac{l}{g}$ are coprime, there exist $a, b \in \mathbb{Z}$

with $a\frac{k}{g} + b\frac{l}{g} = 1$. For $x \in \mathbb{Z}$ represent an element in T. Then there exists $y \in \mathbb{Z}$ with $kx = ly$. In particular, $\frac{k}{g}x = \frac{l}{g}y$, and it follows that

$$x = \left(a\frac{k}{g}x + b\frac{l}{g}y\right) = a\frac{l}{g}y + b\frac{l}{g}x = f(ay + bx),$$

showing that f is surjective. If $f(x) = 0$, then $\frac{l}{g}x = ly$ for some $y \in \mathbb{Z}$. Dividing by $\frac{l}{g}$, we obtain $x = gy$, so $x = 0 \in \mathbb{Z}/g$.

1.2. **The Künneth formula.** The Tor-functor(s) allow us to measure precisely how much the map

$$\mu \colon H(C) \otimes_R H(D) \to H(C \otimes_R D)$$

fails to be an isomorphism, at least under some favourable conditions. To reduce the size of the formulas we have to write, we introduce the following shorthand. Given a functor $F \colon \mathcal{A} \times \mathcal{A}' \to \mathcal{B}$ which is additive in both variables, define for chain complexes C in $\mathcal{A}$ and D in $\mathcal{A}'$ a new chain complex $F(C, D)$ by

$$F(C, D)_n := \bigoplus_{p+q=n} F(C_p, D_q)$$

$$d^{F(C,D)} \colon F(C_p, D_q) \xrightarrow{(F(d_p^C, \mathrm{id}), (-1)^p F(\mathrm{id}, d_q^D))} F(C_{p-1}, D_p) \oplus F(C_p, D_{q-1}).$$

This generalises the construction of the tensor product of chain complexes, which is the special case $F = - \otimes_R -$.

1.2.1. **Definition.** A ring R is *hereditary* if submodules of projective modules are projective.

1.2.2. **Theorem** (Künneth formula)**.** *Let R be a hereditary ring and let C and D be chain complexes over R.*

If C is degreewise projective, then there exists for each $n \geq 0$ a natural short exact sequence

$$0 \to H(C) \otimes_R H(D) \to H(C \otimes_R D) \to \mathrm{Tor}_1^R(H(C), H(D))[1] \to 0.$$

Proof. Let Z and B denote the graded modules of cycles and boundaries in C, respectively. Then we have a short exact sequence

$$0 \to Z \to C \to B[1] \to 0.$$

Since B is degreewise projective, $\mathrm{Tor}_1^R(B_q, -) = 0$ for all q by Lemma 1.1.20 (4) and Remark 1.1.19. Hence tensoring with D yields a short exact sequence

$$0 \to Z \otimes_R D \to C \otimes_R D \to B[1] \otimes_R D \to 0.$$

Observing that Z is also degreewise projective and hence flat, we apply Lemma 1.1.3 to obtain the exact sequence

$$B \otimes_R H(D) \xrightarrow{\partial} Z \otimes_R H(D) \to H(C \otimes_R D) \to (B \otimes_R H(D))[1] \xrightarrow{\partial[1]} (Z \otimes_R H(D))[1]$$

We claim that the kernel and cokernel of the boundary map ∂ yield the required terms. Tracing through the definition of the boundary map, one finds that ∂ is induced by the inclusion $B \to Z$. It follows directly that

$$\mathrm{coker}(\partial) \cong H(C) \otimes_R H(D),$$

and the induced map $\mathrm{coker}(\partial) \to H(C \otimes_R D)$ is equal to μ by construction. Moreover, $0 \to B \to Z \to H(C) \to 0$ is a projective resolution, so Theorem 1.1.17 implies that

$$\ker(\partial) \cong \mathrm{Tor}_1^R(H(C), H(D)).$$

This finishes the proof. $\square$

1.2.3. **Lemma.** *Let R be a hereditary ring. For every chain complex D over R, there exist a degreewise projective chain complex P and quasi-isomorphisms $P \xrightarrow{\sim} D$ and $P \xrightarrow{\sim} H(D)$.*

Proof. Let $Z_n \subseteq D_n$ and $B_n \subseteq D_n$ be the submodules of n-cycles and n-boundaries, respectively. Choose an epimorphism $p_n \colon P_n \twoheadrightarrow H_n(D)$ and denote by Q_n the kernel of p_n. Since R is hereditary, Q_n is also projective. By projectivity, the map p_n lifts along the epimorphism $Z_n \twoheadrightarrow H_n(D)$ to a map $f_n \colon P_n \to Z_n$. This yields an induced map $g'_n \colon Q_n \to B_n$, which lifts along $d^D_{n+1} \colon D_{n+1} \twoheadrightarrow B_n$ to a map $g_n \colon Q_n \to D_{n+1}$. The morphisms

$$\begin{pmatrix} 0 & \mathrm{inc} \\ 0 & 0 \end{pmatrix} \colon P_{n+1} \oplus Q_n \to P_n \oplus Q_{n-1}$$

define a chain complex, and $f_n + g_{n-1} \colon P_n \oplus Q_{n-1} \to D_n$ is a chain map. By construction, this chain map induces an isomorphism on homology. Similarly, $p_n + 0 \colon P_n \oplus Q_{n-1} \to H_n(D)$ defines a quasi-isomorphism $P \to H(D)$. □

1.2.4. **Proposition.** *The short exact sequence of Theorem 1.2.2 splits.*

Proof. Using Lemma 1.2.3, pick degreewise projective chain complexes P and Q which come with quasi-isomorphisms $C \xleftarrow{\sim} P \xrightarrow{\sim} H(C)$ and $D \xleftarrow{\sim} Q \xrightarrow{\sim} H(D)$. Then we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H(P) \otimes_R H(Q) \arrow[r] \arrow[d, "\cong"'] & H(P \otimes_R Q) \arrow[r] \arrow[d] & \mathrm{Tor}_1^R(H(P), H(Q))[1] \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H(C) \otimes_R H(D) \arrow[r] & H(C \otimes_R D) \arrow[r] & \mathrm{Tor}_1^R(H(C), H(D))[1] \arrow[r] & 0
\end{tikzcd}$$

with exact rows. As indicated, the outer vertical maps are isomorphisms, so the middle map is also an isomorphism. Hence it suffices to show that the upper row splits.

The given chain maps induce a chain map $P \otimes_R Q \xrightarrow{\alpha \otimes \beta} H(C) \otimes_R H(D)$, and therefore a map $r \colon H(P \otimes_R Q) \to H(C) \otimes_R H(D)$. The resulting diagram

$$\begin{tikzcd}
H(P) \otimes_R H(Q) \arrow[rr, "\cong", "H(\alpha \otimes \beta)"'] \arrow[dr, "\mu"'] & & H(C) \otimes_R H(D) \\
& H(P \otimes_R Q) \arrow[ur, "r"'] &
\end{tikzcd}$$

commutes, as one checks on representatives: for an element $[p] \otimes [q] \in H(P) \otimes_R H(Q)$, we have

$$(r \circ \mu)([p] \otimes [q]) = r([p \otimes q]) = \alpha(p) \otimes \alpha(q) = H(\alpha \otimes \beta)([p] \otimes [q]).$$

Hence r is the required retraction. □

1.2.5. **Remark.** Proposition 1.2.4 can be very useful since it tells us that we do not have to worry about extension problems when using the Künneth formula to determine the homology of tensor products. However, note that we do not claim that there exists a natural splitting of the sequence: the proof of Proposition 1.2.4, which involves a choice of P, certainly cannot produce a natural splitting. In fact, there is provably no natural choice of splitting; we may discuss this in one of the exercises.

By specialising to the case where D is concentrated in degree 0, we obtain the following.

1.2.6. **Theorem** (Universal coefficient theorem for homology)**.** *Let R be a hereditary ring, let C be a chain complex over R, and let M be an R-module. If C is degreewise projective, there exists a natural short exact sequence*

$$0 \to H(C) \otimes_R M \to H(C \otimes_R M) \to \mathrm{Tor}_1^R(H(C), M)[1] \to 0.$$

Moreover, this sequence splits.

Finally, specialising further to the case $C = C^{\mathrm{sing}}(X)$ for some topological space X, we obtain the universal coefficient theorem for singular homology.

1.2.7. **Corollary.** *Let X be a topological space and let M be an R-module over a hereditary ring R. For each $k \geq 0$, there exists a natural short exact sequence*

$$0 \to H_k(X; R) \otimes_R M \to H_k(X; M) \to \mathrm{Tor}_1^R(H_{k-1}(X; R), M) \to 0,$$

where we set $H_{-1}(X; R) = 0$. Moreover, this sequence splits.

Proof. Since M is an R-module, we have

$$C^{\mathrm{sing}}(X; M) = C^{\mathrm{sing}}(X) \otimes M \cong (C^{\mathrm{sing}}(X) \otimes R) \otimes_R M \cong C^{\mathrm{sing}}(X; R) \otimes_R M.$$

Now apply Theorem 1.2.6 □

1.2.8. **Example.** Recall from Topology I that the integral homology of the real projective space $\mathbb{R}P^n$ is given by

$$H_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}/2 & 0 < k < n \text{ and } k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem 1.2.7, we can determine $H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$ from this: consider the short exact sequence

$$0 \to H_k(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{Z}/2 \xrightarrow{\mu} H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{k-1}(\mathbb{R}\mathrm{P}^n), \mathbb{Z}/2) \to 0.$$

For $k > n+1$, both outer terms are zero. For $k = n+1$, either both outer terms are obviously zero (if n is even), or we obtain an isomorphism

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong \mathrm{Tor}_1^{\mathbb{Z}}(H_n(\mathbb{R}\mathrm{P}^n), \mathbb{Z}/2) = 0$$

for n odd.

Therefore, we can concentrate on $k \leq n$. Note that μ is an isomorphism whenever k is odd or $k = 0$, and that the second map is an isomorphism whenever $k \neq 0$ is even. For $k = 0$, we immediately get $H_0(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. For odd k, the isomorphisms $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ and $\mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ imply that $H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

For k even and non-zero, we conclude from $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ that $H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ as well.

In total, we have

$$H_k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

2. The Eilenberg–Zilber theorem

We can use the Künneth formula Theorem 1.2.2 to describe the homology of a tensor product of free chain complexes over $\mathbb{Z}$. Our next goal is to determine the singular homology of a product of spaces from this. For that, we are missing a comparison between the singular chains of a product space and the tensor product of the individual singular chain complexes.

2.1. **The Eilenberg–Zilber and Alexander–Whitney maps.** For technical reasons, we introduce the following concept.

2.1.1. **Definition.** An *augmented chain complex* over $\mathbb{Z}$ is a positive chain complex C together with a chain map $\varepsilon\colon C \to \mathbb{Z}[0]$.

The augmented chain complexes over $\mathbb{Z}$ assemble into a category $\mathrm{Ch}^+(\mathbb{Z})$ whose morphisms are given by commutative squares

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\varepsilon_C"'] & D \arrow[d, "\varepsilon_D"] \\ \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \end{tikzcd}$$

Given two augmented chain complexes (C, ε_C) and (D, ε_D), one can form their tensor product $(C \otimes D, \varepsilon_C \otimes \varepsilon_D)$ since $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$.

2.1.2. **Example.** The singular chains functor $C^{\mathrm{sing}}\colon \mathrm{Top} \to \mathrm{Ch}(\mathbb{Z})$ refines to a functor

$$(C^{\mathrm{sing}}, \varepsilon)\colon \mathrm{Top} \to \mathrm{Ch}^+(\mathbb{Z})$$

valued in augmented chain complexes by defining

$$\varepsilon_X\colon C_0^{\mathrm{sing}}(X) \to \mathbb{Z}, \quad \big(x\colon \Delta^0 \to X\big) \mapsto 1.$$

From this, we obtain the two functors

$$(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2) \quad \text{and} \quad (C^{\mathrm{sing}}(-_1 \times -_2), \varepsilon)\colon \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ch}^+(\mathbb{Z}).$$

In order to state the main theorem of this section, we introduce the following notation: Given two functors (F, ε_F), $(G, \varepsilon_G) \in \mathrm{Fun}(\mathcal{T}, \mathrm{Ch}^+(\mathbb{Z}))$, we denote by

$$[(F, \varepsilon_F), (G, \varepsilon_G)]$$

the collection of all morphisms $(F, \varepsilon_F) \to (G, \varepsilon_G)$ modulo chain homotopy in

$$\mathrm{Ch}(\mathrm{Fun}(\mathcal{T}, \mathrm{Ab})) \cong \mathrm{Fun}(\mathcal{T}, \mathrm{Ch}(\mathbb{Z})),$$

where we consider an augmented chain complex $\varepsilon\colon C \to \mathbb{Z}$ as a chain complex

$$\ldots \to C_2 \xrightarrow{d_2^C} C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \to \ldots$$

concentrated in degrees ≥ -1.

In the following, we denote by

$$-_1\colon \mathrm{Top} \times \mathrm{Top} \to \mathrm{Top} \quad \text{and} \quad -_2\colon \mathrm{Top} \times \mathrm{Top} \to \mathrm{Top}$$

the functors which project onto the first and second component, respectively.

2.1.3. **Theorem.** *Evaluation at the initial object identifies each of*

(1) $\big[(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2), (C^{\mathrm{sing}}(-_1 \times -_2), \varepsilon)\big]$,
(2) $\big[(C^{\mathrm{sing}}(-_1 \times -_2), \varepsilon), (C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2)\big]$,
(3) $\big[(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2), (C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2)\big]$,
(4) $\big[(C^{\mathrm{sing}}(-_1 \times -_2), \varepsilon), (C^{\mathrm{sing}}(-_1 \times -_2), \varepsilon)\big]$,
(5) $\big[(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2), \varepsilon_1 \otimes \varepsilon_2), (C^{\mathrm{sing}}(-_2) \otimes C^{\mathrm{sing}}(-_1), \varepsilon_2 \otimes \varepsilon_1)\big]$,
(6) $\big[(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2) \otimes C^{\mathrm{sing}}(-_3), \varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3), (C^{\mathrm{sing}}(-_1 \times -_2 \times -_3), \varepsilon)\big]$,
(7) $\big[(C^{\mathrm{sing}}(* \times -), \varepsilon), (C^{\mathrm{sing}}(-), \varepsilon)\big]$

with $\mathrm{Nat}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, *where* $\underline{\mathbb{Z}}$ *denotes the respective constant functor with value* $\mathbb{Z}$.

Before we give the proof, recall the following technical lemma from Topology I. Here, the functor $\sigma_{\leq n}\colon \mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}(\mathbb{Z})$ is the "brutal truncation" which replaces everything above degree n by 0.

2.1.4. **Lemma.** *Let $\mathcal{A}$ be an abelian category, let P be a degreewise projective chain complex in $\mathcal{A}$, and let M be an arbitrary chain complex in $\mathcal{A}$. If*

$$\mathrm{Hom}_{\mathcal{A}}(P_{k+1}, H_k(M)) = 0 \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{A}}(P_k, H_k(M)) = 0$$

for all $k \geq n$, then any chain map $\sigma_{\leq n}(\varphi)\colon \sigma_{\leq n}(P) \to \sigma_{\leq n}(M)$ admits an extension to a chain map $\varphi\colon P \to M$ which is unique up to chain homotopy.

Proof of Theorem 2.1.3. As in the proof of homotopy invariance for singular homology in Topology I, we allow ourselves to think of $\mathrm{Fun}(\mathrm{Top} \times \mathrm{Top}, \mathrm{Ab})$ and $\mathrm{Fun}(\mathrm{Top} \times \mathrm{Top} \times \mathrm{Top}, \mathrm{Ab})$ as abelian categories.

Let $F\colon \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$ be an arbitrary functor. For $p, q \geq 0$, we observe that that the universal property of the free abelian group functor $\mathbb{Z}[-]$ and the Yoneda lemma imply that

$$\begin{aligned}
&\mathrm{Nat}(C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2), F) \\
&\quad = \mathrm{Nat}(\mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta^p_{\mathrm{Top}}, -_1)] \otimes \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta^q_{\mathrm{Top}}, -_2)], F) \\
&\quad \cong \mathrm{Nat}(\mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}\times\mathrm{Top}}((\Delta^p_{\mathrm{Top}}, \Delta^q_{\mathrm{Top}}), -)], F) \\
&\quad \cong \mathrm{Nat}(\mathrm{Hom}_{\mathrm{Top}\times\mathrm{Top}}((\Delta^p_{\mathrm{Top}}, \Delta^q_{\mathrm{Top}}), -)], F) \\
&\quad \cong F(\Delta^p_{\mathrm{Top}}, \Delta^q_{\mathrm{Top}}),
\end{aligned}$$

where we consider F as a set-valued functor in the penultimate line. Similarly, we obtain

$$\mathrm{Nat}(C_n^{\mathrm{sing}}(-_1 \times -_2), F) \cong F(\Delta^n_{\mathrm{Top}}, \Delta^n_{\mathrm{Top}})$$

for all $n \geq 0$,

$$\mathrm{Nat}(C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2) \otimes C_r^{\mathrm{sing}}(-_3), F) \cong F(\Delta^p_{\mathrm{Top}}, \Delta^q_{\mathrm{Top}}, \Delta^r_{\mathrm{Top}})$$

for all $p, q, r \geq 0$, and

$$\mathrm{Nat}(C_n^{\mathrm{sing}}(* \times -), F) \cong F(\Delta^n_{\mathrm{Top}})$$

for all $n \geq 0$.

This implies directly that $C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2)$, $C_n^{\mathrm{sing}}(-_1 \times -_2)$, $C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2) \otimes C_r^{\mathrm{sing}}(-_3)$ and $C_n^{\mathrm{sing}}(* \times X)$ are projective for all $p, q, r, n \geq 0$: given a lifting problem

$$\begin{tikzcd}
 & F \arrow[d, two heads, "\pi"] \\
C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2) \arrow[r, "\varphi"] \arrow[ur, dotted] & G
\end{tikzcd}$$

with π an epimorphism, we find a lift as indicated by choosing a preimage of the element in $G(\Delta^p_{\mathrm{Top}}, \Delta^q_{\mathrm{Top}})$ corresponding to the transformation φ. The same kind of argument works for $C_n^{\mathrm{sing}}(-_1 \times -_2)$, $C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2) \otimes C_r^{\mathrm{sing}}(-_3)$ and $C_n^{\mathrm{sing}}(* \times -)$.

Moreover, the constant functor $\underline{\mathbb{Z}}$ is also projective since

$$\underline{\mathbb{Z}} \cong \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}\times\mathrm{Top}}((\varnothing, \varnothing), -)],$$

which implies $\mathrm{Nat}(\underline{\mathbb{Z}}, F) \cong F(\varnothing, \varnothing)$. In particular, $\mathrm{Nat}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ as claimed.

Note that $\sigma_{\leq -1}(C) \cong \underline{\mathbb{Z}}$ for any of the functors featuring in the theorem. Thus

$$\mathrm{Nat}(\sigma_{\leq -1}(C), \sigma_{\leq -1}(D)) \cong \mathbb{Z}$$

for any pair of these functors. We now want to apply Lemma 2.1.4 to show that each of these transformations extends to a transformation $C \Rightarrow D$ which is unique up to chain homotopy. Observing that the augmented singular chain complex computes the reduced homology of a space, this follows in the first case since we have

$$\mathrm{Nat}(C_p^{\mathrm{sing}}(-_1) \otimes C_q^{\mathrm{sing}}(-_2), \widetilde{H}_{p+q}(-_1 \times -_2)) \cong \widetilde{H}_{p+q}(\Delta^p_{\mathrm{Top}} \times \Delta^q_{\mathrm{Top}}) = 0$$

for all $p, q \geq 0$. In the second case, the augmented singular chain complex $C^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}}) \to \mathbb{Z}$ is chain contractible, hence $C^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}}) \otimes C^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}}) \xrightarrow{\varepsilon\otimes\varepsilon} \mathbb{Z}$ is also chain contractible. This implies that

$$\mathrm{Nat}(C^{\mathrm{sing}}_n(-_1 \times -_2), H_n(C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2))) \cong H_n(C^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}}) \otimes C^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}})) = 0$$

as required. The remaining cases follow by completely analogous arguments. □

2.1.5. **Definition.**

(1) Define the *Eilenberg–Zilber transformation* to be the (up to chain homotopy unique) natural transformation

$$\mathrm{EZ}\colon C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2) \to C^{\mathrm{sing}}(-_1 \times -_2)$$

corresponding to $1 \in \mathbb{Z}$ by virtue of Theorem 2.1.3.

(2) Define the *Alexander-Whitney transformation* to be the (up to chain homotopy unique) natural transformation

$$\mathrm{AW}\colon C^{\mathrm{sing}}(-_1 \times -_2) \to C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2)$$

corresponding to $1 \in \mathbb{Z}$ by virtue of Theorem 2.1.3.

2.1.6. **Proposition.**

(1) *There are chain homotopies* $\mathrm{AW} \circ \mathrm{EZ} \simeq \mathrm{id}$ *and* $\mathrm{EZ} \circ \mathrm{AW} \simeq \mathrm{id}$.

(2) *Let* $t\colon -_1 \times -_2 \Rightarrow -_2 \times -_1$ *and* $\tau\colon C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2) \Rightarrow C^{\mathrm{sing}}(-_2) \otimes C^{\mathrm{sing}}(-_1)$ *be the respective flip automorphism. Then the diagram*

$$\begin{tikzcd} C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2) \arrow[r, "\mathrm{EZ}"] \arrow[d, "\tau"'] & C^{\mathrm{sing}}(-_1 \times -_2) \arrow[d, "C^{\mathrm{sing}}(t)"] \\ C^{\mathrm{sing}}(-_2) \otimes C^{\mathrm{sing}}(-_1) \arrow[r, "\mathrm{EZ}"] & C^{\mathrm{sing}}(-_2 \times -_1) \end{tikzcd}$$

commutes up to chain homotopy.

(3) *The diagram*

$$\begin{tikzcd} C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2) \otimes C^{\mathrm{sing}}(-_3) \arrow[r, "\mathrm{EZ}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{EZ}"'] & C^{\mathrm{sing}}(-_1 \times -_2) \otimes C^{\mathrm{sing}}(-_3) \arrow[d, "\mathrm{EZ}"] \\ C^{\mathrm{sing}}(-_1) \otimes C^{\mathrm{sing}}(-_2 \times -_3) \arrow[r, "\mathrm{EZ}"] & C^{\mathrm{sing}}(-_1 \times -_2 \times -_3) \end{tikzcd}$$

commutes up to chain homotopy.

(4) *The diagram*

$$\begin{tikzcd} C^{\mathrm{sing}}(*) \otimes C^{\mathrm{sing}}(-) \arrow[r, "\mathrm{EZ}"] \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C^{\mathrm{sing}}(* \times -) \arrow[d, "C^{\mathrm{sing}}(\mathrm{pr})"] \\ \mathbb{Z}[0] \otimes C^{\mathrm{sing}}(-) \arrow[r, "\cong"] & C^{\mathrm{sing}}(-) \end{tikzcd}$$

commutes up to chain homotopy, where $\mathrm{pr}\colon * \times - \Rightarrow \mathrm{id}$ *denotes the natural transformation given by the obvious projection maps.*

Proof. On augmented chain complexes, both compositions induce the identity map on $\underline{\mathbb{Z}}$ in degree -1. Since the same is true for the identity transformation, the first assertion follows from Theorem 2.1.3 (3) and (4).

The other assertions follow from Theorem 2.1.3 (5), (6) and (7) by an analogous argument. □

2.2. **The singular homology of product spaces.** At this point, we have all the tools available to describe the singular homology of product spaces.

2.2.1. **Theorem** (Künneth formula for singular homology)**.** *Let X and Y be topological spaces, let R be a hereditary ring and let M be an R-module. Then there exists a natural short exact sequence*

$$0 \to H(X;R) \otimes_R H(Y;M) \to H(X \times Y;M) \to \operatorname{Tor}_1^R(H(X;R), H(Y;M))[1] \to 0.$$

Moreover, this sequence splits.

Proof. Since there exists a natural chain homotopy equivalence

$$C^{\mathrm{sing}}(X \times Y;M) \simeq C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(Y) \otimes M \cong C^{\mathrm{sing}}(X;R) \otimes_R C^{\mathrm{sing}}(Y;M)$$

by Proposition 2.1.6, the theorem follows from the algebraic Künneth formula (Theorem 1.2.2). □

2.2.2. **Corollary.** *Let X and Y be topological spaces. If R is a field, then*

$$H(X;R) \otimes_R H(Y;R) \cong H(X \times Y;R).$$

2.2.3. **Example.** Let $T^n := \prod_{i=1}^n S^1$ denote the n-torus. We claim that

$$H_k(T^n) \cong \mathbb{Z}^{\binom{n}{k}}.$$

This follows by induction from the Künneth formula. For $n = 1$, this is known from Topology I. Writing $T^{n+1} \cong T^n \times S^1$, we observe that all Tor-terms vanish since $H(S^1)$ is free in all degrees, so

$$H_k(T^{n+1}) \cong \bigoplus_{p+q=k} H_p(T^n) \otimes H_q(S^1) \cong H_k(T^n) \oplus H_{k-1}(T^n) \cong \mathbb{Z}^{\binom{n}{k}} \oplus \mathbb{Z}^{\binom{n}{k-1}} \cong \mathbb{Z}^{\binom{n+1}{k}}.$$

Later on, it will also be convenient to have relative versions of the Eilenberg–Zilber and Alexander–Whitney maps at our disposal.

2.2.4. **Definition.** An *excisive triad* (X, A, B) is a topological space X together with two subspaces $A \subseteq X$ and $B \subseteq X$ such that $C^{\mathrm{sing}}(A) + C^{\mathrm{sing}}(B) \to C^{\mathrm{sing}}(A \cup B)$ is a chain homotopy equivalence.

2.2.5. **Example.**

(1) For every pair of spaces (X, A), the triple $(X, A, \varnothing)$ is an excisive triad.
(2) If A and B are open subspaces of X, then (X, A, B) is an excisive triad.
(3) If X is a CW-complex and A and B are subcomplexes of X, then (X, A, B) is an excisive triad. This can be deduced from the preceding example by choosing open collars for A and B which deformation retract to A and B, respectively.

2.2.6. **Definition.** Let (X, A) and (Y, B) be pairs of topological spaces. Then define

$$(X, A) \times (Y, B) := (X \times Y, X \times B \cup Y \times A).$$

2.2.7. **Lemma.** *Let $0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules and let $0 \to A' \to B' \to C' \to 0$ be an exact sequence of right R-modules. Denote by $A \otimes_R B' + B \otimes_R A'$ the image of the obvious map $A \otimes_R B' \oplus B \otimes_R A' \to B \otimes_R B'$. Then*

$$0 \to A \otimes_R B' + B \otimes_R A' \to B \otimes_R B' \to C \otimes_R C' \to 0$$

is exact.

Proof. By definition, the first map is injective and the composite of the two middle maps is zero. The right map is surjective since tensoring with a fixed module preserves cokernels. Consider the commutative diagram

$$\begin{tikzcd}
A \otimes_R A' \arrow[r] \arrow[d] & A \otimes_R B' \arrow[r] \arrow[d, "i"] & A \otimes_R C' \arrow[r] \arrow[d] & 0 \\
B \otimes_R A' \arrow[r, "j"] \arrow[d] & B \otimes_R B' \arrow[r] \arrow[d] & B \otimes_R C' \arrow[r] \arrow[d] & 0 \\
C \otimes_R A' \arrow[r] \arrow[d] & C \otimes_R B' \arrow[r] \arrow[d] & C \otimes_R C' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

in which all rows and columns are exact. If $x \in B \otimes_R B'$ maps to zero in $C \otimes_R C'$, its image in $B \otimes_R C'$ lifts to an element in $A \otimes_R C'$, which in turn lifts to an element $a \in A \otimes_R B'$. Then $x - i(a)$ maps to zero in $B \otimes_R C'$, so there exists a preimage b under j. Now $i(a) + j(b) = x$. □

2.2.8. **Lemma.** *Let (X, A) and (Y, B) be pairs of topological spaces. There exists a natural commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & {\begin{matrix} C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(B) \\ +C^{\mathrm{sing}}(A) \otimes C^{\mathrm{sing}}(Y) \end{matrix}} \arrow[r] \arrow[d, "\mathrm{EZ}_{X,B} + \mathrm{EZ}_{A,Y}"'] & C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(Y) \arrow[r] \arrow[d, "\mathrm{EZ}_{X,Y}"] & C^{\mathrm{sing}}(X,A) \otimes C^{\mathrm{sing}}(Y,B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{\mathrm{sing}}(X \times B) + C^{\mathrm{sing}}(Y \times A) \arrow[r] \arrow[d] & C^{\mathrm{sing}}(X \times Y) \arrow[r] \arrow[d, "\mathrm{id}"] & \frac{C^{\mathrm{sing}}(X \times Y)}{C^{\mathrm{sing}}(X \times B) + C^{\mathrm{sing}}(Y \times A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{\mathrm{sing}}(X \times B \cup A \times Y) \arrow[r] & C^{\mathrm{sing}}(X \times Y) \arrow[r] & C^{\mathrm{sing}}((X,A) \times (Y,B)) \arrow[r] & 0
\end{tikzcd}$$

with the following properties:

(1) *all rows are exact;*
(2) *the upper vertical maps are chain homotopy equivalences;*
(3) *if $(X \times Y, X \times B, A \times Y)$ is an excisive triad, the lower vertical maps are chain homotopy equivalences.*

Proof. Exactness of the first row follows form Lemma 2.2.7, and the exactness of the other two rows is clear.

As a consequence of Proposition 2.1.6, the Alexander–Whitney transformation induces a commutative square

$$\begin{tikzcd}
{\begin{matrix} C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(B) \\ +C^{\mathrm{sing}}(A) \otimes C^{\mathrm{sing}}(Y) \end{matrix}} \arrow[r] & C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(Y) \\
C^{\mathrm{sing}}(X \times B) + C^{\mathrm{sing}}(Y \times A) \arrow[r] \arrow[u, "\mathrm{AW}_{X,B} + \mathrm{AW}_{A,Y}"] & C^{\mathrm{sing}}(X \times Y) \arrow[u, "\mathrm{AW}_{X,Y}"']
\end{tikzcd}$$

such that there are chain homotopies $\mathrm{AW}_{X,Y} \circ \mathrm{EZ}_{X,Y} \simeq \mathrm{id}$ and $\mathrm{EZ}_{X,Y} \circ \mathrm{AW}_{X,Y} \simeq \mathrm{id}$ which restrict to chain homotopies

$$(\mathrm{AW}_{X,B} + \mathrm{AW}_{A,Y}) \circ (\mathrm{EZ}_{X,B} + \mathrm{EZ}_{A,Y}) \simeq \mathrm{id}$$
$$\text{and} \quad (\mathrm{EZ}_{X,B} + \mathrm{EZ}_{A,Y}) \circ (\mathrm{AW}_{X,B} + \mathrm{AW}_{A,Y}) \simeq \mathrm{id}$$

since the chain homotopies from Proposition 2.1.6 (1) are natural. This implies that the upper right vertical map is also a chain homotopy equivalence.

The third assertion follows from the definition of an excisive triad. □

To give at least one direct application of these considerations, we obtain a relative version of the Künneth theorem.

2.2.9. **Theorem.** *Let (X, A) and (Y, B) be pairs of topological spaces such that $(X \times Y, X \times B, A \times Y)$ is an excisive triad. Let R be a hereditary ring and let M be an R-module. Then there exists a natural short exact sequence*

$$0 \to H(X, A; R) \otimes_R H(Y, B; M) \to H((X, A) \times (Y, B); M) \\ \to \operatorname{Tor}_1^R(H(X, A; R), H(Y, B; M))[1] \to 0.$$

Moreover, this sequence splits.

3. Singular cohomology

The primary invariants that allow us to distinguish homotopy types so far have been the fundamental group and the singular homology groups. Unfortunately, it is easy to come up with questions about homotopy types that we cannot answer using only these invariants: is $\mathbb{C}P^2$ homotopy equivalent to $S^2 \vee S^4$? Is $\mathbb{C}P^3$ homotopy equivalent to to $S^2 \times S^4$?

We might try to refine our invariants by equipping them with additional structure. The map $H(X) \otimes H(Y) \to H(X \times Y)$ appearing the Künneth theorem defines an "exterior product" (ie lax monoidal transformation) on singular homology. In particular, we have product maps $H(X) \otimes H(X) \to H(X \times X)$. However, there is no good way to turn this into a multiplication map on $H(X)$ since we lack a natural map from $X \times X$ to X (unless we assume that X is a topological monoid or something similar). But there does exist a natural transformation going the other way, namely the diagonal map $\Delta\colon X \to X \times X$. If homology was contravariant, we could compose the "exterior" multiplication with the map induced by Δ and hope to get a ring structure. Singular homology isn't contravariant, of course, but we can try to tweak our definitions a bit so we obtain a contravariant functor. For example, we could take duals (ie apply the functor $\operatorname{Hom}(-, \mathbb{Z})$) to reverse the functoriality of the singular chain complex, and that it precisely what happens next.

3.1. Definition and basic properties.

Recall that for every pair of chain complexes C and D, there exists a chain complex $\underline{\operatorname{Hom}}(C, D)$ of abelian groups.

3.1.1. **Definition.** Let (X, A) be a pair of topological spaces and let M be an abelian group. Define

$$C_{\text{sing}}(X, A; M) := \underline{\operatorname{Hom}}(C^{\text{sing}}(X, A), M[0]).$$

If A is empty, we will write $C_{\text{sing}}(X; M)$ instead of $C_{\text{sing}}(X, \varnothing; M)$, and similarly, we will suppress M from the notation if $M = \mathbb{Z}$.

3.1.2. **Remark.**

(1) In Definition 3.1.1, we write $M[0]$ to emphasise that we regard M as a chain complex concentrated in degree 0.

(2) Since $M[0]$ is concentrated in degree 0, we have

$$C_{\text{sing}}(X, A; M)_n = \operatorname{Hom}(C^{\text{sing}}_{-n}(X, A), M)$$

for all n. In particular, $C_{\text{sing}}(X, A; M)$ is concentrated in non-positive degrees.

Many authors like to reverse the indexing and write

$$C^n_{\text{sing}}(X, A; M) := C_{\text{sing}}(X, A; M)_{-n},$$

but this forces one to consider *cochain complexes*. Since this is merely a question of indexing conventions (the category of cochain complexes is isomorphic to the category of chain complexes), the mathematical content is unchanged. When working on the chain level, we will prefer to stick to chain complexes. However, it will still be convenient to adopt the reverse indexing convention when passing to homology.

3.1.3. **Definition.** Let (X, A) be a pair of topological spaces and let M be an abelian group. The *n-th singular cohomology of (X, A) with coefficients in M* is

$$H^n(X, A; M) := H_{-n}(C_{\text{sing}}(X, A; M)).$$

This gives rise to functors

$$H^n\colon \mathrm{Top}_2^{\mathrm{op}} \to \mathrm{Ab},$$

where Top_2 is our notation for the category of pairs of topological spaces. Singular cohomology enjoys many properties analogous to similar homology. Seeing this will be quite straightforward after some preliminary observations. First of all, let us record the exactness properties of the Hom-functors.

3.1.4. **Lemma.** *Let $\mathcal{A}$ be an abelian category and let A and B be objects of $\mathcal{A}$.*

(1) *The functor*
$$\mathrm{Hom}_{\mathcal{A}}(A, -)\colon \mathcal{A} \to \mathrm{Ab}$$
is left exact. It is exact if and only if A is projective.

(2) *The functor*
$$\mathrm{Hom}_{\mathcal{A}}(-, B)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$$
is left exact. It is exact if and only if B is injective.

Proof. Let $0 \to A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2 \to 0$ be an exact sequence in $\mathcal{A}$. Then

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, A_0) \xrightarrow{-\circ i} \mathrm{Hom}_{\mathcal{A}}(A, A_1) \xrightarrow{-\circ p} \mathrm{Hom}_{\mathcal{A}}(A, A_2)$$

is exact since i is a kernel of p. Every morphism $g\colon A \to A_2$ gives rise to a lifting problem

$$\begin{tikzcd} & A_1 \arrow[d, two heads, "p"] \\ A \arrow[ur, dotted, "f"] \arrow[r, "g"] & A_2 \end{tikzcd}$$

If A is projective, each such lifting problem has a solution, showing that $- \circ p$ is surjective. Conversely, if $- \circ p$ is surjective for every epimorphism p, this proves that A is projective.

Since $\mathrm{Hom}_{\mathcal{A}}(-, B) = \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(B, -)$, the second assertion follows from the first. □

Moreover, we will want to know that taking Hom-complexes preserves chain maps and chain homotopies.

3.1.5. **Remark.** Let C and D be chain complexes over R. By direct inspection of the formulas we provided in Section 1, we observe that

(1) the set of 0-cycles in $\underline{\mathrm{Hom}}_R(C, D)$ is identical to the set of chain maps from C to D;
(2) two 0-cycles (ie chain maps) are homologous if and only if they are chain homotopic: a 1-chain h with $d^{\underline{\mathrm{Hom}}_R(C,D)}(h) = \varphi_1 - \varphi_0$ is exactly the same as a chain homotopy $\varphi_0 \simeq \varphi_1$.

In the following, we will make regular use of this observation.

3.1.6. **Construction.** Consider the following data:

- C a chain complex of (S,R)-bimodules;
- D a chain complex of (T,R)-bimodules;
- E a chain complex of (U,R)-bimodules.

We claim that there exist (U,S)-linear composition maps

$$\kappa\colon \underline{\mathrm{Hom}}_R(D,E)\otimes_T \underline{\mathrm{Hom}}_R(C,D)\to \underline{\mathrm{Hom}}_R(C,E).$$

In fact, the naive formula

$$\kappa\colon \underline{\mathrm{Hom}}_R(D,E)_p\otimes_T\underline{\mathrm{Hom}}_R(C,D)_q\to \underline{\mathrm{Hom}}_R(C,E)_{p+q},\ (\psi_k)_k\otimes(\varphi_k)_k\mapsto(\psi_{k+q}\circ\varphi_k)_k$$

(we are not adding additional indices to κ to unclutter the following formulas a bit) is well-defined and satisfies

$$\begin{aligned}(\kappa\circ d^{\otimes})((\psi_k)_k\otimes(\varphi_k)_k) &= \kappa\left(d^{\underline{\mathrm{Hom}}(D,E)}((\psi_k)_k)\otimes(\varphi_k)_k+(-1)^p(\psi_k)_k\otimes d^{\underline{\mathrm{Hom}}(C,D)}((\varphi_k)_k)\right)\\
&=\kappa\Big(\left(d^E\circ\psi_k+(-1)^{p+1}\psi_{k-1}\circ d^D\right)_k\otimes(\varphi_k)_k\\
&\qquad+(-1)^p(\psi_k)_k\otimes\left(d^D\circ\varphi_k+(-1)^{q+1}\varphi_{k-1}\circ d^C\right)_k\Big)\\
&=\Big(\left(d^E\circ\psi_{k+q}+(-1)^{p+1}\psi_{k-1+q}\circ d^D\right)\circ\varphi_k\\
&\qquad+(-1)^p\psi_{k+q-1}\circ\left(d^D\circ\varphi_k+(-1)^{q+1}\varphi_{k-1}\circ d^C\right)\Big)_k\\
&=\left(d^E\circ\psi_{k+q}\circ\varphi_k+(-1)^{p+q+1}\psi_{k-1+q}\circ\varphi_{k-1}\circ d^C\right)_k\\
&=d^{\underline{\mathrm{Hom}}(C,E)}((\psi_{k+q}\circ\varphi_k)_k)\\
&=(d^{\underline{\mathrm{Hom}}(C,E)}\circ\kappa)((\psi_k)_k\otimes(\varphi_k)_k),\end{aligned}$$

so κ defines a chain map.

If R is commutative, we can regard every chain complex over R as a chain complex of (R,R)-bimodules. Then the resulting composition operation is associative and unital in the sense that

$$\kappa(\chi\otimes\kappa(\psi\otimes\varphi))=\kappa(\kappa(\chi\otimes\psi)\otimes\varphi)\quad\text{and}\quad\kappa(\mathrm{id}_D\otimes\varphi)=\varphi=\kappa(\varphi\otimes\mathrm{id}_C)$$

for $\varphi\in\underline{\mathrm{Hom}}_R(C,D)$, $\psi\in\underline{\mathrm{Hom}}_R(D,E)$ and $\psi\in\underline{\mathrm{Hom}}_R(E,F)$.

Let D be a chain complex of (T,R)-bimodules and let E be a chain complex of (U,R)-bimodules. Then we can recover the (U,R)-linear evaluation map

$$\mathrm{ev}\colon \underline{\mathrm{Hom}}_R(D,E)\otimes_T D\to E$$

from the composition

$$\underline{\mathrm{Hom}}_R(D,E)\otimes_T D\cong\underline{\mathrm{Hom}}_R(D,E)\otimes_T\underline{\mathrm{Hom}}_R(R[0],D)\xrightarrow{\kappa}\underline{\mathrm{Hom}}_R(R[0],E)\cong E.$$

Let now X be a chain complex of (U,T)-bimodules, let Y be a chain complex of (T,S)-bimodules, and let Z be a chain complex of (V,S)-bimodules. Then the (V,S)-linear map $\mathrm{ev}\colon\underline{\mathrm{Hom}}_S(Y,Z)\otimes_T Y\to Z$ induces an isomorphism

$$\alpha\colon\underline{\mathrm{Hom}}_T(X,\underline{\mathrm{Hom}}_S(Y,Z))\to\underline{\mathrm{Hom}}_S(X\otimes_T Y,\underline{\mathrm{Hom}}_S(Y,Z)\otimes_T Y)\xrightarrow{\mathrm{ev}_*}\underline{\mathrm{Hom}}_S(X\otimes_T Y,Z).$$

We stated this isomorphism already in (1.0.1) in a special case. Checking that α is an isomorphism boils down to observing that the corresponding statement is correct on the level of categories of bimodules.

Taking $X=\underline{\mathrm{Hom}}_R(C,D)$, $Y=\underline{\mathrm{Hom}}_R(D,E)$ and $Z=\underline{\mathrm{Hom}}_R(C,E)$, we obtain a chain map

$$\overline{\kappa}:=\alpha^{-1}(\kappa)\colon\ \underline{\mathrm{Hom}}_R(C,D)\to\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_R(D,E),\underline{\mathrm{Hom}}_R(C,E)).$$

Then every chain map $\varphi\colon C\to D$ induces a 0-cycle

$$\varphi^*:=\overline{\kappa}(\varphi)\in\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_R(D,E),\underline{\mathrm{Hom}}_R(C,E))_0.$$

Unwinding the definitions, this is precisely the chain map

$$\varphi^*\colon \underline{\mathrm{Hom}}_R(D,E) \to \underline{\mathrm{Hom}}_R(C,E)$$

sending $(\psi_k)_k$ to $(\psi_k \circ \varphi_k)_k$.

3.1.7. **Corollary.** *Let C, D and E be chain complexes over R. If h is a chain homotopy $\varphi_0 \simeq \varphi_1\colon C \to D$, then restriction along h induces an element*

$$h^* \in \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_R(D,E), \underline{\mathrm{Hom}}_R(C,E))_1$$

satisfying $d^{\underline{\mathrm{Hom}}(\dots)}(h^*) = \varphi_1^* - \varphi_0^*$. In other words, h^* is a chain homotopy

$$\varphi_0^* \simeq \varphi_1^*\colon \underline{\mathrm{Hom}}_R(D,E) \to \underline{\mathrm{Hom}}_R(C,E).$$

Proof. Since h is a chain homotopy $\varphi_0 \simeq \varphi_1$, it defines a chain $h \in \underline{\mathrm{Hom}}_R(C,D)_1$ satisfying $d^{\underline{\mathrm{Hom}}_R(C,D)}(h) = \varphi_1 - \varphi_0$. Define $h^* := \overline{\kappa}(h)$. Since $\overline{\kappa}$ is a chain map, we have

$$d^{\underline{\mathrm{Hom}}(\dots)}(h^*) = \overline{\kappa}(d^{\underline{\mathrm{Hom}}_R(C,D)}(h)) = \varphi_1^* - \varphi_0^*. \qquad \square$$

3.1.8. **Theorem.** *The singular cohomology functors*

$$\{H^n(-;M)\}_{n\geq 0}\colon \mathrm{Top}_2^{\mathrm{op}} \to \mathrm{Ab}$$

have the following properties:

(1) *$H^n(\varnothing;M) = 0$ for all n;*
(2) *$H^0(*;M) \cong M$ and $H^n(*;M) = 0$ for $n \geq 1$;*
(3) *the canonical map*

$$H^n(\coprod_{i\in I}(X_i,A_i);M) \to \prod_{i\in I} H^n(X_i,A_i;M)$$

is an isomorphism for every collection of pairs $(X_i,A_i)_{i\in I}$, all $n \geq 0$ and all M;

(4) *$H^0(X;M) \cong M^{\pi_0(X)}$ for all every weakly locally path-connected space X;*
(5) *if $f,g\colon (X,A) \to (Y,B)$ are homotopic maps of pairs, then*

$$H^n(f) = H^n(g)\colon H^n(Y,B;M) \to H^n(X,A;M);$$

(6) *for every pair of topological spaces (X,A), there exist natural maps*

$$\partial^n\colon H^n(A;M) \to H^{n+1}(X,A;M)$$

such that the sequence

$$H^n(X,A;M) \to H^n(X;M) \to H^n(A;M) \xrightarrow{\partial^n} H^{n+1}(X,A;M) \to H^{n+1}(X;M)$$

is exact for all n;

(7) *for every pair (X,A) and every subspace $U \subseteq A$ with $\overline{U} \subseteq \mathrm{int}(A)$ the maps*

$$H^n(X,A;M) \to H^n(X\setminus U, A\setminus U;M)$$

induced by the inclusion map are isomorphisms;

(8) *for every topological space X and all ordered pairs (A,B) of subspaces $A,B \subseteq X$ with $\mathrm{int}(A)\cup\mathrm{int}(B) = X$ there exist natural maps*

$$\partial^n_{A,B}\colon H^n(A\cap B;M) \to H^{n+1}(X;M)$$

such that the sequence

$$H^n(X;M) \xrightarrow{(i_A^*,i_B^*)} H^n(A;M)\oplus H^n(B;M) \xrightarrow{j_A^*-j_B^*} H^n(A\cap B;M)$$
$$\xrightarrow{\partial^n_{A,B}} H^{n+1}(X;M) \xrightarrow{(i_A^*,i_B^*)} H^{n+1}(A;M)\oplus H^{n+1}(B;M)$$

is exact;

(9) *for every pointed space (X, x), there is a natural isomorphism*

$$H^n(\Sigma X, N; M) \cong H^{n+1}(X, x; M),$$

where N denotes the north pole of the suspension ΣX;

Proof. Since $C^{\text{sing}}(\varnothing) = 0$, we also have $C_{\text{sing}}(\varnothing) = 0$, proving (1). From Topology I, we know that $C^{\text{sing}}(*) \simeq \mathbb{Z}[0]$, so

$$C_{\text{sing}}(*, M[0]) \simeq C_{\text{sing}}(\mathbb{Z}[0], M[0]) \cong M[0]$$

by Corollary 3.1.7. This shows (2).

Let X be a topological space. Then $X \cong \coprod_{Y \in \pi_0(X)} Y$. Hence we obtain

$$\begin{aligned} C_{\text{sing}}(X; M) = \underline{\text{Hom}}(C^{\text{sing}}(X), M[0]) &\cong \underline{\text{Hom}}(C^{\text{sing}}(\sqcup_{Y \in \pi_0(X)} Y), M[0]) \\ &\cong \underline{\text{Hom}}\left(\bigoplus_{Y \in \pi_0(X)} C^{\text{sing}}(Y), M[0] \right) \\ &\cong \prod_{Y \in \pi_0(X)} \underline{\text{Hom}}(C^{\text{sing}}(Y), M[0]) \\ &\cong \prod_{Y \in \pi_0(X)} C_{\text{sing}}(Y; M). \end{aligned}$$

Since taking homology commutes with products, we obtain

$$H^n(X; M) \cong \prod_{Y \in \pi_0(X)} H^n(Y; M).$$

This proves (3).

The canonical map $\coprod_{C \in \pi_0(X)} C \to X$ is a homeomorphism if (and only if) X is weakly locally path-connected. In light of (3), it is enough to consider the case that X is path-connected for (4). Since

$$C_{\text{sing}}(X; M)_0 = \text{Hom}(C_0^{\text{sing}}(X), M) \cong \text{Hom}(\mathbb{Z}[X], M) \cong \prod_{x \in X} M,$$

we can think of elements in this group as sequences $(m_x)_x$. Similarly,

$$C_{\text{sing}}(X; M)_1 \cong \prod_{\sigma \in \text{Hom}_{\text{Top}}(\Delta^1_{\text{Top}}, X)} M,$$

and, under these identifications, the boundary map sends $(m_x)_x$ to

$$(m_{\sigma \circ \delta^0} - m_{\sigma \circ \delta^1})_{\sigma \colon \Delta^1 \to X} .$$

If $(m_x)_x$ is a cycle, path-connectedness of X implies that $m_x = m_y$ for all $x, y \in X$. Evidently, any such sequence is also a cycle. Hence $H^0(X; M) \cong M$.

Consider assertion (5). From Topology I, we know that f and g induce chain homotopic maps $C^{\text{sing}}(f) \simeq C^{\text{sing}}(g)$, so the assertion follows from Corollary 3.1.7.

By Lemma 3.1.4, every pair of topological spaces (X, A) induces an exact sequence

$$0 \to C_{\text{sing}}(X, A; M) \to C_{\text{sing}}(X; M) \to C_{\text{sing}}(A; M) \to 0$$

since the singular chain complex is free in each degree. Hence we obtain the maps ∂^n and the associated long exact sequence in (6) by passing to homology.

For (7), we again rely on Topology I and Corollary 3.1.7 to conclude that the induced map $C_{\text{sing}}(X, A) \to C_{\text{sing}}(X \setminus U, A \setminus U; M)$ is a chain homotopy equivalence.

As in the case of singular homology, assertion (8) follows from assertions (6) and (7), and assertion (9) is a consequence of (8) and (5). $\square$

3.1.9. **Remark.** In the following, we will stop writing $H^n(f)$ for the homomorphism induced by f on the n-th singular cohomology group, and instead write f^* for the induced map (thus suppressing the degree from notation altogether). The superscript indicates that the map is induced via a contravariant functor, so the composition rule reads

$$f^*g^* = (gf)^*$$

for composable maps $f\colon X \to Y$ and $g\colon Y \to Z$.

Next, we want to address questions analogous to the ones we asked (and answered) for singular homology in Sections 1 and 2:

- Can we compute $H^n(X; M)$ from calculations with $\mathbb{Z}$-coefficients?
- What is the singular cohomology of a product space?

3.2. The universal coefficient theorem for cohomology.

3.2.1. **Construction.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then $F^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is right exact. If $\mathcal{A}$ has enough injectives (ie every object $A \in \mathcal{A}$ admits a monomorphism $A \to I$ for some injective object I), Theorem 1.1.17 provides a left derived functor $\{\mathrm{L}F_n^{\mathrm{op}}\}_{n\geq 0}$ of F^{op}. Setting

$$\mathrm{R}F^n := (\mathrm{L}(F^{\mathrm{op}})_n)^{\mathrm{op}},$$

we obtain a *cohomological δ-functor* $\{\mathrm{R}F^n\}_{n\geq 0}$: this means that we have a sequence of functors $\mathrm{R}F^n\colon \mathcal{A} \to \mathcal{B}$ together with a choice of natural boundary map

$$\partial^n\colon \mathrm{R}F^n(C) \to \mathrm{R}F^{n+1}(A)$$

for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that the sequences

$$\mathrm{R}F^n(A) \to \mathrm{R}F^n(B) \to \mathrm{R}F^n(C) \xrightarrow{\partial^n} \mathrm{R}F^{n+1}(A) \to \mathrm{R}F^{n+1}(B)$$

are exact. Moreover, this cohomological δ-functor comes equipped with a natural isomorphism $\mathrm{R}F^0 \cong F$.

The cohomological δ-functor $\{\mathrm{R}F^n\}_{n\geq 0}$ is *universal* in the sense that for every cohomological δ-functor $\{G^n\}_{n\geq 0}$, every natural transformation $\mathrm{R}F^0 \Rightarrow G^0$ extends uniquely to a morphism of cohomological δ-functors $\{\mathrm{R}F^n\}_{n\geq 0} \to \{G^n\}_{n\geq 0}$; this is immediate from the universal property of $\{\mathrm{L}(F^{\mathrm{op}})_n\}_{n\geq 0}$.

3.2.2. **Definition.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be left exact. The universal cohomological δ-functor $\{\mathrm{R}F^n\}_{n\geq 0}$, together with the natural isomorphism $\mathrm{R}F^0 \cong F$, from Construction 3.2.1 is called the *right derived functor* of F.

3.2.3. **Definition.** Let R be a ring and let N be an R-module. Define

$$\{\mathrm{Ext}^n_R(-, N)\}_{n\geq 0}$$

as the right derived functor of the left exact functor

$$\mathrm{Hom}_R(-, N)\colon R\text{-}\mathrm{Mod}^{\mathrm{op}} \to \mathrm{Ab}.$$

3.2.4. **Remark.** Applying universality, each of the functors Ext^n_R refines to a bifunctor

$$\mathrm{Ext}^n_R\colon R\text{-}\mathrm{Mod}^{\mathrm{op}} \times R\text{-}\mathrm{Mod} \to \mathrm{Ab}.$$

Theorem 1.1.17 provides an explicit formula for $\mathrm{Ext}^n_R(M, N)$: starting with the left exact functor $\mathrm{Hom}_R(-N)$, consider the right exact functor $\mathrm{Hom}_R(-, N)^{\mathrm{op}}\colon R\text{-}\mathrm{Mod} \to \mathrm{Ab}^{\mathrm{op}}$. Choosing a projective resolution $P_\bullet \to M$, we obtain

$$\mathrm{L}\left(\mathrm{Hom}_R(-, N)^{\mathrm{op}}\right)_n(M) \cong H_n(\mathrm{Hom}_R(P_\bullet, N)),$$

where the latter term is the homology of a chain complex in $\mathrm{Ab}^{\mathrm{op}}$. If we wish to express this as the homology of a chain complex of abelian groups, we have to apply the trick of inverting the indexing again, which gives us the formula

$$\mathrm{Ext}^n_R(M,N) \cong H_{-n}(\underline{\mathrm{Hom}}_R(P_\bullet, N[0])).$$

3.2.5. **Proposition.**

(1) *Let $N \to I_\bullet$ be an injective resolution of N. Considering $I_\bullet$ as a chain complex concentrated in non-positive degrees, we have*

$$\mathrm{Ext}^n_R(M,N) \cong H_{-n}(\mathrm{Hom}_R(M[0], I_\bullet)).$$

(2) *The natural maps*

$$\mathrm{Ext}^n_R(\bigoplus_{i\in I} M_i, N) \to \prod_{i\in I} \mathrm{Ext}^n_R(M_i, N)$$

and

$$\mathrm{Ext}^n_R(M, \prod_{i\in I} N_i) \to \prod_{i\in I} \mathrm{Ext}^n_R(M, N_i)$$

are isomorphisms.

(3) *if every R-module admits a projective resolution of length $\leq k$, then*

$$\mathrm{Ext}^n_R(M,N) = 0$$

for all $n > k$;

(4) *if every R-module admits an injective resolution of length $\leq k$, then*

$$\mathrm{Ext}^n_R(M,N) = 0$$

for all $n > k$;

(5) *for every projective R-module P, we have*

$$\mathrm{Ext}^n_R(P,-) = 0$$

for all $n \geq 1$;

(6) *for every injective R-module I, we have*

$$\mathrm{Ext}^n_R(-,I) = 0$$

for all $n \geq 1$;

(7) *if submodules of projective modules over R are projective, then $\mathrm{Ext}^n_R(M,N) = 0$ for all $n \geq 2$;*

(8) *if quotients of injective modules over R are injective, then $\mathrm{Ext}^n_R(M,N) = 0$ for all $n \geq 2$;*

(9) *if R is an integral domain and $r \neq 0$, then*

$$\mathrm{Ext}^0_R(R/(r), N) \cong \mathrm{Hom}_R(R/(r), N) \cong \{y \in N \mid ry = 0\}$$

and

$$\mathrm{Ext}^1_R(R/(r), N) \cong N/rN;$$

(10) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/l) \cong \mathbb{Z}/\gcd(k,l)$ *for $k \neq 0$.*

Proof. The vanishing assertions (3), (5) and (7) are immediate from the formula $\mathrm{Ext}^n_R(M,N) \cong H_{-n}(\underline{\mathrm{Hom}}_R(P_\bullet, N[0]))$ for some projective resolution $P_\bullet \to M$ of M. If I is injective and $P_\bullet \to M$ is a projective resolution, then $\underline{\mathrm{Hom}}_R(P_\bullet, I[0])$ is still exact by Lemma 3.1.4. This implies (6).

Observe now that

$$\mathrm{Hom}_R(M, \prod_{i\in I} N_i) \to \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$$

is a natural isomorphism, and that $\prod_{i\in I}\colon \mathrm{Fun}(I, R\text{-Mod}) \to R\text{-Mod}$ is an exact functor. Hence $\{\prod_{i\in I} \mathrm{Ext}^n_R(-, N_i)\}_{n\geq 0}$ is a cohomological δ-functor extending

$\operatorname{Hom}_R(-, \prod_{i\in I} N_i)$. Since $\operatorname{Ext}^n_R(-, N_i)$ vanishes on projectives for $n \geq 1$, it follows from Proposition 1.1.13 that $\{\prod_{i\in I} \operatorname{Ext}^n_R(-, N_i)\}_{n\geq 0}$ is a universal cohomological δ-functor, and therefore has to be isomorphic to $\{\operatorname{Ext}^n_R(-, \prod_{i\in I} N_i)\}_{n\geq 0}$. This proves the second part of (2).

We proceed to prove (1). So let M be an R-module and choose a projective resolution $P_\bullet \to M$. If

$$0 \to N_0 \to N_1 \to N_2 \to 0$$

is an exact sequence, then the induced sequence

$$0 \to \underline{\operatorname{Hom}}_R(P_\bullet, N_0[0]) \to \underline{\operatorname{Hom}}_R(P_\bullet, N_1[0]) \to \underline{\operatorname{Hom}}_R(P_\bullet, N_2[0]) \to 0$$

is also exact by Lemma 3.1.4. Passing to homology, we obtain natural boundary maps

$$\partial^n \colon \operatorname{Ext}^n_R(M, N_2) \to \operatorname{Ext}^{n+1}_R(M, N_0)$$

which give rise to long exact sequences of Ext-groups. This upgrades $\{\operatorname{Ext}^n_R(M, -)\}_{n\geq 0}$ to a cohomological δ-functor. If I is an injective module, $\operatorname{Ext}^n_R(M, I) = 0$ by assertion (6). Since the category of R-modules has enough injectives, it follows from Proposition 1.1.13 that this defines a universal cohomological δ-functor. Applying Construction 3.2.1 to $\operatorname{Hom}_R(-, N)$, it follows that

$$\operatorname{Ext}^n_R(M, N) \cong H_{-n}(\underline{\operatorname{Hom}}_R(M[0], I_\bullet))$$

for some injective resolution $N \to I_\bullet$ (the indexing comes out like this exactly as in Remark 3.2.4 since Theorem 1.1.17 describes Ext^n_R as the homology of a chain complex in $\mathrm{Ab}^{\mathrm{op}}$).

For the first part of (2), we can now argue as before using that

$$\operatorname{Hom}_R(\bigoplus_{i\in I} M_i, N) \to \prod_{i\in I} \operatorname{Hom}_R(M_i, N)$$

is an isomorphism. Moreover, the vanishing assertions (4) and (8) are now obvious.

We are left with proving (9) and (10). Since $0 \to R \xrightarrow{\cdot r} R \to R/(r) \to 0$ is a projective resolution if r is not a zero divisor, we have to compute the homology of the chain complex

$$0 \to N \xrightarrow{\cdot r} N \to 0$$

concentrated in degrees 0 and -1 (since $\operatorname{Hom}_R(R, N) \cong N$ by evaluating at $1 \in R$). Hence

$$\operatorname{Ext}^0_R(R/(r), N) \cong \{y \in N \mid ry = 0\} \quad \text{and} \quad \operatorname{Ext}^0_R(R/(r), N) \cong N/rN.$$

Applying this in the case $R = \mathbb{Z}$ and $k \neq 0$, we have

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/l) \cong (\mathbb{Z}/l)/(k \cdot \mathbb{Z}/l).$$

Set $g := \gcd(k, l)$. Note that the projection map $p\colon \mathbb{Z}/l \to \mathbb{Z}/g$ satisfies $p(k) = g \cdot \frac{k}{g} = 0$. If $p(x) = 0$, then $x = gy$ for some $y \in \mathbb{Z}$. Writing $1 = a\frac{k}{g} + b\frac{l}{g}$, we have $x = aky + bly = aky \in \mathbb{Z}/l$, so p induces an isomorphism $(\mathbb{Z}/l)/(k \cdot \mathbb{Z}/l) \cong \mathbb{Z}/g$. $\square$

The Ext-functors will appear when we try to describe the homology of the complexes $\underline{\operatorname{Hom}}_R(C, D)$. As preparation for the more general statement, we record the following lemma.

3.2.6. **Lemma.** *Let P be a degreewise projective graded R-module and let D be a chain complex over R. Then the map*

$$H(\underline{\operatorname{Hom}}_R(P, D)) \to \underline{\operatorname{Hom}}_R(P, H(D))$$

induced by evaluation is an isomorphism.

Proof. Since $P \cong \bigoplus_n P_n[n]$ as a chain complex, we have

$$\underline{\mathrm{Hom}}_R(P, C) \cong \prod_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_R(P_n[n], C)$$

for every chain complex C. As homology commutes with products, we may restrict to the case of a graded module $P[n]$ concentrated in degree n for some projective R-module P. Then $\underline{\mathrm{Hom}}_R(P[n], D)_k \cong \mathrm{Hom}_R(P, D_{n+k})$. Since $\mathrm{Hom}_R(P, -)$ is exact by Lemma 3.1.4, it follows that $\ker(d^{\underline{\mathrm{Hom}}_R(P[n],D)}) = \mathrm{Hom}_R(P[n], \ker(d^D))$ and $\mathrm{img}(d^{\underline{\mathrm{Hom}}_R(P[n],D)}) = \mathrm{Hom}_R(P[n], \mathrm{img}(d^D))$, and therefore also

$$H(\underline{\mathrm{Hom}}_R(P[n], D)) \cong \underline{\mathrm{Hom}}_R(P[n], H(D)). \qquad \square$$

In analogy to the situation of the Künneth formula, define for a biadditive functor

$$F\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$$

and chain complexes C and D over $\mathcal{A}$ and $\mathcal{A}'$, respectively, a new chain complex $F(C, D)$ by

$$F(C, D)_n := \prod_{p \in \mathbb{Z}} F(C_p, D_{p+n})$$

$$d^{F(C,D)}\colon \prod_{p \in \mathbb{Z}} F(C_p, D_{p+n}) \xrightarrow{\prod_p d^C \circ - + (-1)^{n+1} - \circ d^D} \prod_{p \in \mathbb{Z}} F(C_p, D_{p+n-1})$$

Note that this recovers our definition of $\underline{\mathrm{Hom}}$ (even though our notation is not quite consistent).

3.2.7. **Theorem.** *Let R be a hereditary ring and let C and D be chain complexes of R-modules. If C is degreewise projective, then there exists a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H(C), H(D))[-1] \to H(\underline{\mathrm{Hom}}_R(C, D)) \xrightarrow{\mathrm{ev}} \underline{\mathrm{Hom}}_R(H(C), H(D)) \to 0.$$

Moreover, this sequence splits.

Proof. The proof is very similar to the proof of the Künneth formula. Let Z and B denote the graded modules of cycles and chains in C, respectively. Then $0 \to Z \to C \to B[1] \to 0$ is exact, and both Z and B are degreewise projective. Hence

$$0 \to \underline{\mathrm{Hom}}_R(B[1], D) \to \underline{\mathrm{Hom}}_R(C, D) \to \underline{\mathrm{Hom}}_R(Z, D) \to 0$$

is exact by Lemma 3.1.4 and because products of exact sequences are exact. Taking homology and using that $\underline{\mathrm{Hom}}_R(Z[1], D) \cong \underline{\mathrm{Hom}}_R(Z, D)[-1]$, we obtain an exact sequence

$$H(\underline{\mathrm{Hom}}_R(Z, D))[-1] \xrightarrow{\partial[-1]} H(\underline{\mathrm{Hom}}_R(B, D))[-1] \to H(\underline{\mathrm{Hom}}_R(C, D))$$
$$\to H(\underline{\mathrm{Hom}}_R(Z, D)) \xrightarrow{\partial} H(\underline{\mathrm{Hom}}_R(B, D)).$$

Now apply Lemma 3.2.6 to obtain the isomorphic exact sequence

$$\underline{\mathrm{Hom}}_R(Z, H(D))[-1] \xrightarrow{\partial[-1]} \underline{\mathrm{Hom}}_R(B, H(D)))[-1] \to H(\underline{\mathrm{Hom}}_R(C, D))$$
$$\to \underline{\mathrm{Hom}}_R(Z, H(D)) \xrightarrow{\partial} \underline{\mathrm{Hom}}_R(B, H(D)).$$

As in the proof of the Künneth theorem, the boundary map ∂ is given by restriction along the inclusion map $B \to Z$. Hence the natural short exact sequence

$$0 \to \mathrm{coker}(\partial[-1]) \to H(\underline{\mathrm{Hom}}_R(C, D)) \to \ker(\partial) \to 0$$

is isomorphic to the claimed sequence because $0 \to B \to Z \to H(C) \to 0$ is a projective resolution of $H(C)$.

To show that the sequence splits, use Lemma 1.2.3 to choose quasi-isomorphisms $C \xleftarrow{\sim} P \xrightarrow{\sim} H(C)$ and $D \xleftarrow{\sim} Q \xrightarrow{\sim} H(D)$ with P and Q degreewise projective. Then the exact sequence in question becomes identified with the short exact sequence

$$0 \to \mathrm{Ext}^1_R(H(P), H(Q))[-1] \to H(\underline{\mathrm{Hom}}_R(P, Q)) \xrightarrow{\mathrm{ev}} \underline{\mathrm{Hom}}_R(H(P), H(Q)) \to 0,$$

and the quasi-isomorphism $Q \to H(D)$ in turn identifies this with the short exact sequence

$$0 \to \mathrm{Ext}^1_R(H(P), H(D))[-1] \to H(\underline{\mathrm{Hom}}_R(P, H(D))) \xrightarrow{\mathrm{ev}} \underline{\mathrm{Hom}}_R(H(P), H(D)) \to 0.$$

The chosen quasi-isomorphism $P \xrightarrow{\sim} H(C)$ induces a chain map $\alpha\colon P \to H(P)$ satisfying $H(\alpha) = \mathrm{id}$. Then the restriction map α^* is a section to ev: if $\varphi\colon H(P)[-k] \to H(D)$ is a homomorphism, precomposition with α induces a chain map $P[-k] \xrightarrow{\varphi\circ\alpha[-k]} H(D)$. Since ev sends this chain map to the map $H(P)[-k] \to H(D)$ induced by $\varphi \circ \alpha[-k]$ in homology, $\mathrm{ev}(\varphi \circ \alpha[-k]) = \varphi$. □

3.2.8. **Corollary.** *Let X be a topological space, let R be a hereditary ring and let M be an R-module. Then there exists for each $n \geq 0$ a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X; R), M) \to H^n(X; M) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_R(H_n(X; R), M) \to 0.$$

Moreover, this sequence splits.

Proof. By definition, we have $H^n(X; M) \cong H_{-n}(\underline{\mathrm{Hom}}_R(C^{\mathrm{sing}}(X; R), M))$. Since

$$\big(\mathrm{Ext}^1_R(H(C), M[0])[-1]\big)_{-n} = \mathrm{Ext}^1_R(H_{n-1}(X; R), M),$$

we obtain the short exact sequence as a special case of Theorem 3.2.7. □

3.3. Products on singular cohomology.

As announced at the beginning of this section, the main new feature of singular cohomology is that it admits an additional ring structure which is not present on homology. From here on out, we assume that R is a **commutative** ring.

3.3.1. **Construction.** Consider the following data:

- C and D chain complexes of abelian groups;
- M and N chain complexes of R-modules.

Then we obtain natural chain maps

$$\lambda\colon \underline{\mathrm{Hom}}(C, M) \otimes_R \underline{\mathrm{Hom}}(D, N) \to \underline{\mathrm{Hom}}(C \otimes D, M \otimes_R N)$$

by taking the adjoint of the chain maps

$$\begin{aligned}&\underline{\mathrm{Hom}}(C, M) \otimes_R \underline{\mathrm{Hom}}(D, N) \otimes C \otimes D\\ &\qquad \xrightarrow{\tau} (\underline{\mathrm{Hom}}(C, M) \otimes C) \otimes_R (\underline{\mathrm{Hom}}(D, M) \otimes D)\\ &\qquad \xrightarrow{\mathrm{ev}_C \otimes \mathrm{ev}_D} M \otimes_R N.\end{aligned}$$

In particular, we have $\lambda(\varphi \otimes \psi)(x \otimes y) = (-1)^{|x||\psi|}\varphi(x) \otimes \psi(y)$.

3.3.2. **Definition.** The *cross product* is the natural transformation

$$\times\colon H^*(-_1; M) \otimes_R H^*(-_2; N) \Rightarrow H^*(-_1 \times -_2; M \otimes_R N)$$

of functors $\mathrm{Top} \times \mathrm{Top} \to \mathrm{grAb}$ whose component at (X, Y) is the map

$$\begin{aligned}H(C_{\mathrm{sing}}(X; M)) \otimes_R H(C_{\mathrm{sing}}(Y; N)) &\xrightarrow{\mu} H(C_{\mathrm{sing}}(X; M) \otimes_R C_{\mathrm{sing}}(Y; N))\\ &\cong H(\underline{\mathrm{Hom}}(C^{\mathrm{sing}}(X), M[0]) \otimes_R \underline{\mathrm{Hom}}(C^{\mathrm{sing}}(Y), N[0]))\\ &\xrightarrow{H(\lambda)} H(\underline{\mathrm{Hom}}(C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(Y), M \otimes_R N))\\ &\xrightarrow{H(AW^*)} H(\underline{\mathrm{Hom}}(C^{\mathrm{sing}}(X \times Y), M \otimes_R N))\\ &\cong H^*(X \times Y; M \otimes_R N)\end{aligned}$$

3.3.3. **Definition.** Define
$$\eta \in H^0(*;R) \cong \operatorname{Hom}(H_0(*),R)$$
to be the unique class corresponding to the homomorphism $H_0(*) \to R$ induced by the map $C^{\mathrm{sing}}(*) \xrightarrow{\varepsilon} \mathbb{Z}[0] \xrightarrow{1_R} R[0]$.

3.3.4. **Lemma.** *Let $f\colon X' \to X$, $g\colon Y' \to Y$ and $h\colon Z' \to Z$ be continuous maps, and let L, M and N be R-modules. Consider $\alpha \in H^p(X;L)$, $\beta \in H^q(Y;M)$ and $\gamma \in H^r(Z;N)$.*

(1) *the cross product is natural: $f^*\alpha \times g^*\beta = (f \times g)^*(\alpha \times \beta)$;*
(2) *the cross product is associative: $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$;*
(3) *the cross product is unital: $\eta \times \alpha = \mathrm{pr}_1^* \alpha$ and $\alpha \times \eta = \mathrm{pr}_2^* \alpha$, where $\mathrm{pr}_1\colon * \times X \to X$ and $\mathrm{pr}_2\colon X \times * \to X$ denote the "projection" maps;*
(4) *the cross product is graded-commutative: $t^*(\alpha \times \beta) = (-1)^{|\alpha||\beta|}\beta \times \alpha$, where $t\colon Y \times X \to X \times Y$ is the flip map.*

Proof. The first assertion spells out what it means for $\times$ to be a natural transformation. The other assertions follow from the definitions of μ and λ together with Proposition 2.1.6 by carefully unwinding all definitions. In order to reduce the size of the following diagrams, let us introduce the following abbreviations:

(1) our notation will suppress the coefficient modules L, M, N, $L \otimes_R M$, ... completely; for each term, there is only one coefficient module that makes sense;
(2) $C(X) := C^{\mathrm{sing}}(X)$, and similarly for other spaces;
(3) if A is a chain complex, we write $A^\vee := \underline{\operatorname{Hom}}(A, \text{appropriate coefficient module})$.

To check associativity, we claim that the following diagram commutes:

$$\begin{tikzcd}
H(C(X)^\vee) \otimes H(C(Y)^\vee) \otimes H(C(Z)^\vee) \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"] & H(C(X)^\vee \otimes C(Y)^\vee) \otimes H(C(Z)^\vee) \arrow[r, "H(\lambda)\otimes\mathrm{id}"] \arrow[d, "\mu"] & H((C(X)\otimes C(Y))^\vee) \otimes H(C(Z)^\vee) \arrow[r, "H(\mathrm{AW}^*)\otimes\mathrm{id}"] \arrow[d, "\mu"] & H(C(X\times Y)^\vee) \otimes H(C(Z)^\vee) \arrow[d, "\mu"] \\
H(C(X)^\vee) \otimes H(C(Y)^\vee \otimes C(Z)^\vee) \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\otimes H(\lambda)"] & H(C(X)^\vee \otimes C(Y)^\vee \otimes C(Z)^\vee) \arrow[r, "H(\lambda\otimes\mathrm{id})"] \arrow[d, "H(\mathrm{id}\otimes\lambda)"] & H((C(X)\otimes C(Y))^\vee \otimes C(Z)^\vee) \arrow[r, "H(\mathrm{AW}^*\otimes\mathrm{id})"] \arrow[d, "H(\lambda)"] & H(C(X\times Y)^\vee \otimes C(Z)^\vee) \arrow[d, "H(\lambda)"] \\
H(C(X)^\vee) \otimes H((C(Y)\otimes C(Z))^\vee) \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\otimes H(\mathrm{AW}^*)"] & H(C(X)^\vee \otimes (C(Y)\otimes C(Z))^\vee) \arrow[r, "H(\lambda)"] \arrow[d, "H(\mathrm{id}\otimes\mathrm{AW}^*)"] & H((C(X)\otimes C(Y)\otimes C(Z))^\vee) \arrow[r, "H(\mathrm{AW}^*\otimes\mathrm{id})"] \arrow[d] & H((C(X\times Y)\otimes C(Z))^\vee) \arrow[d, "H(\mathrm{AW}^*)"] \\
H(C(X)^\vee) \otimes H(C(Y\times Z)^\vee) \arrow[r, "\mu"] & H(C(X)^\vee \otimes C(Y\times Z)^\vee) \arrow[r, "H(\lambda)"] & H((C(X)\otimes C(Y\times Z))^\vee) \arrow[r, "H(\mathrm{AW}^*)"] & H((C(X\times Y\times Z)^\vee)
\end{tikzcd}$$

The bottom right square commutes by virtue of Proposition 2.1.6. All other squares commute by inspection.

To see that the cross product is graded-commutative, we verify that the diagram

$$\begin{tikzcd}
H(C(X)^\vee) \otimes H(C(Y)^\vee) \arrow[r, "\mu"] \arrow[d, "\tau"] & H(C(X)^\vee \otimes C(Y)^\vee) \arrow[r, "H(\lambda)"] \arrow[d, "H(\tau)"] & H(C(X)\otimes C(Y))^\vee) \arrow[r, "H(\mathrm{AW}^*)"] \arrow[d, "H(\tau^*)"] & H(C(X\times Y)^\vee) \arrow[d, "t^*"] \\
H(C(Y)^\vee) \otimes H(C(X)^\vee) \arrow[r, "\mu"] & H(C(Y)^\vee \otimes C(X)^\vee) \arrow[r, "H(\lambda)"] & H(C(Y)\otimes C(X))^\vee) \arrow[r, "H(\mathrm{AW}^*)"] & H(C(Y\times X)^\vee)
\end{tikzcd}$$

commutes. The left square commutes because

$$\begin{aligned}
(\mu\circ\tau)([x]\otimes[y]) &= \mu((-1)^{|x||y|}[y]\otimes[x]) = (-1)^{|x||y|}[y\otimes x] \\
&= [\tau(x\otimes y)] = H(\tau)([x\otimes y]) = (H(\tau)\circ\mu)([x]\otimes[y]).
\end{aligned}$$

Commutativity of the middle square follows similarly by unwinding definitions (this is easier to check using the chain map adjoint to λ, see Construction 3.3.1), and the right square commutes by Proposition 2.1.6.

For the assertion about unitality, we similarly consider the diagram

$$\begin{tikzcd}
H(C(*)^\vee) \otimes H(C(X)^\vee) \arrow[r, "\mu"] & H(C(*)^\vee \otimes C(X)^\vee) \arrow[r, "H(\lambda)"] & H(C(*)\otimes C(X))^\vee) \arrow[r, "H(\mathrm{AW}^*)"] & H(C(*\times X)^\vee) \\
H(\mathbb{Z}[0]^\vee) \otimes H(C(X)^\vee) \arrow[r, "\cong"] \arrow[u, "H(\varepsilon^\vee)\otimes\mathrm{id}"] & H(\mathbb{Z}[0]^\vee \otimes C(X)^\vee) \arrow[r, "\mathrm{id}"] \arrow[u, "H(\varepsilon^\vee\otimes\mathrm{id})"] & H((\mathbb{Z}[0]^\vee \otimes C(X))^\vee) \arrow[r, "\cong"] \arrow[u, "H((\varepsilon\otimes\mathrm{id})^\vee)"] & H(C(X)^\vee) \arrow[u, "\mathrm{pr}_1^*"', "\cong"]
\end{tikzcd}$$

The left and middle square commute by direct inspection, and the right square commutes by virtue of Proposition 2.1.6. Now observe that the composition along the top row computes the cross product, and the leftmost vertical arrow sends the class represented by $\mathrm{id}_{\mathbb{Z}}$ in $H(\mathbb{Z}[0]^\vee)$ to η. Since composition along the bottom row sends $[\mathrm{id}_{\mathbb{Z}}] \otimes \alpha$ to α, the assertion about left unitality follows. Right unitality is immediate since we have already checked that the cross product is graded-commutative. □

Inspecting the definition of the cross product, we observe that it can almost be identified with the map μ. The only map which is potentially not an isomorphism is $H(\lambda)$. If $H(\lambda)$ was an isomorphism, we would immediately obtain a Künneth theorem for cohomology from the algebraic Künneth formula. This motivates the next lemma.

3.3.5. **Lemma.** *Let C, D, M and N be chain complexes over a noetherian hereditary ring R. Assume that*

(1) *C and D are degreewise projective;*
(2) *$H(C)$ and $H(D)$ are bounded below;*
(3) *M and N are bounded above;*
(4) *$H(C)$ and M are degreewise finitely generated.*

Then $\lambda\colon \underline{\mathrm{Hom}}_R(C, M) \otimes \underline{\mathrm{Hom}}_R(D, N) \to \underline{\mathrm{Hom}}_R(C \otimes_R D, M \otimes_R N)$ is a chain homotopy equivalence.

Proof. Lemma 1.2.3 (including its proof) shows that there exist degreewise projective chain complexes P and Q and quasi-isomorphisms $\varphi\colon P \xrightarrow{\sim} C$ and $\psi\colon Q \xrightarrow{\sim} D$ such that P and Q are bounded below, and P can additionally be chosen to be degreewise finitely generated. Since C and D are assumed to be degreewise projective themselves, φ and ψ are chain homotopy equivalences. Consequently, the vertical maps in the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_R(C, M) \otimes \underline{\mathrm{Hom}}_R(D, N) \arrow[r, "\lambda"] \arrow[d, "\varphi^* \otimes \psi^*"'] & \underline{\mathrm{Hom}}_R(C \otimes_R D, M \otimes_R N) \arrow[d, "(\varphi\otimes\psi)^*"] \\
\underline{\mathrm{Hom}}_R(P, M) \otimes \underline{\mathrm{Hom}}_R(Q, N) \arrow[r, "\lambda"] & \underline{\mathrm{Hom}}_R(P \otimes_R Q, M \otimes_R N)
\end{tikzcd}$$

are chain homotopy equivalences, and we may concentrate on the lower horizontal map. Let us decode what the domain and target of this map are in each degree n. For the domain, we get

$$\begin{aligned}
&(\underline{\mathrm{Hom}}_R(P, M) \otimes \underline{\mathrm{Hom}}_R(Q, N))_n \\
&\quad = \bigoplus_{k+l=n} \left(\prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(P_p, M_{p+k}) \right) \otimes \left(\prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(Q_q, N_{q+l}) \right) \\
&\quad \cong \bigoplus_{k+l=n} \bigoplus_{p,q} \mathrm{Hom}_R(P_p, M_{p+k}) \otimes \mathrm{Hom}_R(Q_q, N_{q+l}) \\
&\quad \cong \bigoplus_{r+s=p+q+n} \mathrm{Hom}_R(P_p, M_r) \otimes_R \mathrm{Hom}_R(Q_q, N_s),
\end{aligned}$$

where the first isomorphism uses the boundedness assumptions, and the second reindexes the direct sums by $r = p + k$ and $s = q + l$. For the target, we similarly obtain

$$\begin{aligned}
&(\underline{\mathrm{Hom}}_R(P \otimes_R Q, M \otimes_R N))_n \\
&\quad = \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(\bigoplus_{p+q=k} P_p \otimes_R Q_q, \bigoplus_{r+s=p+q+n} M_r \otimes_R N_s) \\
&\quad \cong \bigoplus_{r+s=p+q+n} \mathrm{Hom}_R(P_p, M_r) \otimes_R \mathrm{Hom}_R(Q_q, N_s)
\end{aligned}$$

because the boundedness assumptions on P and Q imply that we only need to consider finitely many indices.

Therefore, it is enough to show that λ is an isomorphism if P, Q, M and N are R-modules such that P and Q are projective and P and M are finitely generated. Exhibiting P and Q as direct summands of free modules R^m and $R[Y]$, we see that the instance of λ for P, Q M and N is a retract of the homomorphism

$$\lambda\colon \mathrm{Hom}_R(R^m, M) \otimes \mathrm{Hom}_R(R[Y], N) \to \mathrm{Hom}_R(R^m \otimes_R R[Y], M \otimes_R N).$$

This map is identified with

$$\bigoplus_m M \otimes_R \mathrm{Hom}_R(R[Y], N) \xrightarrow{\bigoplus_m \lambda} \bigoplus_X \mathrm{Hom}_R(R[Y], M \otimes_R N).$$

Since M is finitely generated and R is noetherian, there exists a finite presentation

$$R^a \to R^b \to M \to 0$$

of M. Since $- \otimes \mathrm{Hom}_R(R[Y], N)$, $- \otimes_R N$ and $\mathrm{Hom}_R(R[Y], -)$ are right exact (the last one by Lemma 3.1.4 because $R[Y]$ is projective), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
R^a \otimes \mathrm{Hom}_R(R[Y], N) & \longrightarrow & R^b \otimes \mathrm{Hom}_R(R[Y], N) & \longrightarrow & M \otimes \mathrm{Hom}_R(R[Y], N) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_R(R[Y], R^a \otimes_R N) & \to & \mathrm{Hom}_R(R[Y], R^b \otimes_R N) & \to & \mathrm{Hom}_R(R[Y], M \otimes_R N) & \to & 0
\end{array}$$

The left and middle vertical maps are isomorphisms by inspection, so it follows that the right vertical map is also an isomorphism. □

There are other sets of assumption that would allow us to conclude that λ is a chain homotopy equivalence. Lemma 3.3.5 is formulated in auch a way that we can deduce a Künneth formula for singular cohomology from it.

We will also require the following.

3.3.6. **Proposition** (Chase). *Let R be an arbitrary commutative ring. Then the following are equivalent:*

(1) *Products of flat modules are flat.*
(2) *Every finitely generated submodule of a free module is finitely presented.*

In particular, products of flat (left/right) modules are flat over (left/right) noetherian rings.

Proof. Let us first show that (2) implies (1). Let $(M_\alpha)_\alpha$ be a family of flat left R-modules and set $M := \prod_\alpha M_\alpha$. Then $\prod_\alpha (M_\alpha \otimes_R -)$ is an exact functor, and $M \otimes_R -$ is a right exact functor. Note that we have a natural transformation $M \otimes_R - \Rightarrow \prod_\alpha (M_\alpha \otimes_R -)$.

If N is finitely generated, choose an exact sequence $0 \to K \to R^n \to N \to 0$. Applying the two functors above yields a commutative diagram

$$\begin{tikzcd}
 & M \otimes_R K \arrow[r] \arrow[d] & M \otimes_R R^n \arrow[r] \arrow[d] & M \otimes_R N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_\alpha (M_\alpha \otimes_R K) \arrow[r] & \prod_\alpha (M_\alpha \otimes_R R^n) \arrow[r] & \prod_\alpha (M_\alpha \otimes_R N) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The middle vertical arrow is an isomorphism, so the right vertical map is an epimorphism. In particular, if N is finitely presented and we have chosen the exact sequence $0 \to K \to R^n \to N \to 0$ such that K is finitely generated, the left vertical map is also an epimorphism. This implies that the right vertical map is an isomorphism for N finitely presented. Under assumption (2), K is then also finitely presented, so the left vertical map is also an isomorphism. This shows that the upper row is exact, so $\mathrm{Tor}_1^R(M, N) = 0$ for every finitely presented module N.

Returning to the case that N is finitely generated, we can write

$$K \cong \operatorname*{colim}_{L \subseteq K \text{ fin. gen.}} L$$

as a directed colimit over its finitely generated submodules. Then

$$N \cong \operatorname*{colim}_{L \subseteq K \text{ fin. gen.}} R^n/L$$

is a filtered colimit of finitely presented modules. Since Tor_1^R commutes with filtered colimits, we have $\mathrm{Tor}_1^R(M, N) = 0$ for all finitely generated modules, and an analogous argument allows us to conclude that $\mathrm{Tor}_1^R(M, N) = 0$ for all modules N. This is equivalent to asserting that $M \otimes_R -$ is exact, so M is flat.

Let us now prove the converse direction. Consider a free module $R[B]$ and a finitely generated submodule $M \subseteq R[B]$. If $x_1, \ldots, x_k$ generate M, then each x_i can be written in the form $\sum_{b \in B} r_{i,b} b$ with all but finitely many of the $r_{i,b}$ being zero. Hence the set $\{b \in B \mid r_{i,b} \neq 0 \text{ for some } i\}$ is finite, so we may assume without loss of generality that B is finite.

Our goal is to show that kernel K of the tautological epimorphism $\pi \colon R[x_1, \ldots, x_k] \to M$ is finitely generated. For $y \in K$, write $y = \sum_{i=1}^k \lambda_i(y) x_i$ and set

$$z_i := (\lambda_i(y))_{y \in K} \in \prod_{y \in K} R.$$

Expanding $x_i = \sum_{b \in B} \xi_{i,b} b$ in $R[B]$, we obtain

$$0 = \sum_{i=1}^k \lambda_i(y) \cdot \sum_{b \in B} \xi_{i,b} b = \sum_{b \in B} \left(\sum_{i=1}^k \lambda_i(y) \xi_{i,b} \right) b,$$

so each of the cofficients is zero. We interpret this equation as the assertion that the diagram

$$\begin{tikzcd}
R[B] \arrow[d, "\Xi"'] \arrow[dr, "0"] & \\
R[z_1, \ldots, z_k] \arrow[r] & \prod_{y \in K} R
\end{tikzcd}$$

commutes, where Ξ is given by the matrix $(\xi_{i,b})_{1 \leq i \leq k, b \in B}$, and the lower horizontal map is the tautological one. The above diagram represents a morphism in the category $E(\prod_{y \in K} R)$ introduced in the proof of Proposition 1.1.9. Since $\prod_{y \in K} R$ is assumed to be flat, the proof of Proposition 1.1.9 informs us that $E(\prod_{y \in K} R)$ is filtered. Consequently, the above diagram extends to a diagram

$$\begin{tikzcd}
R[B] \arrow[d, "\Xi"'] \arrow[dr, "0"] & \\
R[z_1, \dots, z_k] \arrow[r] \arrow[d, "\mu=(\mu_{j,i})_{j,i}"'] & \prod_{y \in K} R \\
R^l \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

such that the composite map $\mu \circ \Xi \colon R[B] \to R^l$ is zero. Define a homomorphism $R^l \to R[x_1, \dots, x_k]$ by sending the j-th standard basis vector to $\sum_{i=1}^k \mu_{j,i} x_i$. Then

$$\pi(\sum_{i=1}^k \mu_{j,i} x_i) = \sum_{i=1}^k \mu_{j,i} \cdot \sum_{b \in B} \left(\sum_{i=1}^k \mu_{j,i} \xi_{i,b} \right) b.$$

Each coefficient is a matrix coefficient of the composite $\mu \circ \Xi$, so $\sum_{i=1}^k \mu_{j,i} x_i \in K$.

It now suffices to check that the induced map $\kappa \colon R^l \to K$ is surjective. Let $y \in K$. Commutativity of the lower part of the above diagram translates to the equation

$$\lambda_i(y) = \sum_{j=1}^l \varphi_j(y) \mu_{j,i}$$

for all $1 \le i \le k$, $1 \le j \le l$ and $y \in K$, where φ_j denotes the j-th component of φ. Consequently, we obtain

$$y = \sum_{i=1}^k \lambda_i(y) x_i = \sum_{i=1}^k \left(\sum_{j=1}^l \varphi_j(y) \mu_{j,i} \right) x_i = \sum_{j=1}^l \varphi_j(y) \cdot \sum_{i=1}^k \mu_{j,i} x_i,$$

which shows that y lies in the image of κ. □

3.3.7. **Lemma.** *Let R be a principal ideal domain. Then an R-module is flat if and only if it is torsionfree.*

Proof. One direction is immediate from Lemma 1.1.20 (6). If M is a torsionfree module, then the same statement shows that $\operatorname{Tor}_1^R(M, R/(r)) = 0$ for all $r \in R$. By the classification of finitely generated modules and Lemma 1.1.20 (1), this implies $\operatorname{Tor}_1^R(M, N) = 0$ for all finitely generated modules, and therefore $\operatorname{Tor}_1^R(M, N) = 0$ for all modules by Lemma 1.1.20 (2) since every module is a directed colimit of its finitely generated submodules. Hence M is flat. □

3.3.8. **Theorem** (Künneth formula for singular cohomology)**.** *Let X and Y be topological spaces, let R be a principal ideal domain and let M be an R-module. If $H(X; R)$ is degreewise finitely generated, then there exists a short exact sequence*

$$0 \to H^*(X; R) \otimes_R H^*(Y; M) \xrightarrow{\times} H^*(X \times Y; M)$$
$$\to \operatorname{Tor}_1^R(H^*(X; R), H^*(Y; M))[-1] \to 0.$$

Moreover, this sequence splits.

Proof. Our assumptions guarantee that the cross product map is isomorphic to the map

$$\mu \colon H(C_{\text{sing}}(X; R)) \otimes_R H(C_{\text{sing}}(Y; M)) \to H(C_{\text{sing}}(X; R) \otimes_R C_{\text{sing}}(Y; M))$$

by Lemma 3.3.5.

Observing that

$$C_{\text{sing}}(X;)_n \cong \operatorname{Hom}_R(C^{\text{sing}}_{-n}(X; R), R[0])$$

is a product of free modules, Proposition 3.3.6 implies that $C_{\mathrm{sing}}(X;R)$ is degreewise flat because R is noetherian. Let Z and B denote the subcomplexes of cycles and boundaries in $C_{\mathrm{sing}}(X;R)$. Two applications of Lemma 3.3.7 imply that Z and B are degreewise flat. By inspection of the proof of Theorem 1.2.2, one sees that these assumptions are enough to make the proof work (using that Tor can be computed using flat resolutions), so we obtain the desired short exact sequence from the algebraic Künneth formula. The splitting arises again from Lemma 1.2.3.

The indexing shift comes from our indexing convention for cohomology. □

3.3.9. **Remark.** To remove any confusion about the correct indexing, the short exact sequence of Theorem 3.3.8 in degree n reads

$$0 \to \bigoplus_{p+q=n} H^p(X;M) \otimes_R H^q(Y;N) \xrightarrow{\times} H^n(X \times Y; M \otimes_R N)$$

$$\to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(H^p(X;M), H^q(Y;N)) \to 0.$$

3.3.10. **Corollary.** *If either*

(1) *R is a field and $H(X;R)$ is degreewise finitely generated or*
(2) *R is a principal ideal domain and $H(X;R)$ is degreewise finitely generated free,*

then

$$\times\colon H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$$

is an isomorphism.

Proof. The assumptions imply that that the Tor-terms appearing in Theorem 3.3.8 vanish. □

With the "external" cross product defined, we can now equip the cohomology of a space with a product structure by pulling back the cross product along the diagonal map.

3.3.11. **Definition.** Let R be a ring and let X be a topological space. Define the *cup product* as the composition

$$\cup\colon H^*(X;R)\otimes_R H^*(X;R) \xrightarrow{\times} H^*(X\times X;R\otimes_R R) \xrightarrow{\Delta^*} H^*(X;R\otimes_R R) \cong H^*(X;R).$$

3.3.12. **Remark.** We have used the cross product to define the cup product. It is also possible to extract the cross product from the cup product: given topological spaces X and Y, let $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$ be the projection maps. Then we have for $\alpha \in H^p(X;R)$ and $\beta \in H^q(Y;R)$

$$\mathrm{pr}_X^* \alpha \cup \mathrm{pr}_Y^* \beta = \Delta^*(\mathrm{pr}_X^* \alpha \times \mathrm{pr}_Y^* \beta) = \Delta^*(\mathrm{pr}_X \times \mathrm{pr}_Y)^*(\alpha \times \beta) = \alpha \times \beta$$

by Lemma 3.3.4 (1) because $(\mathrm{pr}_X \times \mathrm{pr}_Y) \circ \Delta = \mathrm{id}_{X\times Y}$.

Next up, we define the appropriate category in which the cohomology of spaces, equipped with the cup product, lives.

3.3.13. **Definition.** A *graded-commutative R-algebra* $(H, \cdot, \eta)$ is a graded R-module $H = \bigoplus_{n\in\mathbb{N}} H^n$ together with a morphism

$$\cdot\colon H \otimes_R H \to H$$

and an element $\eta \in H^0$ such that the following holds:

(1) the product is associative: $(\alpha_1 \cdot \alpha_2) \cdot \alpha_3 = \alpha_1 \cdot (\alpha_2 \cdot \alpha_3)$;
(2) the product is unital: $\eta \cdot \alpha = \alpha = \alpha \cdot \eta$;
(3) the product is graded-commutative: $\alpha_1 \cdot \alpha_2 = (-1)^{|\alpha_1|\alpha_2} \alpha_2 \cdot \alpha_1$ for all α_1, α_2 of homogeneous degree.

A *morphism* of graded-commutative R-algebras $\varphi\colon (H,\cdot,\eta) \to (H',\cdot',\eta')$ is a morphism of graded R-modules $\varphi\colon H \to H'$ such that $\varphi(\alpha_1 \cdot \alpha_2) = \varphi(\alpha_1)\cdot' \varphi(\alpha_2)$ and $\varphi(\eta) = \eta'$.

The category of graded-commutative R-algebras and their morphisms will be denoted by $\mathrm{grCAlg}_{\mathrm{R}}$.

Recall that we defined a distinguished element $\eta \in H^0(*;R)$ which acts as a unit element for the cross product. Since the projection map $\mathrm{pr}\colon X \to *$ induces for each topological space X a morphism $H^*(*;R) \to H^*(X;R)$, we obtain an element in $\eta_X := \mathrm{pr}^*\eta \in H^0(X;R)$. By abuse of notation, we will denote this element also by η.

3.3.14. **Proposition.** *Sending a topological space X to the triple $(H^*(X;R),\cup,\eta)$ and continuous maps to the induced map in cohomology defines a functor*

$$H^*(-;R)\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{grCAlg}_{\mathrm{R}}.$$

Proof. We know that $H^*(-;R)$ is a functor with values in graded R-modules, so we only have to show that the cup product equips $H^*(X;R)$ with the structure of a graded-commutative ring, and that each continuous map $f\colon X \to Y$ induces a morphism of graded R-algebras.

Fix a topological space X and let $\alpha \in H^p(X;R)$, $\beta \in H^q(X;R)$ and $\gamma \in H^r(X;R)$ be elements in $H^*(X;R)$. Using Lemma 3.3.4 and the relation $(\Delta\times\mathrm{id})\circ\Delta = (\mathrm{id}\times\Delta)\circ\Delta$, we obtain

$$\begin{aligned}
(\alpha\cup\beta)\cup\gamma &= \Delta^*(\Delta^*(\alpha\times\beta)\times\gamma)\\
&= \Delta^*(\Delta\times\mathrm{id})^*((\alpha\times\beta)\times\gamma)\\
&= \Delta^*(\Delta\times\mathrm{id})^*(\alpha\times(\beta\times\gamma)\\
&= \Delta^*(\mathrm{id}\times\Delta)^*(\alpha\times(\beta\times\gamma))\\
&= \Delta^*(\alpha\times\Delta^*(\beta\times\gamma))\\
&= \Delta^*(\alpha\times(\beta\cup\gamma))\\
&= \alpha\cup(\beta\cup\gamma).
\end{aligned}$$

Moreover, Lemma 3.3.4 also implies

$$\begin{aligned}
\alpha\cup\beta &= \Delta^*(\alpha\times\beta)\\
&= \Delta^*((-1)^{|\alpha||\beta|}t^*(\beta\times\alpha))\\
&= (-1)^{|\alpha||\beta|}\beta\cup\alpha,
\end{aligned}$$

where $t\colon X\times X \to X\times X$ is the flip map, because $t\circ\Delta = \Delta$. To see that η is in fact a unit for the multiplication, we obtain from Lemma 3.3.4 (1) the commutative diagram

$$\begin{tikzcd}
H^0(*;R)\otimes H^p(X;R) \arrow[r, "\times"] \arrow[d, "\mathrm{pr}^*\otimes\mathrm{id}"'] & H^p(X;R) \arrow[d, "(\mathrm{pr}\times\mathrm{id})^*"'] \arrow[dr, "\mathrm{id}"] & \\
H^0(X;R)\otimes H^p(X;R) \arrow[r, "\times"] & H^p(X\times X;R) \arrow[r, "\Delta^*"] & H^p(X;R)
\end{tikzcd}$$

and Lemma 3.3.4 (3) shows that the instance of the cross product on the top line sends $\eta\otimes\alpha$ to α. $\square$

3.3.15. **Remark.** Let $(H,\cdot,\eta)$ be a graded-commutative R-algebra. If $\alpha \in H^{2n+1}$ is an odd-degree element, then $\alpha\cup\alpha = (-1)^{(2n+1)^2}\alpha\cup\alpha$ by graded-commutativity, so

$$2(\alpha\cup\alpha) = 0.$$

Unless R has characteristic 2, this already imposes some restrictions on the products in graded-commutative R-algebras.

3.3.16. **Example.**

(1) Denote by $P_R(x)$ the polynomial ring over R in one variable x. For even d, we can turn $P_R(x)$ into a graded-commutative R-algebra by declaring

$$P_R(x)^k := \begin{cases} Rx^{\frac{k}{d}} & d \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases}$$

If R has characteristic 2, this definition also makes sense for odd d, see Remark 3.3.15.

(2) In analogy to the preceding example, there are also truncated polynomial rings $P_R(x)/x^{n+1}$ for a generator in even degree d (and also for odd d if R has characteristic 2).

(3) For arbitrary $d \geq 1$, the exterior algebra $\Lambda_R(x)$ on a generator in degree d has

$$\Lambda_R(x)^k := \begin{cases} R & k = 0, d, \\ 0 & \text{otherwise.} \end{cases}$$

Note that there is only one choice of graded-commutative R-algebra structure on this graded module. If d is even (or the characteristic of R is 2), there is an isomorphism of graded-commutative rings

$$\Lambda_R(x) \cong P_R(x)/x^2.$$

3.3.17. **Example.** We have

$$H^*(S^n) \cong \Lambda_{\mathbb{Z}}(x)$$

with $|x| = n$.

3.3.18. **Definition.** Let H and H' be graded-commutative R-algebras. Define the tensor product $H \otimes_R H'$ by

$$(H \otimes H')^n := \bigoplus_{p+q=n} H^p \otimes_R (H')^q$$

together with the unit map

$$R \cong R \otimes_R R \xrightarrow{\eta \otimes \eta'} H \otimes_R H'$$

and the multiplication map

$$(H \otimes_R H') \otimes_R (H \otimes_R H') \cong (H \otimes_R H) \otimes_R (H' \otimes_R H') \xrightarrow{\cdot \otimes \cdot'} H \otimes_R H'$$

which sends $x \otimes x'$ and $y \otimes y'$ to $(-1)^{|x'||y|} xy \otimes x'y'$.

3.3.19. **Remark.** The tensor product allows us to easily make sense of polynomial and exterior algebras with more than one generator, eg

$$P_R(x_1, \dots, x_k) \cong P_R(x_1) \otimes \dots \otimes P_R(x_k).$$

3.3.20. **Lemma.** *The cross product map*

$$\times \colon H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$$

is a morphism of graded-commutative R-algebras.

Proof. Let $\alpha_1 \in H^p(X; R)$, $\alpha_2 \in H^{p'}(X; R)$ and $\beta_1 \in H^q(Y; R)$, $\beta_2 \in H^{q'}(Y; R)$. Then

$$(\alpha_1 \otimes \alpha_2) \cup (\beta_1 \otimes \beta_2) = (-1)^{|\alpha_2||\beta_1|} (\alpha_1 \cup \alpha_2) \otimes (\beta_1 \cup \beta_2),$$

so we calculate using Lemma 3.3.4 that

$$\begin{aligned}
&(-1)^{|\alpha_2||\beta_1|}(\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2) \\
&\qquad = (-1)^{|\alpha_2||\beta_1|}\Delta_X^*(\alpha_1 \times \alpha_2) \times \Delta_Y^*(\beta_1 \times \beta_2) \\
&\qquad = (-1)^{|\alpha_2||\beta_1|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2) \\
&\qquad = \Delta_{X\times Y}^*(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\
&\qquad = (\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2)
\end{aligned}$$ □

In the fortunate circumstance that all the Tor-terms in the Künneth theorem vanish, Lemma 3.3.20 implies that the Künneth theorem actually describes the entire cohomology algebra of a product of spaces. One instance is the following.

3.3.21. **Corollary.** *For all $n \geq 1$, we have*

$$H^*(T^n) \cong \Lambda_{\mathbb{Z}}(x_1, \dots, x_n)$$

with $|x_i| = 1$ for all $1 \leq i \leq n$.

Proof. By Example 3.3.17, we have $H^*(S^1) \cong \Lambda_{\mathbb{Z}}(x)$ with $|x| = 1$. Combining Theorem 3.3.8 and Lemma 3.3.20, we obtain

$$H^*(T^n) \cong \bigotimes_{i=1}^n \Lambda_{\mathbb{Z}}(x_i) \cong \Lambda_{\mathbb{Z}}(x_1, \dots, x_n)$$

with $|x_i| = 1$. □

Calculating the cohomology ring of a space is typically a non-trivial task. We will give a slightly ad-hoc argument to determine the cohomology rings of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$; later on, we will obtain the same calculation as a corollary of Poincaré duality. This alone will suffice to give a number of applications which hopefully illustrate the usefulness of the cup product.

3.3.22. **Theorem.** *Let $n \geq 1$.*

(1) *Let $w \in H^1(\mathbb{R}P^n; \mathbb{F}_2)$ be the unique non-trivial element. Then*

$$H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong P_{\mathbb{F}_2}(w)/(w^{n+1}).$$

Under this identification, the inclusion map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n+1}$ induces the projection $P_{\mathbb{F}_2}(w)/(w^{n+2}) \to P_{\mathbb{F}_2}(w)/(w^{n+1})$.

(2) *Let $c \in H^2(\mathbb{C}\mathrm{P}^n)$ be a generator. Then*

$$H^*(\mathbb{C}\mathrm{P}^n) \cong P_{\mathbb{Z}}(c)/(c^{n+1}).$$

Under this identification, the inclusion map $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^{n+1}$ induces the projection $P_{\mathbb{Z}}(c)/(c^{n+2}) \to P_{\mathbb{Z}}(c)/(c^{n+1})$.

For the proof of the theorem, it will be convenient, although somewhat technical, to observe that we can also define relative cup products.

3.3.23. **Definition.** Let (X, A) and (Y, B) be pairs of spaces such that $(X \times Y, X \times B, A \times Y)$ is an excisive triad and let M and N be R-modules. Define the *relative cross product*

$$\times \colon H^*(X, A; M) \otimes_R H^*(Y, B; N) \to H^*((X, A) \times (Y, B); M \otimes_R N)$$

as the composition

$$\begin{aligned} & H(C_{\text{sing}}(X,A;M)) \otimes_R H(C_{\text{sing}}(Y,B;N)) \\ & \xrightarrow{\mu} H(C_{\text{sing}}(X,A;M) \otimes_R C_{\text{sing}}(Y,B;N)) \\ & \xrightarrow{H(\lambda)} H(\underline{\text{Hom}}(C^{\text{sing}}(X,A) \otimes C^{\text{sing}}(Y,B), (M \otimes_R N)[0])) \\ & \xrightarrow{\cong} H(C_{\text{sing}}((X,A) \times (Y,B); M \otimes_R N), \end{aligned}$$

where the final isomorphism arises from Lemma 2.2.8.

3.3.24. **Remark.** The properties of the cross product from Lemma 3.3.4 carry over in straightforward fashion to the relative case. In the following, we will only need to know that the relative cross product is natural in maps of pairs, which is obvious from the definition and Lemma 2.2.8.

The proof of Theorem 3.3.8 also carries over to obtain a Künneth theorem for pairs of spaces.

3.3.25. **Theorem.** *Let (X,A) and (Y,B) be pairs of spaces, let R be a principal ideal domain and let M be an R-module. If $H(X,A;R)$ is degreewise finitely generated and $(X \times Y, X \times B, A \times Y)$ is an excisive triad, then there exists a short exact sequence*

$$\begin{aligned} 0 \to H^*(X,A;R) \otimes_R H^*(Y,B;M) \xrightarrow{\times} H^*((X,A) \times (Y,B); M) \\ \to \text{Tor}_1^R(H^*(X,A;R), H^*(Y,B;M))[-1] \to 0. \end{aligned}$$

Moreover, this sequence splits.

In addition, we can define a relative cup product. This can be done is slightly greater generality than what follows below, but we try to give a definition which is completely parallel to the absolute case.

3.3.26. **Definition.** Let X be a topological space and let $A_1, A_2 \subseteq X$ be subspaces. If $(X \times X, X \times A_2, A_1 \times X)$ is an excisive triad, define

$$\cup\colon H^*(X,A_1;R) \otimes_R H^*(X,A_2;R) \xrightarrow{\times} H^*((X,A_1) \times (X,A_2); R) \xrightarrow{\Delta^*} H^*((X, A_1 \cup A_2); R).$$

3.3.27. **Remark.** By definition, the cup product is natural (in triples (X,A_1,A_2) such that $(X \times X, X \times A_2, A_1 \times X)$ is an excisive triad). It satisfies

$$\alpha \times \beta = \text{pr}_X^* \alpha \cup \text{pr}_Y^* \beta$$

for all $\alpha \in H^p(X,A;R)$ and $\beta \in H^q(Y,B;R)$, where $\text{pr}_X\colon (X \times Y, A \times Y) \to (X,A)$ and $\text{pr}_Y\colon (X \times Y, X \times B) \to (Y,B)$ denote the respective projections.

Let us not dive deeper into spelling out the formal properties of relative cross and cup products, since this is all we will need to know for our argument.

Proof of Theorem 3.3.22. We argue by induction, the case $n = 1$ being clear since $\mathbb{R}\text{P}^1 \cong S^1$.

We identify $\mathbb{R}\text{P}^k$ with the subspace

$$\{[x_0 : \ldots : x_k : 0 : \ldots : 0] \in \mathbb{R}\text{P}^n \mid (x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\}\}$$

and denote the inclusion map by $i_k\colon \mathbb{R}\text{P}^k \to \mathbb{R}\text{P}^n$. The universal coefficient theorem (Corollary 3.2.8) gives rise to the commutative square

$$\begin{array}{ccc} H^k(\mathbb{R}\text{P}^n; \mathbb{F}_2) & \xrightarrow{\cong} & \text{Hom}_{\mathbb{F}_2}(H_k(\mathbb{R}\text{P}^n; \mathbb{F}_2), \mathbb{F}_2) \\ {\scriptstyle i_k^*}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^k(\mathbb{R}\text{P}^k; \mathbb{F}_2) & \xrightarrow{\cong} & \text{Hom}_{\mathbb{F}_2}(H_k(\mathbb{R}\text{P}^k; \mathbb{F}_2), \mathbb{F}_2) \end{array}$$

We know from Topology I that the right vertical map is an isomorphism, so the left vertical map is an isomorphism as well.

In particular, the inductive assumption describes the entire product structure of $H^*(\mathbb{RP}^n;\mathbb{F}_2)$ in degrees $\leq n-1$ since i_k^* is a homomorphism of graded-commutative rings.

Define now for $k \leq n$

$$\mathbb{RP}^{n,k} := \{[0:\ldots:0:x_k:\ldots:x_n] \in \mathbb{RP}^n \mid (x_k,\ldots,x_n) \in \mathbb{R}^{n-k+1} \setminus \{0\}\}$$

and note that this subspace is homeomorphic to $\mathbb{RP}^{n-k}$. In analogy to $\mathbb{RP}^{n,k}$, define for $k < n$

$$\mathbb{R}^{n,k} := \{(0,\ldots,0,x_{k+1},\ldots,x_n) \in \mathbb{R}^n \mid (x_{k+1},\ldots,x_n) \in \mathbb{R}^{n-k}\},$$

which is abstractly homeomorphic to $\mathbb{R}^{n-k}$, but embedded into $\mathbb{R}^n$ in a specific way.

For $k < n$, consider the following commutative diagram:

$$\begin{tikzcd}
H^k(\mathbb{RP}^n;\mathbb{F}_2) \arrow[d, "\cong"'] & H^k(\mathbb{RP}^n,\mathbb{RP}^n \setminus \mathbb{RP}^{n,k};\mathbb{F}_2) \arrow[l] \arrow[r, "c_{n,k}^*"] \arrow[d, "i_k^*"] & H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^{n,k};\mathbb{F}_2) \arrow[d, "j_k^*"] \\
H^k(\mathbb{RP}^k;\mathbb{F}_2) & H^k(\mathbb{RP}^k,\mathbb{RP}^k \setminus \mathbb{RP}^{k,k};\mathbb{F}_2) \arrow[l] \arrow[r, "c_{k,k}^*"] & H^k(\mathbb{R}^k,\mathbb{R}^k \setminus \{0\};\mathbb{F}_2)
\end{tikzcd}$$

The vertical maps and horizontal maps on the left are induced by the appropriate inclusions. The right horizontal maps are induced by the following maps:

$$c_{n,k} \colon \mathbb{R}^n \to \mathbb{RP}^n, \quad (x_1,\ldots,x_n) \mapsto [x_1:\ldots:x_k:1:x_{k+1}:\ldots:x_n]$$

Note that $c_{k,k}$ is a characteristic map for the top-dimensional cell in $\mathbb{RP}^k$.

We claim that all maps in the above diagram are isomorphisms. To this end, observe first that for $k \leq n$

$$(\mathbb{RP}^n \setminus \mathbb{RP}^{n,k}) \times [0,1] \to \mathbb{RP}^k, \quad ([x_0:\ldots:x_n],t) \mapsto [x_0:\ldots:x_{k-1}:tx_k:\ldots:tx_n]$$

defines a deformation retraction onto $\mathbb{RP}^{k-1}$ (considered as a subspace of $\mathbb{RP}^n$ via i_{k-1}). This proves that both the horizontal map on the bottom left of the diagram and i_k^* are isomorphisms. The map $c_{k,k}^*$ is an isomorphism by excision. Finally, $j_k \colon (\mathbb{R}^k,\mathbb{R}^k \setminus \{0\}) \to (\mathbb{R}^k,\mathbb{R}^n \setminus \mathbb{R}^{n,k})$ is a homotopy equivalence of pairs, so it follows that all other maps are isomorphisms as well.

By an analogous argument, we obtain a zig-zag of isomorphisms

$$H^{n-k}(\mathbb{RP}^n;\mathbb{F}_2) \xleftarrow{\cong} H^{n-k}(\mathbb{RP}^n,\mathbb{RP}^n \setminus \mathbb{RP}^k;\mathbb{F}_2) \xrightarrow{\cong} H^{n-k}(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^k;\mathbb{F}_2).$$

Naturality of the (relative) cup product now allows us to identify the cup product

$$\cup \colon H^k(\mathbb{RP}^n;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-k}(\mathbb{RP}^n;\mathbb{F}_2) \to H^n(\mathbb{RP}^n;\mathbb{F}_2)$$

with the cup product

$$\cup \colon H^k(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^{n,k};\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-k}(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^k;\mathbb{F}_2) \to H^n(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};\mathbb{F}_2).$$

Identifying

$$(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^{n,k}) \cong (\mathbb{R}^k \times \mathbb{R}^{n-k},(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{n-k})$$

and

$$(\mathbb{R}^n,\mathbb{R}^n \setminus \mathbb{R}^k) \cong (\mathbb{R}^k \times \mathbb{R}^{n-k},\mathbb{R}^k \times (\mathbb{R}^{n,k} \setminus \{0\})$$

we apply Remark 3.3.27 to obtain the commutative diagram

$$\begin{tikzcd}
H^k(\mathbb{R}^k,\mathbb{R}^k \setminus \{0\};\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-k}(\mathbb{R}^{n-k},\mathbb{R}^{n-k} \setminus \{0\};\mathbb{F}_2) \arrow[r, "\times", "\cong"'] \arrow[d, "\mathrm{pr}_k^* \otimes \mathrm{pr}_{n-k}^*"', "\cong"] & H^n(\mathbb{R}^n,\mathbb{R}^n \setminus \{0\};\mathbb{F}_2) \\
\begin{array}{c} H^k(\mathbb{R}^k \times \mathbb{R}^{n-k},(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{n-k};\mathbb{F}_2) \\ \otimes_{\mathbb{F}_2} H^k(\mathbb{R}^k \times \mathbb{R}^{n-k},\mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\});\mathbb{F}_2) \end{array} \arrow[ur, "\cup"'] &
\end{tikzcd}$$

The top horizontal arrow is an isomorphism by Theorem 3.3.25, and the vertical map is an isomorphism since the projection maps are homotopy equivalences. It follows that the diagonal map given by the relative cup product is an isomorphism. We conclude that

$$H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong P_{\mathbb{F}_2}(w)/(w^{n+1})$$

as claimed.

Repeating the above argument with $\mathbb{C}$ replacing $\mathbb{R}$, $2n$ and $2k$ replacing n and k and $\mathbb{Z}$ replacing $\mathbb{F}_2$, one obtains the second part of the theorem. □

3.4. **Applications.** Computing the cup product structure on projective spaces was not entirely trivial, so let us reap some benefits from the work we invested.

3.4.1. **Corollary.** *$\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to $S^2 \vee S^4$.*

Proof. If it was, $H^*(S^2)$ would be a retract of $H^*(\mathbb{C}\mathrm{P}^2)$. In particular, every degree 2 class would square to zero, which contradicts Theorem 3.3.22. □

3.4.2. **Lemma.** *Let $i\colon A \to X$ be a cofibration and let $f_0, f_1\colon A \to Y$ be homotopic maps. Then $X \cup_{f_0} Y \simeq X \cup_{f_1} Y$.*

Proof. Exercise! □

In light of Lemma 3.4.2, we can infer from Corollary 3.4.1 that the attaching map of the 4-cell in $\mathbb{C}\mathrm{P}^2$ is not nullhomotopic. One can try to turn this observation into an invariant which aims to detect non-trivial maps $S^{2n-1} \to S^n$.

3.4.3. **Construction.** Let $f\colon S^{2n-1} \to S^n$ be a continuous map, $n \geq 2$. Taking f to be the attaching map of a $2n$-cell, we define the CW-complex $X(f)$ by the pushout

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[d] & S^n \arrow[d] \\ D^{2n} \arrow[r] & X(f) \end{tikzcd}$$

By inspection of the cellular chain complex and the universal coefficient theorem, we obtain

$$H^k(X(f);\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & k = 0, n, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Let α_f be the generator of $H^n(X(f);\mathbb{F}_2)$.

3.4.4. **Definition.** Let $f\colon S^{2n-1} \to S^n$ be a continuous map. Define its *Hopf invariant* to be

$$h(f) := \alpha \cup \alpha \in H^{2n}(X(f);\mathbb{F}_2) \cong \mathbb{Z}/2.$$

We want to show that the Hopf invariant allows us to detect non-trivial maps. This rests on the following observation, which we will prove in the exercises.

3.4.5. **Proposition.** *Let $f\colon S^{2n-1} \to S^n$ be a continuous map, $n \geq 2$.*

1. *If f has Hopf invariant 1, then f is not nullhomotopic.*
2. *If n is odd, then $h(f) = 0$.*
3. *The attaching map $\eta\colon S^3 \to S^2$ of the 4-cell in $\mathbb{C}\mathrm{P}^2$ has Hopf invariant 1.*

Proof. Suppose f is nullhomotopic. Then Lemma 3.4.2 implies that $X(f) \simeq S^n \vee S^{2n}$. Hence $H^*(S^n;\mathbb{F}_2)$ is a retract of $H^*(X(f);\mathbb{F}_2)$, which implies that the generator in degree n squares to zero. So $h(f) = 0$.

The commutative diagram

$$\begin{array}{ccc} H^k(X(f);\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(H_k(X(f)),\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^k(X(f);\mathbb{F}_2) & \xrightarrow{\cong} & \mathrm{Hom}(H_k(X(f)),\mathbb{F}_2) \end{array}$$

arising from the universal coefficient theorem shows that the non-trivial classes in $H^*(X(f);\mathbb{F}_2)$ come from integral classes. Let $\beta \in H^n(X(f);\mathbb{Z})$ be an integral lift of the generator $\alpha \in H^n(X(f);\mathbb{F}_2)$. If n is odd, we have $2\beta \cup \beta = 0$ by Remark 3.3.15. As $H^{2n}(X(f);\mathbb{Z})$ is torsionfree, it follows that $\beta \cup \beta = 0$, and hence $\alpha \cup \alpha = 0$.

Theorem 3.3.22 directly implies that $h(\eta) = 1$ because $\mathbb{C}\mathrm{P}^2 \cong X(\eta)$. □

3.4.6. **Remark.** In analogy to η, there exist maps $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$ which serve as attaching maps for the top-dimensional cell in $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, respectively. As in Theorem 3.3.22, one can show that the cohomology rings of these spaces are truncated polynomial algebras, which implies that $h(\nu) = 1$ and $h(\sigma) = 1$. In particular, these maps are not nullhomotopic.

It is a theorem due to Adams that η, ν and σ are the only maps $S^{2n-1} \to S^n$ with Hopf invariant one.

Next, we fulfill a promise made last term and prove the general version of the Borsuk–Ulam theorem.

3.4.7. **Theorem** (Borsuk–Ulam). *Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. Then there exists some $x \in S^n$ with $f(x) = f(-x)$.*

Proof. The cases $n \leq 2$ were already discussed last term, so let us assume $n \geq 3$. Assume to the contrary that $f(x) \neq f(-x)$ for all $x \in S^n$. Then

$$g\colon S^n \to S^{n-1}, \quad x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a well-defined continuous map satisfying $g(x) = -g(-x)$. Consequently, it induces a map

$$h\colon \mathbb{R}P^n \to \mathbb{R}\mathrm{P}^{n-1}.$$

Let $\lambda\colon [0,1] \to \mathbb{R}\mathrm{P}^n$ represent the non-trivial element in $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$. Then its lift $\mu\colon [0,1] \to S^n$ satisfies $\mu(0) = -\mu(1)$: by covering theory, a loop in $\mathbb{R}\mathrm{P}^n$ lifts to a loop in S^n if and only if it is nullhomotopic. Then

$$g(\mu(0)) = g(-\mu(1)) = -g(\mu(1)),$$

so $g \circ \mu\colon [0,1] \to S^{n-1}$ is a lift of $h \circ \lambda$ with different endpoints. Since $n - 1 \geq 2$, it follows as before that $h \circ \lambda$ represents a generator of $\pi_1(\mathbb{R}\mathrm{P}^{n-1}) \cong \mathbb{Z}/2$, so $\pi_1(h)$ is an isomorphism.

It follows that $H_1(h)$ is also an isomorphism. By considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}^1_{\mathbb{Z}}(H^0(\mathbb{R}\mathrm{P}^{n-1}),\mathbb{Z}/2) & \to & H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}/2) & \to & \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{R}\mathrm{P}^{n-1}),\mathbb{Z}/2) & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle H^1(h)} & & \downarrow{\scriptstyle H_1(h)^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(H^0(\mathbb{R}\mathrm{P}^{n}),\mathbb{Z}/2) & \longrightarrow & H^1(\mathbb{R}\mathrm{P}^{n};\mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{R}\mathrm{P}^{n}),\mathbb{Z}/2) & \longrightarrow & 0 \end{array}$$

arising from Corollary 3.2.8, it follows that $H^1(h)$ is also an isomorphism because $H^0(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$ is free.

Let $v \in H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $w \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ be the generators provided by Theorem 3.3.22. Since $H^*(h)$ is a morphism of graded rings, we conclude from

Theorem 3.3.22 that

$$0 = H^n(h)(y^n) = H^1(y)^n = x^n \neq 0,$$

which is a contradiction. □

Finally, we consider the question in which cases $\mathbb{R}^n$ admits the structure of a division algebra.

3.4.8. **Definition.** A *real division algebra* is a bilinear map

$$\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

such that for all $a \neq 0$ in $\mathbb{R}^n$ and all $b \in \mathbb{R}^n$ there exist $x, y \in \mathbb{R}^n$ with $\mu(x,a) = b$ and $\mu(a,y) = b$.

3.4.9. **Example.** The multiplication maps of $\mathbb{R}$, $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{O} \cong \mathbb{R}^8$ are real division algebras.

3.4.10. **Remark.** If $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a division algebra and $a \in \mathbb{R}^n$ is non-zero, $\mu(a,-)\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\mu(-,a)\colon \mathbb{R}^n \to \mathbb{R}^n$ are surjective, and therefore isomorphisms. In particular, $\mu(a,b) = 0$ if and only if $b = 0$.

3.4.11. **Theorem.** *If $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a real division algebra, then $n = 2^k$ for some $k \geq 0$.*

Proof. Note that μ is necessarily continuous. Since the multiplication of any two non-zero elements is non-zero, the map

$$f\colon S^{n-1} \times S^{n-1} \to S^{n-1}, \quad (x,y) \mapsto \frac{\mu(x,y)}{\|\mu(x,y)\|}$$

is well-defined and continuous.

Since $f(-x,y) = -f(x,y) = f(x,-y)$, the composite

$$S^{n-1} \times S^{n-1} \xrightarrow{f} S^{n-1} \xrightarrow{p} \mathbb{R}\mathrm{P}^{n-1}$$

of f with the canonical projection induces a continuous map

$$g\colon \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}.$$

Our goal is to determine the effect of the map g on cohomology.

We begin by observing that the commutative diagram

$$\begin{tikzcd}
S^{n-1} \times S^{n-1} \arrow[r, "f"] \arrow[d, "p \times p"'] & S^{n-1} \arrow[d, "p"] \\
\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \arrow[r, "g"] & \mathbb{R}\mathrm{P}^{n-1}
\end{tikzcd}$$

allows us to determine the effect of g on fundamental groups: let $\lambda\colon [0,1] \to \mathbb{R}\mathrm{P}^{n-1}$ represent the non-trivial element in $\pi_1(\mathbb{R}\mathrm{P}^{n-1})$. Then λ lifts to a path $\widetilde{\lambda}\colon [0,1] \to S^{n-1}$ satisfying $\widetilde{\lambda}(0) = -\widetilde{\lambda}(1)$. Letting c denote the constant path at the basepoint c, we have

$$f(\widetilde{\lambda}(0), c) = \frac{\mu(\widetilde{\lambda}(0), c(0))}{\|\mu(\widetilde{\lambda}(0), c(0))\|} = -\frac{\mu(\widetilde{\lambda}(1), c(1))}{\|\mu(\widetilde{\lambda}(1), c(1))\|} = -f(\widetilde{\lambda}(1), c),$$

so $g \circ (\lambda, c)$ represents the non-trivial element in $\pi_1(\mathbb{R}\mathrm{P}^{n-1})$. Reversing the roles of λ and c, we find that the same is true for $g \circ (c, \lambda)$. It follows that g induces the addition map

$$\pi_1(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1}) \cong \pi_1(\mathbb{R}\mathrm{P}^{n-1}) \times \pi_1(\mathbb{R}\mathrm{P}^{n-1}) \xrightarrow{+} \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

Since all fundamental groups in question are abelian, this also determines the effect of g on H_1.

The universal coefficient theorem yields a commutative square

$$\begin{tikzcd}
H^1(\mathbb{RP}^{n-1};\mathbb{F}_2) \arrow[r, "\cong"] \arrow[d, "g^*"] & \mathrm{Hom}_{\mathbb{F}_2}(H_1(\mathbb{RP}^{n-1};\mathbb{F}_2),\mathbb{F}_2) \arrow[d, "\Delta"] \\
H^1(\mathbb{RP}^{-1} \times \mathbb{RP}^{n-1};\mathbb{F}_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{F}_2}(H_1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1};\mathbb{F}_2),\mathbb{F}_2)
\end{tikzcd}$$

Our choice of basepoints induces inclusion maps $i_1\colon \mathbb{RP}^{n-1} \times * \to \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$ and $i_2\colon * \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$. Applying the universal coefficient theorem once more, these give rise to commutative squares

$$\begin{tikzcd}
H^1(\mathbb{RP}^{-1} \times \mathbb{RP}^{n-1};\mathbb{F}_2) \arrow[r, "\cong"] \arrow[d, "i_k^*"'] & \mathrm{Hom}_{\mathbb{F}_2}(H_1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1};\mathbb{F}_2),\mathbb{F}_2) \arrow[d, "\mathrm{pr}_k"] \\
H^1(\mathbb{RP}^{-1};\mathbb{F}_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{F}_2}(H_1(\mathbb{RP}^{n-1};\mathbb{F}_2),\mathbb{F}_2)
\end{tikzcd}$$

Tracing the generator $w \in H^1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ through these diagrams shows that

$$\begin{aligned} g^*(w) = w \otimes 1 + 1 \otimes w \in H^1(\mathbb{RP}^{n-1};\mathbb{F}_2) &\otimes H^0(\mathbb{RP}^{n-1};\mathbb{F}_2) \\ &\oplus H^0(\mathbb{RP}^{n-1};\mathbb{F}_2) \otimes H^1(\mathbb{RP}^{n-1};\mathbb{F}_2). \end{aligned}$$

It now follows that

$$0 = g^*(w^n) = (w \otimes 1 + 1 \otimes w)^n = \sum_{k=0}^{n} \binom{n}{k} w^k \otimes w^{n-k} = \sum_{k=1}^{n-1} \binom{n}{k} w^k \otimes w^{n-k}$$

since $w^n = 0$. Hence $\binom{n}{k} = 0 \in \mathbb{F}_2$ for all $1 \leq k \leq n-1$.

From $\binom{n}{k} = 0 \in \mathbb{F}_2$ for all $1 \leq k \leq n-1$, it follows that $(1+x)^n = 1 + x^n$ in the polynomial ring $P_{\mathbb{F}_2}(x)$. The binary expansion $n = \sum_{k \in I} 2^k$ on the other hand implies that

$$(1+x)^n = \prod_{k \in I} (1+x)^{2^k} = \prod_{k \in I} (1 + x^{2^k})$$

because squaring is an additive operation in characteristic two. Expanding the product, we obtain

$$(1+x)^n = \sum_{J \subseteq I} x^{\sum_{k \in J} 2^k}.$$

Since binary expansions are unique, the exponents in this sum are pairwise different. Therefore, it follows from $1 + x^n = (1+x)^n$ that I contains exactly one element, which means $n = 2^k$ for some k. □

3.4.12. **Remark.** Kervaire and Milnor have shown that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only real division algebras that exist. As in the case of the Hopf invariant, proving this requires more machinery.

4. Poincaré duality

The overall goal of this section is to describe the (co)homology of manifolds. We will show that being a compact manifold puts severe restrictions on the (co)homology, which makes it much more computable than the (co)homology of an arbitrary space.

4.1. Topological manifolds.

4.1.1. **Definition.** A *(topological) n-manifold* is a second countable Hausdorff space X such that for each point $x \in X$ there exists an open neighbourhood U of x which is homeomorphic to $\mathbb{R}^n$.

We will call any such open neighbourhood a *coordinate chart around x*.

4.1.2. **Remark.**

(1) Equivalently, X is an n-manifold if for each point $x \in X$ there exists an open neighbourhood U of x which is homeomorphic to an open subset of $\mathbb{R}^n$.
(2) Every manifold is regular and paracompact. In particular, there exist continuous partitions of unity subordinate to any open cover.
(3) In contrast to the notion of a smooth manifold, being a topological manifold is a *property* of a topological space.

4.1.3. **Definition.** A topological n-manifold is *closed* if it is compact.

Definition 4.1.3 may look slightly odd. Many authors take "manifold" to mean a more general concept, namely that of a manifold with boundary, in which case "closed" always signifies "compact and with empty boundary". The terminology introduced in Definition 4.1.1 is more convenient for us, but we will say "closed manifold" to formulate statements in a way which is closer to standard usage.

4.1.4. **Example.**

(1) The empty set $\varnothing$ is a closed n-manifold for every n.
(2) Every countable discrete space is a 0-manifold. It is closed if and only if it is finite.
(3) $\mathbb{R}^n$ is an n-manifold, and so is every open subset of $\mathbb{R}^n$.
(4) S^n is a closed n-manifold.
(5) Every surface Σ_g is a closed 2-manifold, and the same is true for the Klein bottle.
(6) $\mathbb{R}\mathrm{P}^n$ is a closed n-manifold.
(7) $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-manifold.
(8) If X is an n-manifold and $x \in X$, then $X \setminus x$ is also an n-manifold.

It follows from Lemma 4.2.1 at the beginning of the next subsection that the dimension of a non-empty manifold is well-defined.

It will be useful to know from the start that the homology of a compact manifold is finitely generated. For smooth manifolds, this can be derived from the existence of Morse functions, which actually shows that every compact manifold is homotopy equivalent to a finite CW-complex. The analogous statement for compact topological manifolds is also true, but much harder to prove. However, showing that the homology is finitely generated can still be accomplished by elementary arguments.

4.1.6. **Lemma.** *Let $U \subseteq \mathbb{R}^n$ be open and let $\varepsilon > 0$. Then U can be equipped with the structure of an n-dimensional CW-complex such that each open cell has diameter $\leq \varepsilon$.*

Proof. The idea of the proof is fairly simply: start with a cubulation of $\mathbb{R}^n$ which consists of sufficiently small cubes (eg $\mathbb{R}^n = \bigcup_{z \in \frac{\varepsilon}{\sqrt{n}}\mathbb{Z}^n} z + [0, \frac{\varepsilon}{\sqrt{n}}]^n$). If a cube is disjoint from U, forget about it. If a cube is contained in U, keep it. For all remaining cubes, subdivide them into 2^n smaller cubes and iterate this procedure with the newly obtained cubes.

More formally consider the homeomorphism $\frac{\sqrt{n}}{\varepsilon}\cdot \colon \mathbb{R}^n \to \mathbb{R}^n$. It suffices to prove the lemma for the image of U under this homeomorphism and $\varepsilon = 1$, then we can scale back to obtain the general case. Hence assume without loss of generality that $\varepsilon = 1$.

Define

$$C_k = \left\{ \prod_{i=1}^{n} \left[\frac{z_i}{2^k}, \frac{z_i + 1}{2^k} \right] \mid z_i \in \mathbb{Z} \right\}.$$

By induction, let

$$D_0 := \{c \in C_0 \mid c \subseteq \mathbb{R}^n \setminus X\},$$
$$D_{k+1} := D_k \cup \{c \in C_{k+1} \mid c \subseteq \mathbb{R}^n \setminus X \text{ and there is no } c' \in D_k \text{ such that } c \subseteq c'\}.$$

Set $D := \bigcup_k D_k$. This provides $\mathbb{R}^n \setminus X = \bigcup_{c \in D} c$ with a CW-structure, and we even have preferred characteristic maps in the form of

$$\Phi_c \colon D^n \cong c \hookrightarrow \mathbb{R}^n,$$

together with the restrictions of these maps to those faces which do not intersect smaller cubes; consult eg [Hat02, Prop. A.2] to verify that this really defines a CW-structure. □

We can use this lemma to say something about the homology of open subsets in $\mathbb{R}^n$.

4.1.7. **Definition.** For a topological space X, denote by $\mathcal{K}_X$ the poset of compact subsets of X, ordered by inclusion.

Note that $\mathcal{K}_X$ is a directed poset since the union of two compact sets is compact.

4.1.8. **Lemma.** *Let X be a topological space and let M be an abelian group. Then the map*

$$\operatorname*{colim}_{K \in \mathcal{K}_X} H_k(K; M) \to H_k(X; M)$$

induced by the inclusions is an isomorphism.

Proof. Let $\sum_{i=1}^l m_i \sigma_i \in C_k^{\mathrm{sing}}(X; M)$. Then $K := \bigcup_{i=1}^l \sigma_i(\Delta^k_{\mathrm{Top}})$ is a compact subset of X, and clearly $\sum_{i=1}^l a_i \sigma_i$ defines a singular chain in K. Since $C^{\mathrm{sing}}(K; M) \to C^{\mathrm{sing}}(X; M)$ is injective, the resulting chain is a cycle over K if and only if it is a cycle over X.

This implies directly that the comparison map is surjective. Similarly, if a k-cycle $z \in C_k^{\mathrm{sing}}(K; M)$ represents 0 in $H_k(X; M)$, there exists a singular $(k+1)$-chain x with $d_{k+1}(x) = z$. Then $x \in C_{k+1}^{\mathrm{sing}}(L; M)$ for some $L \supseteq K$, so z also represents the trivial element in the colimit. □

4.1.9. **Corollary.** *Let $U \subseteq \mathbb{R}^n$ be open and let M be an abelian group. Then $H_k(U; M) = 0$ for $k \geq n$.*

Proof. Let $a \in H_k(U; M)$. By Lemma 4.1.8, there exists some compact subset $K \subseteq U$ such that $a \in \operatorname{img}(H_k(K; M) \to H_k(U; M))$. Since K is compact, there exists some $\varepsilon > 0$ such that $K_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, K) < \varepsilon\}$ is contained in U. Using Lemma 4.1.6, choose an n-dimensional CW-structure on $\mathbb{R}^n$ all whose cells have diameter $\leq \frac{\varepsilon}{2}$. Then K is contained in a finite subcomplex X of $\mathbb{R}^n$ which also satisfies $X \subseteq U$.

If $k > n$, it is immediate that $a = 0$ since $H_k(X; M) = 0$. For $k = n$, consider the exact sequence

$$H_{n+1}(\mathbb{R}^n, X; M) \xrightarrow{\partial} H_n(X; M) \to H_n(\mathbb{R}^n; M) = 0.$$

With respect to the chosen CW-structure, the left hand term is the homology of an n-dimensional CW-complex relative to a subcomplex, so it vanishes as well. This proves $H_n(X; M) = 0$, which implies $a = 0$ as before. □

After this short detour, we return to the question why the homology groups of closed manifolds are finitely generated.

4.1.10. **Proposition.** *Let $X \subset \mathbb{R}^n$ be compact. The following are equivalent:*

1. *X is a retract of a finite CW-complex Y.*

(2) *X is weakly locally contractible: for every $x \in X$ and every neighbourhood U of x in X, there exists a neighbourhood $V \subseteq U$ of x such that the inclusion map $V \to U$ is nullhomotopic.*
(3) *There exists an open neighbourhood U of X and a retraction $r\colon U \to X$ to the inclusion map $X \to U$.*

Proof. We consider it known that CW-complexes are locally contractible (ie every point has a neighbourhood basis consisting of contractible subsets). Let $X \xrightarrow{i} Y \xrightarrow{r} X$ be a retraction diagram, let $x \in X$ and let U be a neighbourhood of x. Then $r^{-1}(U)$ is a neighbourhood of $i(x)$, so there exists a contractible neighbourhood $V' \subseteq r^{-1}(U)$ of x. Let $h\colon V \times [0,1] \to r^{-1}(U)$ be a nullhomotopy of the inclusion map. It follows that $V := i^{-1}(V')$ is a neighbourhood of x, and

$$V \times [0,1] \xrightarrow{i \times \mathrm{id}} V' \times [0,1] \xrightarrow{h} r^{-1}(U) \xrightarrow{r} U$$

defines a nullhomotopy of the inclusion map $V \to U$. This proves (1) $\Rightarrow$ (2).

Let us now prove (3) $\Rightarrow$ (1). Using Lemma 4.1.6, pick a CW-structure on U. Since X is compact, it is contained in a finite subcomplex Y of this CW-structure. Restricting the retraction map to Y exhibits X as a retract of a finite CW-complex.

The implication (2) $\Rightarrow$ (3) is the most difficult one. Using Lemma 4.1.6, choose a CW-structure on the open subset $\mathbb{R}^n \setminus X$.

We claim that the following holds:

(4.1.11)

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in X\ \forall\, c \text{ open cell in } \mathbb{R}^n \setminus X\colon c \cap B_\delta(x) \neq \varnothing \Rightarrow \operatorname{diam}(c) < \varepsilon$$

Suppose to the contrary that there exists $\varepsilon > 0$ such that for every $\delta > 0$ there are $x \in X$ and an open cell c such that $c \cap B_\delta(x) \neq \varnothing$ and $\operatorname{diam}(c) \geq \varepsilon$. Then we can find sequences $(\delta_k)_k$, $(x_k)_k$ and $(c_k)_k$ such that the open cells c_k are pairwise distinct, $(\delta_k)_k$ is strictly decreasing with $\delta_k \xrightarrow{k\to\infty} 0$, $c_k \cap B_{\delta_k}(x_k) \neq \varnothing$ and $\operatorname{diam}(c_k) \geq \varepsilon$. Since X is compact, the sequence $(x_k)_k$ has an accumulation point x. After rechoosing the sequence $(\delta_k)_k$ appropriately, we may assume that $x_k = x$ for all k. Then the sequence of balls $B_{\delta_k}(x)$ is a descending chain, so we find an infinite discrete set of points $(y_k)_k$ in $B_{\delta_0}(x)$. As the closure of this ball is compact, this is a contradiction.

We proceed now inductively to define a subcomplex Y of $\mathbb{R}^n \setminus X$ and a map $r'\colon Y \to X$ as follows: let $Y^{(0)}$ be the entire 0-skeleton of X. Pick $r'_0\colon Y^{(0)} \to X$ such that $d(y, r'_0(y)) = d(y, X)$.

Given $r'_k\colon Y^{(k)} \to X$, let $Y^{(k+1)}$ have $Y^{(k)}$ as its k-skeleton and add precisely those $(k+1)$-cells e whose boundary lies in $Y^{(k)}$ and such that $(r'_k \circ \Phi_e)|_{S^k}$ is nullhomotopic in X. Choose some extension $r'_{k+1}\colon Y^{(k+1)} \to X$ with the property that

$$\operatorname{diam}((r'_{k+1} \circ \Phi_e)(D^{k+1}) \leq 2 \cdot \inf\{\operatorname{diam} \rho(D^{k+1}) \mid \rho \text{ extends } (r'_k \circ \Phi_e)|_{S^k}\}.$$

Set $r' := r'_n$ and $Y := Y_n$. Then define $r\colon Y \cup X \to X$ by setting

$$r(y) := \begin{cases} r'(y) & y \in Y, \\ y & y \in X. \end{cases}$$

This is evidently a retraction to the inclusion map $X \to Y \cup X$. We have to show that r is continuous and that $Y \cup X$ contains a neighbourhood of X.

Let $\varepsilon > 0$ and $x \in X$ be arbitrary. Since X is weakly locally contractible, we can choose a sequence of balls

$$B_{\varepsilon_{2n+1}}(x) \supseteq B_{\varepsilon_{2n}}(x) \supseteq \ldots \supseteq B_{\varepsilon_1}(x) \supseteq B_{\varepsilon_0}(x)$$

such that

(1) $\varepsilon_{2n+1} = \varepsilon$;
(2) $\varepsilon_{2k} < \frac{\varepsilon_{2k+1}}{5}$;
(3) for every $k \leq n$, every map $S^k \to B_{\varepsilon_{2k+1}}(x) \cap X$ becomes nullhomotopic in $B_{\varepsilon_{2k+2}}(x) \cap X$.

Apply (4.1.11) to find $\delta > 0$ such that every open cell c with $c \cap B_\delta(x) \neq \varnothing$ satisfies $\operatorname{diam}(c) < \frac{\varepsilon_0}{2}$. Without loss of generality, assume $\delta < \frac{\varepsilon_0}{2}$. Consider the subcomplex

$$Z := \bigcup_{\substack{c \text{ open cell in } Y:\\ c \cap B_\delta(x) \neq \varnothing}} c$$

of Y. If $c \cap B_\delta(x) \neq \varnothing$, then

$$c \subseteq B_{\delta + \frac{\varepsilon_0}{2}}(x) \subseteq B_{\varepsilon_0}(x).$$

In particular, the entire 0-skeleton of Z lies in $B_{\varepsilon_0}(x)$. By the triangle inequality, we have for any pair of 0-cells $z, z' \in Z$

$$d(r(z), r(z')) \leq d(r(z), z) + d(z, x) + d(x, z') + d(z', r(z')) \leq 4\varepsilon_0 < \varepsilon_1.$$

So $r(Z^{(0)}) \subseteq B_{\varepsilon_1}(x)$.

By induction, suppose that $r(Y^{(k)}) \subseteq B_{\varepsilon_{2k+1}}(x)$. Since all attaching maps of $(k+1)$-cells become nullhomotopic in $B_{\varepsilon_{2k+2}}(x) \cap X$, the definition of r' ensures that $r(Z^{(k+1)}) \subseteq B_{\varepsilon_{2k+3}}(x)$. It follows that $r(Z) \subseteq B_\varepsilon(x)$.

By definition, $B_\delta(x) \subseteq Z \cup X$, so this arguments simultaneously shows that $Z \subseteq Y$ contains a neighbourhood of x and that r is continuous at x. So $r\colon Y \cup X \to X$ is a continous retraction whose domain contains a neighbourhood of X. □

4.1.12. **Proposition.** *Let X be a compact n-manifold. Then X satisfies the equivalent conditions of Proposition 4.1.10. In particular, the homology of X is concentrated in finitely many degrees and finitely generated in each degree.*

Proof. The "in particular" part follows from the corresponding assertion for finite CW-complexes. Since X is locally contractible, we only have to show that it can be embedded as a compact subspace of some Euclidean space. Pick a finite collection of charts $\{\varphi_i\colon U_i \xrightarrow{\cong} \mathbb{R}^n\}_{i=1,\dots,r}$ such that $X = \bigcup_{i=1}^r U_i$. By collapsing the complement of U_i in X to a point, we obtain induced maps

$$\psi_i\colon X \to (\mathbb{R}^n)^+ \cong S^n \subseteq \mathbb{R}^{n+1},$$

where $(-)^+$ denotes the one point-compactification. Then

$$\psi\colon X \xrightarrow{(\psi_i)_i} \prod_{i=1}^r \mathbb{R}^{n+1} \cong \mathbb{R}^{r(n+1)}$$

is a continuous and injective map. Since X is compact and the target is Hausdorff, ψ is a homeomorphism onto its image, which is a compact subset of $\mathbb{R}^{r(n+1)}$. □

4.2. **Orientations.** Let (X, A) be a pair of spaces. The relative homology group $H_n(X, X \setminus A)$ involves only cycles comprised of singular simplices whose image intersects A non-trivially. In this sense, we can think of $H_n(X, X \setminus A)$ as the "local" homology of X around A. Our first observation is that the local homology of a manifold around a point is particularly simple.

4.2.1. **Lemma.** *Let X be an n-manifold and let $x \in X$. Then*

$$H_k(X, X \setminus x; M) \cong \begin{cases} M & k = n, \\ 0 & \textit{else.} \end{cases}$$

Proof. Let $\varphi\colon U \xrightarrow{\cong} \mathbb{R}^n$ be a coordinate chart around x. Applying excision for the open cover $X = U \cup X \setminus x$, we obtain

$$\widetilde{H}_k(X, X\backslash x; M) \xleftarrow{\cong} H_k(U, U\backslash x; M) \xrightarrow[\cong]{h_*} H_k(\mathbb{R}^n, \mathbb{R}^n\backslash h(x); M) \cong H_k(\mathbb{R}^n, \mathbb{R}^n\backslash 0; M).$$

By homotopy invariance, we obtain

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus 0; M) \cong H_k(D^n, S^{n-1}; M),$$

which proves the lemma. □

These local homology groups actually enjoy a certain amount of functoriality in x.

4.2.2. **Construction.** Let X be an n-manifold and let M be an abelian group. Let $\lambda\colon [0,1] \to M$ be a path from x to y in X. Choose a subdivision $0 = t_0 < t_1 < \ldots < t_r = 1$ of the unit interval such that $\lambda_{[t_i,t_{i+1}]}$ maps to a single chart $U_i \xrightarrow[\cong]{\varphi_i} \mathbb{R}^n$. Then define a map α_i through the commutative diagram

$$\begin{tikzcd} H_n(X, X \setminus \lambda(t_i); M) & H_n(X, X \setminus \varphi_i^{-1}(B_i); M) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(X, X \setminus \lambda(t_{i+1}); M) \arrow[from=1-1, to=1-3, bend right, "\alpha_i"'] \end{tikzcd}$$

where $B_i \subseteq \mathbb{R}^n$ is a sufficiently large open ball which contains the image of $\varphi_i \circ \lambda_{|[t_i,t_{i+1}]}$. Then set

$$\alpha_\lambda := \alpha_{r-1} \circ \ldots \circ \alpha_0 \colon H_n(X, X \setminus x; M) \xrightarrow{\cong} H_n(X, X \setminus y; M).$$

We will show momentarily that this gives rise to a functor

$$\nu_X \colon \Pi(X) \to \mathrm{Ab}$$

by setting $\nu_X(x) := H_n(X, X \setminus x; M)$ and $\nu_X([\lambda]) := \alpha_\lambda$.

Note that for any path λ from x to y, the diagram

$$\begin{tikzcd} & H_n(X; M) \arrow[dl, "\rho_x"'] \arrow[dr, "\rho_y"] & \\ H_n(X, X \setminus x; M) \arrow[rr, "\alpha_\lambda"] & & H_n(X, X \setminus y; M) \end{tikzcd}$$

is commutative. Denoting by $\underline{H_n(X; M)}\colon \Pi(X) \to \mathrm{Ab}$ the constant functor with value $H_n(X; M)$, this means that the family of maps $(\rho_x)_{x\in X}$ is a natural transformation

$$\rho\colon \underline{H_n(X; M)} \Rightarrow \nu_X.$$

Proof that ν_X is a functor. We ahve to show that the definition of $\nu_X([\lambda])$ is independent of the choice of subdivision, choice of charts and choice of balls. Once we know this, functoriality is clear since we can concatenate appropriate choices of data to describe the morphism induced by a concatenation of paths.

Step 1: The construction is independent of the choice of balls B_i. If B_i and B_i' are two balls in $\mathbb{R}^n$ containing the image of $\varphi_i \circ \lambda|_{[t_i,t_{i+1}]}$, choose a ball B containing both B_i and B_i' in its interior. Then we obtain a commutative diagram

$$\begin{tikzcd} & H_n(X, X \setminus \varphi_i^{-1}(B); M) \arrow[dl, "\cong"'] \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\ H_n(X, X \setminus \lambda(t_i); M) & H_n(X, X \setminus \varphi_i^{-1}(B_i); M) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(X, X \setminus \lambda(t_{i+1}); M) \arrow[from=2-1, to=2-3, bend right, "\alpha_i"'] \end{tikzcd}$$

which expresses α_i in terms of the ball B. The same argument applies to B_i' in place of B_i, so α_i is independent of the choice of B_i.

Step 2: We may arbitrarily subdivide λ in a given chart. To simplify notation, assume that the image of λ lies in a single chart $\varphi\colon U \xrightarrow{\cong} \mathbb{R}^n$. Let $0 = t_0 < \ldots < t_r = 1$ be a subdivision of $[0,1]$. Then we may choose balls $B_i \subseteq \mathbb{R}^n$ such that B_i contains the image of $\varphi_i \circ \lambda|_{[t_i,t_{i+1}]}$ in its interior, and pick moreover a closed ball $B \subseteq \mathbb{R}^n$ which contains $\bigcup_{i=1}^r B_i$ in its interior. Then we obtain a commutative diagram

$$\begin{tikzcd}
 & & H_n(X, X \setminus \varphi^{-1}(B)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[ddll, bend right, "\cong"'] \arrow[ddrr, bend left, "\cong"] & & \\
 & H_n(X, X \setminus \varphi^{-1}(B_0)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & H_n(X, X \setminus \varphi^{-1}(B_1)) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
H_n(X, X \setminus \lambda(0)) & & H_n(X, X \setminus \lambda(t_1)) & & H_n(X, X \setminus \lambda(t_2))
\end{tikzcd}$$

which shows that the induced morphism $H_n(X, X \setminus \lambda(0)) \to H_n(X, X \setminus \lambda(t_2))$ is independent of the further subdivision. By induction, it follows that $H_n(X, X \setminus \lambda(0)) \to H_n(X, X \setminus \lambda(1))$ is well-defined.

Step 3: The construction is independent of the choice of chart. Assume $\varphi\colon U \xrightarrow{\cong} \mathbb{R}^n$ and $\psi\colon V \xrightarrow{\cong} \mathbb{R}^n$ are charts such that the image of λ is contained in $U \cap V$. Then we may choose a subdivision $0 = t_0 < \ldots < t_r = 1$ of $[0,1]$ and balls $(B_i)_i$ in $\mathbb{R}^n$ such that $\varphi \circ \lambda_{[t_i,t_{i+1}]}$ maps to B_i and $\varphi(B_i))$ is contained in V. By Step 2, the induced morphism with respect to the chart φ is given by a composition of maps of the form

$$H_n(X, X \setminus \lambda(t_i)) \xleftarrow{\cong} H_n(X, X \setminus \varphi^{-1}(B_i)) \xrightarrow{\cong} H_n(X, X \setminus \lambda(t_{i+1})).$$

Since $\psi(\varphi^{-1}(B_i)) \subseteq \mathbb{R}^n$ is bounded, there exists some ball $B_i' \subseteq \mathbb{R}^n$ which contains $\psi(\varphi^{-1}(B_i))$. Then we obtain a commutative diagram

$$\begin{tikzcd}
 & H_n(X, X \setminus \psi^{-1}(B_i'); M) \arrow[dl, "\cong"'] \arrow[d] \arrow[dr, "\cong"] & \\
H_n(X, X \setminus \lambda(t_i); M) & H_n(X, X \setminus \varphi^{-1}(B_i); M) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(X, X \setminus \lambda(t_{i+1}); M)
\end{tikzcd}$$

which shows that the morphism $H_n(X, X \setminus \lambda(t_i); M) \to H_n(X, X \setminus \lambda(t_{i+1}); M)$ defined in terms of the chart φ is identical to the morphism defined in terms of the chart ψ (since we are free to choose whichever ball we like by Step 1).

Step 4: The definition of α_λ is independent globally of the choice of subdivision. If we have two subdivisions of $[0,1]$, we may take their union to obtain a subdivision refining both at the same time. For some choice of chart associated to the first subdivision, we can apply Step 2 to each segment of λ to see that the additional subdivision does not change the definition of α_i. The same argument applies to the second subdivision, so α_λ is well-defined.

Step 5: The definition of α_λ is independent of the homotopy class of λ relative endpoints.

Let $h\colon [0,1] \times [0,1] \to X$ be a homotopy relative endpoints between λ_0 and λ_1. Then subdivide $[0,1] \times [0,1]$ into sufficiently small squares such that each square is contained in a single chart. For a sufficiently large ball in the respective chart, the restriction of the homotopy to an individual sub-square is contained in the image of that ball, so the induced maps are the same. By induction, it follows that $\alpha_{\lambda_0} = \alpha_{\lambda_1}$.

This completes the proof. $\square$

Our first goal is to show that the knowledge of these local homology groups implies the vanishing of the homology of an n-manifold above degree n. This requires a little preparation.

4.2.3. **Corollary.** *Let X be an n-manifold and let M be an abelian group. If X has no compact path component, then the map*

$$\rho_x\colon H_n(X;M) \to H_n(X, X\setminus x; M)$$

is trivial for all $x \in X$.

Proof. Since $H_n(X;M) \cong \bigoplus_{C\in\pi_0(X)} H_n(C;M)$, it suffices to show that $H_n(C) \to H_n(X, X\setminus x; M)$ is trivial for all path components C of X.

If $x \notin C$, then $(C,\varnothing) \to (X, X\setminus x)$ factors over $(X\setminus x, X\setminus x)$, and it follows that the map factors over 0. If $x \in C$, then each class $a \in H_n(X;M)$ lies in the image of the map $H_n(K;M) \to H_n(X;M)$ for some compact subset $K \subseteq X$ by Lemma 4.1.8. Since C is not compact, there exists some $y \in C\setminus K$. Choosing a path λ from x to y, we obtain the commutative diagram

$$\begin{tikzcd} & H_n(X;M) \arrow[dl, "\rho_x"'] \arrow[dr, "\rho_y"] & \\ H_n(X, X\setminus x; M) \arrow[rr, "\alpha_\lambda", "\cong"'] & & H_n(X, X\setminus y; M) \end{tikzcd}$$

Since $(K,\varnothing) \to (X, X\setminus y)$ factors via $(X\setminus y, X\setminus y)$, we also have $\rho_y(a) = 0$. □

4.2.4. **Lemma.** *Let $U \subseteq \mathbb{R}^n$ be open and let M be an abelian group. Then the map*

$$H_n(\mathbb{R}^n, U; M) \to \prod_{x\in\mathbb{R}^n\setminus U} H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x; M)$$

is injective.

Proof. Since $\mathbb{R}^n$ is contractible, the boundary map for the pair $(\mathbb{R}^n, U)$ identifies the map in question with the map

$$\widetilde{H}_{n-1}(U;M) \to \prod_{x\in\mathbb{R}^n\setminus U} \widetilde{H}_{n-1}(\mathbb{R}^n\setminus x; M).$$

Let $a \in \widetilde{H}_{n-1}(U;M)$ lie in the kernel of this map. By Lemma 4.1.8, there exists a compact subset $K \subset U$ such that $a \in \operatorname{img}(\widetilde{H}_{n-1}(K;M) \to \widetilde{H}_{n-1}(U;M)$. Without loss of generality, we may assume that $K = \overline{\operatorname{int} K}$: otherwise, replace K by the closure of an open neighbourhood which is still contained in U.

Let C be a closed ball such that $K \subseteq \operatorname{int} C$. For each point $x \in C\setminus U$, choose a closed ball B_x around x which is disjoint from K. Since $C\setminus U \subseteq C$ is closed, it is compact. Hence there are finitely many points $x_1, \dots, x_r$ such that $C\setminus U \subseteq \bigcup_{i=1}^r B_{x_i}$. Set $D_i := C \cap B_{x_i}$.

Let $b \in \widetilde{H}_{n-1}(K;M)$ be a preimage of a. We will show by induction on r that the inclusion map $K \to \operatorname{int} C \setminus \bigcup_{i=1}^r D_i$ sends b to zero. Since the D_i cover $C\setminus U$, we have a chain of inclusions

$$K \subseteq C \setminus \bigcup_{i=1}^r D_i \subseteq U.$$

Hence this will be sufficient to conclude that $a = 0$.

For $r = 0$, the image of b in $\widetilde{H}_{n-1}(C;M)$ is trivial since $\operatorname{int} C$ is contractible. For the induction step, write

$$\operatorname{int} C \setminus \bigcup_{i=1}^{r+1} D_i = \left(\operatorname{int} C \setminus \bigcup_{i=1}^{r} D_i\right) \cap \left(\mathbb{R}^n \setminus B_{x_{r+1}}\right).$$

Noting that
$$\left(\operatorname{int} C \setminus \bigcup_{i=1}^{r} D_i\right) \cup \left(\mathbb{R}^n \setminus B_{x_{r+1}}\right)$$
is an open subset of $\mathbb{R}^n$, Corollary 4.1.9 implies that the associated Mayer–Vietoris sequence yields an injective map
$$\widetilde{H}_{n-1}(\operatorname{int} C \setminus \bigcup_{i=1}^{r+1} D_i; M) \to \widetilde{H}_{n-1}(\operatorname{int} C \setminus \bigcup_{i=1}^{r} D_i; M) \oplus \widetilde{H}_{n-1}(\mathbb{R}^n \setminus B_{x_{r+1}}; M).$$
The image of b in the first summand of the target is zero by the inductive assumption, and the image of b in the second summand becomes zero by assumption because $\widetilde{H}_{n-1}(\mathbb{R}^n \setminus B_{x_{r+1}}; M) \cong \widetilde{H}_{n-1}(\mathbb{R}^n \setminus x_{r+1}; M)$ and $x_{r+1} \notin U$. □

4.2.5. **Proposition.** *Let X be an n-manifold and let M be an abelian group. Then*

(1) *$H_k(X; M) = 0$ for $k > n$;*
(2) *$H_n(X; M) = 0$ if X has no compact path component.*

Proof. Let $a \in H_k(X; M)$. Again by Lemma 4.1.8, there exists a compact subset $K \subseteq M$ such that $a \in \operatorname{img}(H_k(K; M) \to H_k(X; M))$. Choose finitely many charts $\{U_i \xrightarrow{\cong} \mathbb{R}^n\}_{1 \le i \le r}$ such that $K \subseteq \bigcup_{i=1}^r U_i$. We proceed by induction on r.

If $r = 1$, the subset K is contained in a single chart, and the composite $H_k(K; M) \to H_k(U; M) \to H_k(X; M)$ is zero for $k \ge n$ by virtue of Corollary 4.1.9.

By induction, assume that $K = U \cup V$ such that U is part of a chart $\varphi\colon U \xrightarrow{\cong} \mathbb{R}^n$, the subset $V \subseteq X$ is open, and $H_k(V; M) = 0$ for $k > n$ (respectively $k \ge n$ if no component of X is compact). Consider the exact Mayer–Vietoris sequence
$$H_k(U; M) \oplus H_k(V; M) \to H_k(U \cup V; M) \xrightarrow{\partial} H_{k-1}(U \cap V; M) \to H_{k-1}(U; M) \oplus H_{k-1}(V; M).$$
If $k > n$, the sum on the left is trivial by assumption, and $H_{k-1}(U \cap V; M)$ vanishes by Corollary 4.1.9.

Hence we only need to consider the case that X has no compact component and $k = n$. Since $H_n(U; M)$ and $H_n(V; M)$ still vanish, the boundary map ∂ is a monomorphism and we must show that $H_{n-1}(U \cap V; M) \to H_{n-1}(U; M) \oplus H_{n-1}(V; M)$ is injective. Note that it is enough to show that the corresponding map
$$\widetilde{H}_{n-1}(U \cap V; M) \to \widetilde{H}_{n-1}(U; M) \oplus \widetilde{H}_{n-1}(V; M)$$
in reduced homology is injective.

Since $U \cong \mathbb{R}^n$, we have $\widetilde{H}_{n-1}(U; M) = 0$. Let now $x \in U \setminus U \cap V$ be arbitrary and consider the commutative diagram
$$\begin{tikzcd}
 & H_n(U \cup V; M) \arrow[r] \arrow[d] & H_n(X; M) \arrow[d] \\
H_n(V, U \cap V; M) \arrow[r, "j_*"] \arrow[d, "\partial"'] & H_n(U \cup V, U \cap V; M) \arrow[r] \arrow[dl, "\partial"'] & H_n(X, X \setminus x; M) \\
\widetilde{H}_{n-1}(U \cap V; M) & H_n(U, U \cap V; M) \arrow[l, "\partial"'] \arrow[r] \arrow[u, "i_*"'] & H_n(U, U \setminus x; M) \arrow[u, "\cong"]
\end{tikzcd}$$
in which all arrows are either boundary maps or induced by inclusions.

Let $a \in \ker(\widetilde{H}_{n-1}(U \cap V; M) \to \widetilde{H}_{n-1}(V; M))$. Then there exists $a_V \in H_n(V, V \cap U; M)$ with $\partial(a_V) = a$. Since $\widetilde{H}_{n-1}(U; M) = 0$, there is also an element $a_U \in H_n(U, V \cap U; M)$ with $\partial(a_U) = a$. Then $\partial(i_* a_U - j_* a_V) = 0$, this element lifts to an element $b \in H_n(U \cup V; M)$.

The map $H_n(X;M) \to H_n(X, X \setminus x; M)$ is trivial by Corollary 4.2.3. Hence $i_*a_U - j_*a_V$ maps to zero in $H_n(X, X \setminus x; M)$. The inclusion map $(V, U \cap V) \to (X, X \setminus x)$ factors via $(X \setminus x, X \setminus x)$ (due to $x \notin V$), so the image of j_*a_V in $H_n(X, X \setminus x; M)$ is also trivial. This implies that the image of i_*a_U in that group is trivial. By the commutativity of the lower right square, it follows that the image of a_U in $H_n(U, U \setminus x; M)$ is trivial.

Since $(U, U \cap V) \cong (\mathbb{R}^n, \varphi(U \cap V))$ and $x \in U \setminus U \cap V$ was arbitrary, Lemma 4.2.4 implies that $a_U = 0$. Consequently, $a = 0$. □

As we saw in the proof of Lemma 4.2.1, a choice of coordinate chart around a point x in a manifold X induces an identification $H_n(X, X \setminus x; R) \cong H_n(D^n, S^{n-1}; R) \cong R$. We can think of this as choosing a local orientation around x, and may wonder whether it is possible to choose orientations around all points of X in a compatible manner.

Let us introduce the following auxiliary notation: for (X, A) a pair of topological spaces, set

$$H_n(X|A; M) := H_n(X, X \setminus A; M).$$

4.2.6. **Definition.** Let X be an n-manifold and let $A \subseteq X$ be a subset.

(1) An *R-orientation of X at A* is an element $o_{X,A} \in H_n(X|A; R)$ such that $o_{X,A}$ maps to a generator under the map

$$\rho_x \colon H_n(X|A; R) \to H_n(X|x; R)$$

for every $x \in A$.

(2) An *R-orientation of X* is an element

$$o_X = (o_{X,K})_K \in \lim_{K \in \mathcal{K}_X^{\mathrm{op}}} H_n(X|K; R)$$

such that $o_{X,K}$ is an R-orientation of X at K.

We call X *R-orientable* if there exists an R-orientation. For $R = \mathbb{Z}$, we abbreviate $\mathbb{Z}$-orientation and $\mathbb{Z}$-orientable to *orientation* and *orientable*.

4.2.7. **Remark.** If X is a closed n-manifold, $\mathcal{K}_X^{\mathrm{op}}$ has a least element, namely X itself. In this case, an R-orientation o_X is the same as an element

$$o_X \in H_n(X; R)$$

which maps to a generator in $H_n(X, X \setminus x; R)$ for every $x \in X$.

4.2.8. **Definition.** Let X be an n-manifold and let $A, B \subseteq X$. Define

$$\rho_{A,B} \colon H_n(X|A; R) \to H_n(X|A \cap B; R)$$

to be the map induced by the inclusion.

Since we will in particular be interested in the case that A and B are members of a family $(A_\alpha)_\alpha$ of subsets of X, we also introduce the shorthand

$$\rho_{\alpha,\beta} := \rho_{A_\alpha, A_\beta}.$$

4.2.9. **Definition.** Let $(U_\alpha)_\alpha$ be a collection of open subsets in an n-manifold X. A collection $(s_\alpha)_\alpha$ is a *compatible family of R-orientations* if $s_\alpha \in H_n(X|U_\alpha; R)$ is an R-orientation of M at U_α for every α and $\rho_{\alpha,\beta}(s_\alpha) = \rho_{\beta,\alpha}(s_\beta)$ for all α and β.

4.2.10. **Lemma.** *Let X be an n-manifold and let $K \subseteq X$ be compact.*

(1) *$H_k(X|K; M) = 0$ for $k > n$ and any abelian group M;*

(2) *Suppose that $(U_\alpha)_\alpha$ is an open cover of X and that $(s_\alpha)_\alpha$ is a compatible family of R-orientations.*

Then there exists a unique R-orientation o_X of X such that

$$o_{X,x} = \rho_x(s_\alpha)$$

for all α with $x \in U_\alpha$.

Proof. For the second assertion, it suffices to show that for each compact subset $K \subseteq X$, there exists a unique R-orientation $o_{X,K}$ of X at K such that $\rho_x(o_{X,K}) = \rho_x(s_\alpha)$ for all $x \in K$ and all α with $x \in U_\alpha$. Therefore, both statements reduce to claims about a fixed compact subset $K \subseteq X$, which we can prove in tandem.

Assume without loss of generality that each U_α is the domain of a coordinate chart $\varphi_\alpha \colon U_\alpha \xrightarrow{\cong} \mathbb{R}^n$. We write $K = \bigcup_{i=1}^r K_i$ as a union of finitely many compact subsets $K_i \subseteq X$, each of which is contained in some U_α: for example, each point $x \in X$ is contained in some U_α. Then set $K_x := \varphi_\alpha^{-1}(\overline{B_1}(\varphi_\alpha(x)))$. By compactness, there exist $x_1, \ldots, x_r \in X$ such that $K = \bigcup_{i=1}^r K \cap K_{x_i}$.

We proceed by induction on r. If K is contained in a single chart U_α, excision implies isomorphisms

$$H_k(X|K;R) \cong H_k(U_\alpha|K;R) \cong H_k(\mathbb{R}^n|\varphi_\alpha(K);R).$$

The vanishing assertion for $k > n$ is now a consequence of Corollary 4.1.9. Moreover, the image of s_α under $H_n(X|U_\alpha;R) \to H_n(X|K;R)$ is an R-orientation of X at K. Given the above isomorphism, the uniqueness of this element follows from Lemma 4.2.4.

For the induction step, assume that $K = K_1 \cup K_2$ such that K_1 is the union of r compact subsets, each lying in a single chart, and K_2 is contained in a single chart. Note that $K_1 \cap K_2$ is then also the union of at most r compact subsets, each contained in a single coordinate chart. Consider the Mayer–Vietoris sequence

$$\begin{aligned} H_{k+1}(X|K_1 \cap K_2;R) &\xrightarrow{\partial} H_k(X|K_1 \cup K_2;R) \\ &\to H_k(X|K_1;R) \oplus H_k(X|K_2;R) \to H_k(X|K_1 \cap K_2;R) \end{aligned}$$

of Lemma A.0.2. For $k > n$, the vanishing assertion follows immediately by induction.

For $k = n$, let o_{X,K_1}, o_{X,K_2} and $o_{X,K_1\cap K_2}$ be the unique R-orientations at K_1, K_2 and $K_1 \cap K_2$ provided by the inductive assumption. By uniqueness, o_{X,K_i} maps to $o_{X,K_1\cap K_2}$ under $H_n(X|K_i;R) \to H_n(X|K_1 \cap K_2;R)$.

Remembering that the last map in the above exact sequence takes the difference of the inclusion-induced maps, it follows that (o_{X,K_1}, o_{X,K_2}) maps to zero. Since $H_{n+1}(X|K_1 \cap K_2;R) = 0$, we obtain a unique lift $o_{X,K_1\cup K_2}$ with the desired properties. As each point in $K_1 \cup K_2$ lies in K_1 or K_2, the class $o_{X,K_1\cup K_2}$ is an R-orientation of X at $K_1 \cup K_2$. □

4.2.11. **Theorem.** *Let X be a non-empty n-manifold.*

(1) *There is a bijection*

$$\{R\text{-orientations of } X\} \sim \{\text{isomorphisms } \nu_X \cong \underline{R}\}.$$

In particular, X is R-orientable if and only if ν_X is isomorphic to the constant functor $\underline{R}$. If X is connected and R-orientable, the set of R-orientations of X is an $R^\times$-torsor.

(2) *If X is closed and connected, the following are equivalent:*
  (a) *X is R-orientable;*
  (b) *the natural transformation $\rho \colon \underline{H_n(X;R)} \Rightarrow \nu_X$ is an isomorphism;*
  (c) *$H_n(X;R) \cong R$.*

(3) *If X is orientable, then it is R-orientable for every R.*

(4) *If X is $\mathbb{F}_p$-orientable for some prime $p > 2$, then X is orientable.*

(5) *There is a unique $\mathbb{F}_2$-orientation of X.*

(6) *If X is closed and orientable, then $H_{n-1}(X)$ is torsionfree.*

(7) *If X is closed, connected and not orientable, then $H_n(X) = 0$ and the torsion submodule of $H_{n-1}(X)$ is isomorphic to $\mathbb{Z}/2$.*

Proof. Let $o_X = (o_{X,K})_K$ be an R-orientation of X. Then define

$$\tau_x \colon R \to \nu_X(x) = H_n(X|x;R)$$

as the unique morphism sending 1 to $o_{X,x}$. Since $o_{X,x}$ is assumed to be a generator, each τ_x is an isomorphism, and we only have to show naturality. Therefore, let $\lambda \colon [0,1] \to X$ be a path from x to y. Choose a subdivision $0 = t_0 < \ldots < t_r = 1$ as in Construction 4.2.2. Then it is enough to show that $\alpha_i(o_{X,\lambda(t_i)}) = o_{X,\lambda(t_{i+1})}$ (notation as in Construction 4.2.2). Let B_i be (the preimage of) a sufficiently large ball containing both $\lambda(t_i)$ and $\lambda(t_{i+1})$. Then $\overline{B_i}$ is compact, and $o_{X,\overline{B_i}} \in H_n(X|\overline{B_i};R)$ restricts to both $o_{X,\lambda(t_i)}$ and $o_{X,\lambda(t_{i+1})}$ as required. In total, $\tau \colon \underline{R} \Rightarrow \nu_X$ is an isomorphism.

Suppose conversely that $\tau \colon \underline{R} \Rightarrow \nu_X$ is an isomorphism. Pick a coordinate chart $\varphi_x \colon U_x \xrightarrow{\cong} \mathbb{R}^n$ around each $x \in X$, and consider the open cover by

$$V_x := \varphi_x^{-1}(B_1(\varphi_x(x))).$$

For each $x \in X$, we have isomorphisms

$$\omega_x \colon H_n(X|\overline{V}_x;R) \xrightarrow{\cong} H_n(X|V_x;R) \xrightarrow{\cong} H_n(X|x;R) \underset{\cong}{\xrightarrow{\tau_x^{-1}}} R$$

by homotopy invariance. Let $\overline{s}_x := \omega_x^{-1}(1) \in H_n(X|\overline{V}_x;R)$. Denote by s_x the image of $\overline{s}_x$ in the group $H_n(X|V_x;R)$.

We claim that $(s_x)_x$ is a compatible family of R-orientations. For this, we show that the images of $\overline{s}_x$ and $\overline{s}_y$ agree in $H_n(X|\overline{V}_x \cap \overline{V}_y;R)$ for all $x,y \in X$. Let $z \in \overline{V}_x \cap \overline{V}_y$. Choose a path λ from x to z in $\overline{V}_x$ and a path μ from y to z in $\overline{V}_y$. Then consider the diagram

$$\begin{tikzcd}
H_n(X|\overline{V}_x;R) \arrow[rr, "\cong"] \arrow[dr] & & H_n(X|x;R) \arrow[r, "\tau_x", "\cong"'] \arrow[d, "\alpha_\lambda", "\cong"'] & R \arrow[d, "\mathrm{id}"] \\
& H_n(X|\overline{V}_x \cap \overline{V}_y;R) \arrow[r, "r"] & H_n(X|z;R) \arrow[r, "\tau_z", "\cong"'] & R \\
H_n(X|\overline{V}_y;R) \arrow[rr, "\cong"] \arrow[ur] & & H_n(X|y;R) \arrow[r, "\tau_y", "\cong"'] \arrow[u, "\alpha_\mu", "\cong"'] & R \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

The right part of this diagram commutes because τ is a natural transformation, the left part commutes due to the definition of α_λ and α_μ. This implies that $\overline{s}_x$ and $\overline{s}_y$ map to the same element in $H_n(X|z;R)$. Since $\overline{V}_x \cap \overline{V}_y$ is contained in a single chart U, excision identifies the map r with the map

$$H_n(U|\overline{V}_x \cap \overline{V}_y;R) \to H_n(U|z;R).$$

Using that $U \cong \mathbb{R}^n$ and $z \in \overline{V}_x \cap \overline{V}_y$ is arbitrary, Lemma 4.2.4 implies that $\overline{s}_x$ and $\overline{s}_y$ map to the same element in $H_n(X|\overline{V}_x \cap \overline{V}_y;R)$ as required. Now apply Lemma 4.2.10 to obtain an induced R-orientation o_X of X satisfying $o_{X,x} = \tau_x^{-1}(1)$.

By inspection of the above constructions, we see that this sets up the desired bijection between the set of R-orientations of X and the set of trivialisations of ν_X.

If X is connected, any two isomorphisms $\nu_X \cong \underline{R}$ differ by an R-linear automorphism of R. It follows that the set of R-orientations is an $R^\times$-torsor if it is non-empty and X is connected.

Let $\nu_X^{\mathbb{Z}}$ be the version of ν_X for integral homology. Then observe that $\nu_X \cong \nu_X^{\mathbb{Z}} \otimes R$. Therefore, every isomorphism $\nu_X^{\mathbb{Z}} \cong \underline{\mathbb{Z}}$ induces an isomorphism $\nu_X \cong \underline{R}$, which implies R-orientability.

Suppose X is $\mathbb{F}_p$-orientable for $p > 2$. Observe that for any ring R, we have an equivalence

$$\mathrm{Fun}(\Pi(X), R\text{-Mod}) \simeq \prod_{C \in \pi_0(X)} \mathrm{Fun}(B\pi_1(C, x_C), (\text{-Mod}R)),$$

where x_C denotes some base point in the component C and BG is the category with a single object $*$ and $\mathrm{Hom}_{BG}(*,*) := G$. This allows us to assume without loss of generality that X is connected with some base point x.

Then $\mathbb{F}_p$-orientability amounts to the assertion that the $\pi_1(X,x)$-action on $H_n(X|x;\mathbb{F}_p)$ encoded by the functor $\nu_X|_{B\pi_1(X,x)}$ is trivial. Since $H_n(X|x) \to H_n(X|x;\mathbb{F}_p)$ is surjective by the universal coefficient theorem, any generator $o_x \in H_n(X|x)$ maps to a generator in $H_n(X|x;\mathbb{F}_p)$. For $[\lambda] \in \pi_1(X,x)$, we then obtain a commutative diagram

$$\begin{array}{ccc} H_n(X|x) & \xrightarrow{\nu_X^{\mathbb{Z}}([\lambda])} & H_n(X|x) \\ \downarrow & & \downarrow \\ H_n(X|x;\mathbb{F}_p) & \xrightarrow{\nu_X^{\mathbb{F}_p}([\lambda])} & H_n(X|x;\mathbb{F}_p) \end{array}$$

Consequently, $\nu^{\mathbb{Z}}([\lambda])(o_x)$ is a generator lifting the same element in $H_n(X|x;\mathbb{F}_p)$ because $\nu_X^{\mathbb{F}_p}([\lambda]) = \mathrm{id}$. Since $p > 2$, such a generator is unique, which implies that $\nu_X^{\mathbb{Z}}([\lambda]) = \mathrm{id}$ as well. Hence X is orientable.

Over $\mathbb{F}_2$, note that there is precisely one isomorphism $\mathbb{F}_2 \xrightarrow{\cong} \mathbb{F}_2$, namely the identity. Therefore, every functor $\nu\colon \Pi(X) \to \mathbb{F}_2\text{-Mod}$ satisfying $\nu(x) \cong \mathbb{F}_2$ for all $x \in X$ is isomorphic to $\underline{\mathbb{F}_2}$. In similar fashion, uniqueness of the orientation follows from $\mathbb{F}_2^\times = \{1\}$.

Consider from now on the case that X is closed. Note that the exact sequence

$$H_n(X \setminus x;R) \to H_n(X;R) \xrightarrow{\rho_x} H_n(X|x;R)$$

and Proposition 4.2.5 imply that ρ_x is always injective.

If X is R-orientable, then ρ_x is also surjective, so ρ is an isomorphism. Since $H_n(X|x;R) \cong R$ by Lemma 4.2.1, this in turn implies that $H_n(X;R) \cong R$.

We have to show that $H_n(X;R) \cong R$ implies R-orientability. Assume first that $R = \mathbb{Z}$. Then ρ_x is given by multiplication with a non-zero integer k (after choosing generators). If k was different from ± 1, the universal coefficient theorem for homology would yield a commutative square

$$\begin{array}{ccc} H_n(X) \otimes \mathbb{Z}/k & \longrightarrow & H_n(X;\mathbb{Z}/k) \\ {\scriptstyle 0}\downarrow & & \downarrow \\ H_n(X|x) \otimes \mathbb{Z}/k & \xrightarrow{\cong} & H_n(X|x;\mathbb{Z}/k) \end{array}$$

in which the unlabelled arrows are injective, which is impossible. So $k = \pm 1$, which means that any choice of generator in $H_n(X)$ is an orientation of X.

Since ρ_x is injective, $H_n(X) = 0$ if X is not orientable.

Over $R = \mathbb{F}_p$, we can argue similarly: since ρ_x is an injection into the 1-dimensional $\mathbb{F}_p$-vector space $H_n(X|x;\mathbb{F}_p)$, this map is an isomorphism if $H_n(X;\mathbb{F}_p) \cong \mathbb{F}_p$. Hence any generator in $H_n(X;\mathbb{F}_p)$ defines an $\mathbb{F}_p$-orientation of X.

Let us use this information to deduce the description of $H_{n-1}(X)$. If X is orientable, the map $H_n(X) \otimes \mathbb{F}_p \to H_n(X;\mathbb{F}_p)$ is an isomorphism. Hence $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),\mathbb{F}_p) = 0$ for all primes p. Since $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),\mathbb{F}_p)$ is isomorphic to the p-torsion part of $H_{n-1}(X)$, it follows that $H_{n-1}(X)$ is torsionfree.

If X is not orientable, then X is also not $\mathbb{F}_p$-orientable for $p > 2$. Hence $H_n(X) = 0$ and $H_n(X;\mathbb{F}_p) = 0$ for all $p > 2$. This implies that $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),\mathbb{F}_p) = 0$, so any torsion element in $H_{n-1}(X)$ is a 2-primary torsion elements. Since X is not orientable, there exists a loop λ at some point $x \in M$ which induces multiplication by -1 on $H_n(X|x)$. Hence $\rho_x = -\rho_x$ over any ring R, which implies that ρ_x maps $H_n(X;R)$ injectively to the 2-torsion part of R. Considering $R = \mathbb{Z}/2^k$, it follows that $H_n(X;\mathbb{Z}/2^k)$ injects into $\mathbb{Z}/2$. Since $H_n(X)=0$, the map $H_n(X;\mathbb{Z}/2^k)\to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),\mathbb{Z}/2^k)$ is an isomorphism for all k, so the torsion part of $H_{n-1}(X)$ is a subgroup of $\mathbb{Z}/2$. The case $k=1$ implies that the torsion part of $H_{n-1}(X)$ is non-trivial because $H_n(X;\mathbb{F}_2)\cong\mathbb{Z}/2$.

As an upshot of this discussion, if $H_n(X;R)\cong R$ and R contains an element which is not 2-torsion, then X is orientable, which implies R-orientability. So the only situation left to consider is the case that $H_n(X;R)\cong R$ for a ring R in which every element is 2-torsion. Then R is an $\mathbb{F}_2$-vector space, so the universal coefficient theorem yields a commutative square

$$\begin{tikzcd} H_n(X;\mathbb{F}_2)\otimes_{\mathbb{F}_2} R \arrow[r] \arrow[d, "\cong"] & H_n(X;R) \arrow[d, "\rho_x"] \\ H_n(X|x;\mathbb{F}_2)\otimes_{\mathbb{F}_2} R \arrow[r, "\cong"] & H_n(X|x;R) \end{tikzcd}$$

Consequently, ρ_x is surjective for all x, which is equivalent to R-orientability. □

4.2.12. **Definition.** Let X be a connected n-manifold and let $x\in X$. Define the *orientation character* of X as the composition

$$\omega_X\colon \pi_1(X,x)\xrightarrow{\cong}\operatorname{Hom}_{\Pi(X)}(x,x)^{\mathrm{op}}\to\operatorname{Aut}(H_n(X|x))^{\mathrm{op}}\cong\mathbb{Z}/2.$$

4.2.13. **Corollary.** *Let X be a connected n-manifold and let $x\in X$. Then X is orientable if and only if ω_X is trivial.*

Proof. By Theorem 4.2.11, X is orientable if and only if ν_X is isomorphic to the constant functor $\underline{\mathbb{Z}}$.

Consider $\pi_1(X,x)$ as a category $B\pi_1(X,x)$ having a single object with endomorphism monoid $\pi_1(X,x)$. There is a fully faithful functor $B\pi_1(X,x)^{\mathrm{op}}\to\Pi(X)$.

If $\nu_X\colon\Pi(X)\to\mathrm{Ab}$ is isomorphic to $\underline{\mathbb{Z}}$, this implies that ω_X is trivial. If ω_X is trivial, this implies that $\nu_X|_{B\pi_1(X,x)^{\mathrm{op}}}$ is isomorphic to the constant functor with value $\mathbb{Z}$ on $B\pi_1(X,x)^{\mathrm{op}}$. Since X is connected, the functor $B\pi_1(X,x)^{\mathrm{op}}\to\Pi(X)$ is an equivalence, so $\nu_X\cong\underline{\mathbb{Z}}$. □

4.2.14. **Corollary.** *Let X be a connected n-manifold. If the fundamental group of X contains no subgroup of index 2, then X is R-orientable for every R.*

In particular, every simply-connected manifold is orientable.

Proof. If X is not orientable, then ω_X is non-trivial by Corollary 4.2.13. This implies that $\ker(\omega_X)$ is an index 2 subgroup of $\pi_1(X,x)$. □

4.2.15. **Remark.** Since $\mathbb{Z}/2$ is abelian and $H_1(X)$ is the abelianisation of $\pi_1(X,x)$, the orientation character induces a homomorphism $H_1(X)\to\mathbb{Z}/2$. By the universal coefficient theorem, we have an isomorphism $H^1(X;\mathbb{Z}/2)\cong\operatorname{Hom}(H_1(X),\mathbb{Z}/2)$. In this way, ω_X gives rise to a unique class

$$w_1(X)\in H^1(X;\mathbb{Z}/2),$$

the *first Stiefel–Whitney class* of X. By the preceding discussion, $w_1(X)=0$ if and only if X is orientable.

4.3. **The Poincaré duality theorem.** We have almost everything in place to state and prove the main result of this section. The only missing ingredient is a comparison map between cohomology and homology which will allow us to state the Poincaré duality theorem.

4.3.1. **Definition.** Let (X,A,B) be an excisive triad and let M be an R-module. Define a "diagonal" transformation as follows: the composite

$$C^{\mathrm{sing}}(X)\xrightarrow{C^{\mathrm{sing}}(\Delta)}C^{\mathrm{sing}}(X\times X)\xrightarrow{\mathrm{AW}}C^{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X)$$

is a natural map. By naturality, it restricts to a map

$$C^{\mathrm{sing}}(A) + C^{\mathrm{sing}}(B) \to C^{\mathrm{sing}}(A) \otimes C^{\mathrm{sing}}(X) + C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(B),$$

and thus induces a natural map

$$\frac{C^{\mathrm{sing}}(X)}{C^{\mathrm{sing}}(A) + C^{\mathrm{sing}}(B)} \to C^{\mathrm{sing}}(X, A) \otimes C^{\mathrm{sing}}(X, B).$$

The domain admits a natural map to $C^{\mathrm{sing}}(X, A \cup B)$ which is a chain homotopy equivalence, so precomposing with a chain homotopy inverse yields a map

$$\Delta\colon C^{\mathrm{sing}}(X, A \cup B; R) \to C^{\mathrm{sing}}(X, A) \otimes C^{\mathrm{sing}}(X, B; R).$$

If $A = \varnothing$ or $B = \varnothing$, the map we have to invert in the last step is an isomorphism, and we agree to choose the inverse isomorphism. This makes Δ strictly functorial for triples of the form $(X, A, \varnothing)$ or $(X, \varnothing, B)$.

From this diagonal map, we obtain the *(chain-level) cap product* as the composition

$$\begin{aligned}\cap\colon C_{\mathrm{sing}}(X, A; M) \otimes_R C^{\mathrm{sing}}(X, A \cup B; R) & \\ &\xrightarrow{\mathrm{id} \otimes \Delta} C_{\mathrm{sing}}(X, A; M) \otimes C^{\mathrm{sing}}(X, A) \otimes_R C^{\mathrm{sing}}(X, B; R) \\ &\xrightarrow{\mathrm{ev} \otimes \mathrm{id}} M[0] \otimes_R C^{\mathrm{sing}}(X, B; R) \cong C^{\mathrm{sing}}(X, B; M)\end{aligned}$$

The chain-level cap product induces the *cap product*

$$\begin{aligned}\cap\colon H^*(X, A; M) \otimes_R H(X, A \cup B; R) &= H(C_{\mathrm{sing}}(X, A; M)) \otimes_R H(C^{\mathrm{sing}}(X, A \cup B; R)) \\ &\xrightarrow{\mu} H(C_{\mathrm{sing}}(X, A; M) \otimes_R C^{\mathrm{sing}}(X, A \cup B; R)) \\ &\xrightarrow{H(\cap)} H(C^{\mathrm{sing}}(X, B; M)) = H(X, B; M)\end{aligned}$$

4.3.2. **Remark.**

(1) The most important instance of this definition is the case $A = \varnothing$ or $B = \varnothing$, in which (X, A, B) is automatically an excisive triad. On occasion, it will be convenient to have the general version of the cap product at our disposal.

(2) As usual, it makes sense to figure out the correct indexing on the (co)homology groups. Remembering that cohomology is concentrated in non-positive degrees, the cap product is given in each degree by sums of maps of the form

$$\cap\colon H^p(X, A; M) \otimes_R H_n(X, A \cup B; R) \to H_{n-p}(X, B; M).$$

Given the definition of the cap product, it is to be expected that properties of the cap product rely heavily on properties of the "diagonal" Δ.

4.3.3. **Lemma.**

(1) *Let $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ be a map of excisive triads. Then*

$$\begin{tikzcd} C^{\mathrm{sing}}(X, A_1 \cup A_2) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & C^{\mathrm{sing}}(X, A_1) \otimes C^{\mathrm{sing}}(X, A_2) \arrow[d, "f_* \otimes f_*"] \\ C^{\mathrm{sing}}(Y, B_1 \cup B_2) \arrow[r, "\Delta"] & C^{\mathrm{sing}}(Y, B_1) \otimes C^{\mathrm{sing}}(Y, B_2) \end{tikzcd}$$

commutes up to chain homotopy. The square commutes strictly if $A_1 = \varnothing = B_1$ or $A_2 = \varnothing = B_2$.

(2) *Let X be a topological space. Then there exists a chain homotopy*

$$(\mathrm{id} \otimes \Delta) \circ \Delta \simeq (\Delta \otimes \mathrm{id}) \circ \Delta\colon C^{\mathrm{sing}}(X) \to C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(X) \otimes C^{\mathrm{sing}}(X).$$

(3) *Let X be a topological space. Then*

$$\begin{tikzcd} C^{\mathrm{sing}}(X) \arrow[r, "\Delta"] \arrow[dr, "\cong"'] & C^{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X) \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ & C^{\mathrm{sing}}(X)\otimes\mathbb{Z}[0] \end{tikzcd}$$

commutes up to chain homotopy.

(4) *Let X be a topological space. Then*

$$\begin{tikzcd} C^{\mathrm{sing}}(X) \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & C^{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X) \arrow[d, "\tau"] \\ & C^{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X) \end{tikzcd}$$

commutes up to chain homotopy.

Proof. This is an immediate consequence of Proposition 2.1.6. □

Knowing these basic properties of Δ, we can establish some fundamental statements about the cap product.

4.3.4. **Lemma.** *Let X be a topological space. The adjoint of the map*

$$\cap\colon H^p(X;M)\otimes_R H_p(X;R)\to H_0(X;M)\xrightarrow{\varepsilon} M$$

coincides with the evaluation map

$$\mathrm{ev}\colon H^p(X;M)\to \mathrm{Hom}_R(H_p(X;R),M)$$

of the universal coefficient theorem.

Proof. The diagram

$$\begin{tikzcd} C_{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[dr, "\cong"'] & C_{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X) \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\varepsilon"] & C^{\mathrm{sing}}(X) \arrow[d, "\varepsilon"] \\ & C_{\mathrm{sing}}(X)\otimes C^{\mathrm{sing}}(X)\otimes\mathbb{Z}[0] \arrow[r, "\mathrm{ev}"] & \mathbb{Z}[0] \end{tikzcd}$$

commutes up to chain homotopy by Lemma 4.3.3 (3). Now observe that the composition along the top right corner describes the cap product followed by the augmentation map, while the composition along the bottom of the diagram induces the evaluation map. □

The next lemma establishes a sort of "naturality" for the cap product.

4.3.5. **Lemma.** *Let $f\colon (X,A_1,A_2)\to(Y,B_1,B_2)$ be a map of excisive triads. Then*

$$f_*\circ\cap\circ(f^*\otimes\mathrm{id})\simeq\cap\circ f_*\colon C_{\mathrm{sing}}(Y,B_1;M)\otimes_R C^{\mathrm{sing}}(X,A_1\cup A_2;R)\to C^{\mathrm{sing}}(Y,B_2;R).$$

These maps are equal if $A_1=\varnothing=A_2$ or $B_1=\varnothing=B_2$.

In particular,

$$f_*(f^*\beta\cap s)=\beta\cap f_*s\in H_{n-p}(Y,B_2;M)$$

for $\beta\in H^p(Y,B_1;M)$ and $s\in H_n(X,A_1\cup A_2;R)$.

Proof. Consider the following diagram, in which we abbreviate $C^{\text{sing}}(-)$ to $C(-)$ and $C_{\text{sing}}(-)$ to $C(-)^\vee$:

$$\begin{tikzcd}
C(X,A_1;M)^\vee \otimes_R C(X,A_1\cup A_2;R) \arrow[r, "\mathrm{id}\otimes\Delta"] & C(X,A_1;M)^\vee \otimes_R C(X,A_1)\otimes C(X,A_2;R) \arrow[dr, "\mathrm{ev}"] & \\
C(Y,B_1;M)^\vee \otimes_R C(X,A_1\cup A_2;R) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[u, "f^*\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes f_*"] & C(Y,B_1;M)^\vee \otimes_R C(X,A_1)\otimes C(X,A_2;R) \arrow[u, "f^*\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes f_*\otimes f_*"] & C(X,A_2;R) \arrow[d, "f_*"] \\
C(Y,B_1;M)^\vee \otimes_R C(Y,B_1\cup B_2) \arrow[r, "\mathrm{id}\otimes\Delta"] & C(Y,B_1;M)^\vee \otimes_R C(Y,B_1;M)\otimes C(Y,B_2;R) \arrow[r, "\mathrm{ev}"] & C(Y,B_2;R)
\end{tikzcd}$$

The upper left square evidently commutes, and the right part of the diagram commutes since $\mathrm{ev}\circ(f^*\otimes\mathrm{id}) = \mathrm{ev}\circ(\mathrm{id}\otimes f_*)$. The lower left square commutes up to homotopy by Lemma 4.3.3 (1), and it commutes strictly in case $A_1=\varnothing=A_2$ or $B_1=\varnothing=B_2$. $\square$

Again, it is probably most instructive to consider only the absolute version of this statement on first pass. However, we will have to use the relative version in some of the upcoming proofs.

The next lemma allows for a slightly more conceptual way to phrase Lemma 4.3.5.

4.3.6. **Lemma.** *Let X be a topological space and let $\alpha\in H^p(X;R)$, $\beta\in H^q(X;R)$, $s\in H_n(X;R)$. Then*

$$(\alpha\cup\beta)\cap s = \alpha\cap(\beta\cap s)\in H_{n-p-q}(X;R).$$

Proof. Abbreviate notation by writing $C(-):=C^{\text{sing}}(-;R)$, $(-)^\vee := \underline{\mathrm{Hom}}(-;R[0])$, and dropping the subscripts on the tensor symbols. Consider the following diagram:

$$\begin{tikzcd}
C(X)^\vee\otimes C(X)^\vee\otimes C(X) \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta"] \arrow[d, "\lambda\otimes\mathrm{id}"] & C(X)^\vee\otimes C(X)^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}"] \arrow[d, "\lambda\otimes\mathrm{id}\otimes\mathrm{id}"] & C(X)^\vee\otimes C(X)^\vee\otimes C(X)\otimes C(X)\otimes C(X) \arrow[d, "\lambda\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}"] \\
(C(X)\otimes C(X))^\vee\otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "\mathrm{AW}^*"] & (C(X)\otimes C(X))^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta\otimes\mathrm{id}"] \arrow[d, "\mathrm{AW}^*\otimes\mathrm{id}\otimes\mathrm{id}"] & (C(X)\otimes C(X))^\vee\otimes C(X)\otimes C(X)\otimes C(X) \arrow[d, "\mathrm{AW}^*\otimes\mathrm{EZ}\otimes\mathrm{id}"] \\
C(X\times X)^\vee\otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "\Delta^*\otimes\mathrm{id}"] & C(X\times X)^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta_*\otimes\mathrm{id}"] \arrow[d, "\Delta^*\otimes\mathrm{id}"] & C(X\times X)^\vee\otimes C(X\times X)\otimes C(X) \arrow[d, "\mathrm{ev}\otimes\mathrm{id}"] \\
C(X)^\vee\otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta"] & C(X)^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] & C(X)
\end{tikzcd}$$

Note that the composition along the bottom left corner computes the cup product followed by the cap product.

The left column commutes for obvious reasons, and so do the top right and bottom right square. Remembering that $\Delta=\mathrm{AW}\circ\Delta_*$, the center right square commutes up to chain homotopy since AW and EZ are chain homotopy inverses of each other. Since $\mathrm{AW}\circ\mathrm{EZ}\simeq\mathrm{id}$, we also see that the composition of the vertical maps on the right of the diagram is chain homotopic to

$$C(X)^\vee\otimes C(X)^\vee\otimes C(X)\otimes C(X)\otimes C(X) \xrightarrow{(\mathrm{ev}_{C(X)\otimes C(X)}\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})} C(X).$$

The definition of λ (see Construction 3.3.1) requires the diagram

$$\begin{tikzcd}
C(X)^\vee\otimes C(X)^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[d, "\lambda\otimes\mathrm{id}\otimes\mathrm{id}"] & C(X)^\vee\otimes C(X)\otimes C(X)^\vee\otimes C(X) \arrow[d, "\mathrm{ev}\otimes\mathrm{ev}"] \\
(C(X)\otimes C(X))^\vee\otimes C(X)\otimes C(X) \arrow[r, "\mathrm{ev}"] & \mathbb{Z}[0]
\end{tikzcd}$$

to commute. Therefore, the composition of the vertical maps on the right of the large diagram is chain homotopic to the composition

$$\begin{aligned}
&C(X)^\vee \otimes C(X)^\vee \otimes C(X) \otimes C(X) \otimes C(X) \\
&\xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id}} C(X)^\vee \otimes C(X) \otimes C(X)^\vee \otimes C(X) \otimes C(X) \\
&\xrightarrow{\mathrm{ev}\otimes\mathrm{ev}\otimes\mathrm{id}} C(X)
\end{aligned}$$

Now consider the diagram

$$\begin{tikzcd}
C(X)^\vee \otimes C(X)^\vee \otimes C(X) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "\mathrm{id}\otimes\Delta"'] & C(X)^\vee \otimes C(X)^\vee \otimes C(X) \otimes C(X) \arrow[r, "\mathrm{id}\otimes\mathrm{ev}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\Delta"] & C(X)^\vee \otimes C(X) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
C(X)^\vee \otimes C(X)^\vee \otimes C(X) \otimes C(X) \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}"'] & C(X)^\vee \otimes C(X)^\vee \otimes C(X) \otimes C(X) \otimes C(X) \arrow[r, "\mathrm{id}\otimes\mathrm{ev}\otimes\mathrm{id}"] & C(X)^\vee \otimes C(X) \otimes C(X) \arrow[d, "\mathrm{ev}\otimes\mathrm{id}"] \\
C(X)^\vee \otimes C(X)^\vee \otimes C(X) \otimes C(X) \otimes C(X) \arrow[ur, "\mathrm{id}\otimes\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[r, "\mathrm{id}\otimes\tau\otimes\mathrm{id}\otimes\mathrm{id}"] & C(X)^\vee \otimes C(X) \otimes C(X)^\vee \otimes C(X) \otimes C(X) \arrow[r, "\mathrm{ev}\otimes\mathrm{ev}\otimes\mathrm{id}"] & C(X)
\end{tikzcd}$$

and note that the composition along the bottom left corner in this diagram is chain homotopic to the composition along the top right of the previous large diagram.

The upper right square in this diagram is commutative. The top left part of the diagram commutes up to chain homotopy since

$$(\tau \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \simeq (\Delta \otimes \mathrm{id}) \circ \Delta \simeq (\mathrm{id} \otimes \Delta) \circ \Delta$$

by Lemma 4.3.3 (2) and (4). For the bottom right part of the diagram, let $\alpha \otimes \beta \otimes x \otimes y \otimes z$ be an elementary tensor such that each component has homogeneous degree. If $|\alpha| \neq |x|$ or $|\beta| \neq |y|$, both images of this element in $C(X)$ are zero. Otherwise, they also agree because

$$\begin{aligned}
&((\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}))(\alpha \otimes \beta \otimes x \otimes y \otimes z) \\
&\quad = (-1)^{|x||y|} \alpha(x)\beta(y)z \\
&\quad = (-1)^{|x||\beta|} \alpha(x)\beta(y)z \\
&\quad = ((\mathrm{ev} \otimes \mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id}))(\alpha \otimes \beta \otimes x \otimes y \otimes z).
\end{aligned}$$

Observing that the composition in the top right of the last diagram computes the iterated cap product, this finishes the proof. □

4.3.7. **Remark.** Recalling once more that cohomology lives in non-positive degrees, one can conclude that $H(X;R)$ is a graded left $H^*(X;R)$-module; Lemma 4.3.6 asserts that the scalar multiplication is associative. With this interpretation of the cap product, Lemma 4.3.5 asserts that $f_*\colon H(X) \to H(Y)$ is $H^*(Y)$-linear for any map $f\colon X \to Y$.

As in the case of homology, we introduce the shorthand

$$H^p(X|A;M) := H^p(X, X \setminus A; M)$$

for a pair of topological spaces (X, A).

4.3.8. **Definition.** Let X be a topological space.

(1) Define the *cohomology of X with compact supports* by

$$H^n_c(X;M) := \operatorname*{colim}_{K \in \mathcal{K}_X} H^n(X|K;M).$$

(2) Dually, define the *locally finite homology* of X by

$$H^{\mathrm{lf}}_n(X;M) := \lim_{K \in \mathcal{K}_X^{\mathrm{op}}} H_n(X|K;M).$$

If X is an n-manifold, an R-orientation of X is an element of $H^{\mathrm{lf}}_n(X;R)$. The following observation can make it easier to determine these groups.

4.3.9. **Lemma.** *Let $\mathcal{C}$ be a category and let I and J be directed posets. Suppose that $f\colon J \to I$ is a* cofinal *morphism of posets, ie f is order-preserving and for every $i \in I$ there exists some $j \in J$ with $i \le f(j)$.*

(1) *Let $D\colon I \to \mathcal{C}$ be a diagram such that $D \circ f$ admits a colimit. Then define for each $i \in I$ a morphism $\tau_i\colon D(i) \to \operatorname{colim}_J D \circ f$ by choosing $j \in J$ with $i \le f(j)$ and taking the composite*

$$\tau_i\colon D(i) \to D(f(j)) \to \operatorname*{colim}_J D \circ f,$$

where the second map is the structure morphism of the colimit. Then $(\tau_i)_i$ exhibit $\operatorname{colim}_J D \circ f$ as a colimit of D.

(2) *Let $E\colon I^{\mathrm{op}} \to \mathcal{C}$ be a diagram such that $E \circ f^{\mathrm{op}}$ admits a limit. Then $\lim_{J^{\mathrm{op}}} E \circ f^{\mathrm{op}}$ becomes similarly a limit of E.*

In particular, if X is compact, then $H^n_{\mathrm{c}}(X; M) \cong H^n(X; M)$ and $H^{\mathrm{lf}}_n(X; M) \cong H_n(X; M)$.

Proof. If j and j' are two elements from J such that $i \le f(j)$ and $i \le f(j')$, then there exists some $j'' \in J$ with $j \le j''$ and $j' \le j''$. Then the diagram

$$\begin{tikzcd}
 & D(f(j)) \arrow[d] \arrow[dr] & \\
D(i) \arrow[ur] \arrow[r] \arrow[dr] & D(f(j'')) \arrow[r] & \operatorname{colim}_J D \circ f \\
 & D(f(j')) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commutes, which shows that the definition of τ_i is independent of the choice of j. In particular, for $i \le i'$ in I, we may choose $j \in J$ such that $i' \le f(j)$ to see that $(\tau_i)_i$ defines a cone under the diagram D. Hence it suffices to show that the induced map

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname*{colim}_J D \circ f, y) \to \lim_{i \in I^{\mathrm{op}}} \operatorname{Hom}_{\mathcal{C}}(D(i), y)$$

is a bijection for every $y \in \mathcal{C}$. Since $\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_J D \circ f, y) \cong \lim_{j \in J^{\mathrm{op}}} \operatorname{Hom}_{\mathcal{C}}(D(f(j)), y)$, this reduces the first assertion to the second assertion for the case $\mathcal{C} = \mathrm{Set}$.

For the second assertion, we obtain an analogous transformation by choosing for each $i \in I$ some $j \in J$ with $i \le f(j)$ and setting

$$\sigma_i\colon \lim_{J^{\mathrm{op}}} E \circ f^{\mathrm{op}} \to E(f(j)) \to E(i).$$

This induces a comparison map

$$\operatorname{Hom}_{\mathcal{C}}(x, \lim_{J^{\mathrm{op}}} E \circ f^{\mathrm{op}}) \to \lim_{i \in I^{\mathrm{op}}} \operatorname{Hom}_{\mathcal{C}}(x, E(i)).$$

An element in the target of this map is given by a family of morphisms $(a_i\colon x \to E(i))_{i \in I}$ such that for $i \le i'$, the diagram

$$\begin{tikzcd}
 & E(i) \arrow[d] \\
x \arrow[ur, "a_i"] \arrow[r, "a_{i'}"'] & E(i')
\end{tikzcd}$$

commutes. This implies that $(a_{f(j)}\colon x \to E(f(j)))_{j \in J}$ defines a preimage of $(a_i)_{i \in I}$ under the above map. Injectivity is obvious, so we are done. ∎

4.3.10. **Construction.** If $K \subseteq L \subseteq X$ are compact subsets, the diagram

$$\begin{tikzcd}
H^p(X|K;M) \otimes_R H_n(X|L;R) \arrow[r] \arrow[d] & H^p(X|K;M) \otimes_R H_n(X,X|K;R) \arrow[d, "\cap"] \\
H^p(X|L;M) \otimes_R H_n(X|L;R) \arrow[r, "\cap"] & H_{n-p}(X;M)
\end{tikzcd}$$

commutes by virtue of Lemma 4.3.5. This amounts to the assertion that the maps

$$H^p(X|K;M) \to \mathrm{Hom}(H_n^{\mathrm{lf}}(X;R), H_{n-p}(X;M))$$
$$\alpha \mapsto [(s_K)_K \mapsto \alpha \cap s_K]$$

induce a morphism

$$\cap\colon H_{\mathrm{c}}^p(X;M) \otimes_R H_n^{\mathrm{lf}}(X;R) \to H_{n-p}(X;R).$$

4.3.11. **Theorem** (Poincaré duality)**.** *Let X be an n-manifold and let $o_X \in H_n^{\mathrm{lf}}(X;R)$ be an R-orientation of X. Then*

$$- \cap o_X \colon H_{\mathrm{c}}^p(X;M) \to H_{n-p}(X;M)$$

is an isomorphism for all p.

4.3.12. **Remark.** We are mostly interested in the Poincaré duality theorem for closed manifolds, where we are not required to mention cohomology with compact supports or locally finite homology. These are rather auxiliary concepts that will be useful to formulate the proof of the theorem. However, it will be important to record the functoriality of these invariants:

(1) Suppose $f\colon X \to Y$ is *proper* in the sense that $f^{-1}(L)$ is compact for every compact subset $L \subseteq Y$. Then f induces for each compact subset $L \subseteq Y$ a map
$$H^n(Y, Y \setminus L; M) \to H^n(X, X \setminus f^{-1}(L); M).$$
These maps are natural in L, so we obtain an induced map
$$f^*\colon H_{\mathrm{c}}^n(Y;M) \to H_{\mathrm{c}}^*(X;M).$$
Similarly, we have compatible maps
$$H_n(X, X \setminus f^{-1}(L); M) \to H_n(Y, Y \setminus L; M)$$
which induce
$$f_*\colon H_n^{\mathrm{lf}}(X;M) \to H_n^{\mathrm{lf}}(Y;M).$$

(2) Suppose that $i\colon U \to X$ is the inclusion of an open subset. By excision, we obtain for each compact subset $K \subseteq U$ a morphism
$$H^n(U, U \setminus K; M) \xrightarrow{\cong} H^n(X, X \setminus K; M) \to H_{\mathrm{c}}^n(X;M).$$
Taking the colimit over K yields
$$i_\sharp\colon H_{\mathrm{c}}^n(U;M) \to H_{\mathrm{c}}^n(X;M).$$
Similarly, the excision isomorphisms in homology yield for each compact subset $K \subseteq U$ a map
$$H_n^{\mathrm{lf}}(X;M) \to H_n(X, X \setminus K; M) \xrightarrow{\cong} H_n(U, U \setminus K; M)$$
which assemble to a morphism
$$i^\sharp\colon H_n^{\mathrm{lf}}(X;M) \to H_n^{\mathrm{lf}}(U;M).$$

All of these constructions are functorial for the respective type of map.

4.3.13. **Lemma.** *Let X be a regular*[1] *Hausdorff space and let $s \in H_n^{\mathrm{lf}}(X;R)$.*

(1) *Let $i\colon U \subseteq X$ be the inclusion of an open subset. Then*

$$\begin{tikzcd} H^p_c(U;M) \arrow[r, "i_\sharp"] \arrow[d, "\cap i^\sharp(s)"'] & H^p_c(X;M) \arrow[d, "\cap s"] \\ H_{n-p}(U;M) \arrow[r, "i_*"] & H_{n-p}(X;M) \end{tikzcd}$$

commutes.

(2) *Let (U,V) be an open cover of X. Denote by $i^U : U\cap V \to U$, $i^V : U\cap V \to V$, $j^U : U \to X$, $j^V : V \to X$ and $j^{U\cap V} : U \cap V \to X$ the respective inclusion maps.*

Then the diagram

$$\begin{tikzcd} H^p_c(U\cap V;M) \arrow[r, "(i^U_\sharp, i^V_\sharp)"] \arrow[d, "\cap j^{U\cap V,\sharp}(s)"'] & H^p_c(U;M) \oplus H^p_c(V;M) \arrow[r, "j^U_\sharp - j^V_\sharp"] \arrow[d, "\cap j^{U,\sharp}(s) \oplus \cap j^{V,\sharp}(s)"] & H^p(X;M) \arrow[d, "\cap s"] \\ H_{n-p}(U\cap V;M) \arrow[r, "(i^U_*, i^V_*)"] & H_{n-p}(U;M) \oplus H_{n-p}(V;M) \arrow[r, "j^U_* - j^V_*"] & H_{n-p}(X;M) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\delta"] & H^{p+1}_c(U\cap V;M) \arrow[r, "(i^U_\sharp, i^V_\sharp)"] \arrow[d, "\cap j^{U\cap V,\sharp}(s)"] & H^{p+1}_c(U;M) \oplus H^p(V;M) \arrow[d, "\cap j^{U,\sharp}(s) \oplus \cap j^{V,\sharp}(s)"] \\ {} \arrow[r, "\partial"] & H_{n-p-1}(U\cap V;M) \arrow[r, "(i^U_*, i^V_*)"] & H_{n-p-1}(U;M) \oplus H_{n-p-1}(V;M) \end{tikzcd}$$

commutes, and both rows are exact.

Proof. Every class in $H^p_c(U;M)$ is represented by a class $\alpha \in H^p(U, U \setminus K; M)$ with $K \subseteq U$ compact. The class $i^\sharp(s)$ is represented by $t \in H_n(U, U \setminus K; R)$ satisfying $i_* t = s_K$, where $s_K \in H_n(X, X \setminus K; R)$ is the K-th component of s. Let $\beta \in H^p(X, X \setminus K; M)$ be the unique class satisfying $i^*\beta = \alpha$. Then Lemma 4.3.5 implies that

$$i_*(\alpha \cap i^\sharp(s)) = i_*(i^*\beta \cap t) = \beta \cap i_* t = \beta \cap s_K = i_\sharp(\alpha) \cap s.$$

This proves the first assertion.

For the second assertion, observe first that

$$\begin{aligned} \mathcal{K}_U \times \mathcal{K}_V \to \mathcal{K}_{U\cap V}, &\quad (K,L) \mapsto K \cap L \\ \mathcal{K}_U \times \mathcal{K}_V \to \mathcal{K}_X, &\quad (K,L) \mapsto K \cup L \end{aligned}$$

are both surjective maps: for the first map, this is obvious; for the second, let $C \subseteq X$ be compact and choose for each $x \in K \cap U$ a closed neighbourhood U_x which is contained in U. Similarly, choose for each $x \in K \cap V$ a closed neighbourhood V_x which is contained in V. Then $\{\operatorname{int} U_x\}_{x \in K\cap U} \cup \{\operatorname{int} V_x\}_{x\in K\cap V}$ is an open cover of K, so there exist finitely many points $\{x_i\}_{i=1,\dots,r}$ and $\{y_j\}_{j=1,\dots,s}$ such that $\{U_{x_i}\}_{i=1,\dots,r} \cup \{V_{y_j}\}_{j=1,\dots,s}$ cover K. Then $K := C \cap \bigcup_{i=1}^r U_{x_i}$ is a compact subset contained in U and $L := C \cap \bigcup_{j=1}^s V_{y_j}$ is a compact subset contained in V such that $K \cup L = C$.

In particular, both of these maps are cofinal. This allows us to express the compactly supported cohomology groups in the upper row as colimits indexed by

[1] Reminder: X is regular if every closed subset $C \subseteq X$ and every point $x \in X$ can be separated by open neighbourhoods. This implies that if U is an open neighbourhood of a point x, then there exists a closed neighbourhood V of x which is contained in U: applying the separation condition to x and $X \setminus U$ yields an open neighbourhood W of $X \setminus U$ which is disjoint from some open neighbourhood of x. Then $X \setminus W$ is a closed neighbourhood of x which is contained in U.

the same directed poset. Since filtered colimits preserve exactness, it is now enough to show that for each pair $(K, L) \in \mathcal{K}_U \times \mathcal{K}_V$ we have a commutative diagram

$$\begin{tikzcd}
H_c^p(U \cap V|K \cap L) \arrow[r, "(i_\sharp^U, i_\sharp^V)"] \arrow[d, "\cap j^{U\cap V,\sharp}(s)_{K\cap L}"'] & H_c^p(U|K) \oplus H_c^p(V|L) \arrow[r, "j_\sharp^U - j_\sharp^V"] \arrow[d, "\cap j^{U,\sharp}(s)_K \oplus \cap j^{V,\sharp}(s)_L"] & H^p(X|K \cup L) \arrow[d, "\cap s_{K\cup L}"] \\
H_{n-p}(U \cap V|K \cap L) \arrow[r, "(i_*^U, i_*^V)"] & H_{n-p}(U|K) \oplus H_{n-p}(V|L) \arrow[r, "j_*^U - j_*^V"] & H_{n-p}(X|K \cup L)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H_c^{p+1}(U \cap V|K \cap L) \arrow[r, "(i_\sharp^U, i_\sharp^V)"] \arrow[d, "\cap j^{U\cap V,\sharp}(s)_{K\cap L}"] & H_c^{p+1}(U|K) \oplus H^p(V|L) \arrow[d, "\cap j^{U,\sharp}(s)_K \oplus \cap j^{V,\sharp}(s)_L"] \\
{} \arrow[r, "\partial"] & H_{n-p-1}(U \cap V|K \cap L) \arrow[r, "(i_*^U, i_*^V)"] & H_{n-p-1}(U|K) \oplus H_{n-p-1}(V|L)
\end{tikzcd}$$

with exact rows; we are suppressing the coefficients to save some space. All squares which do not involve boundary maps commute by virtue of the first assertion.

Excision allows us to identify the top row:

$$\begin{tikzcd}
H^p(X|K \cap L) \arrow[r] \arrow[d, "\cong"] & H^p(X|K) \oplus H^p(X|L) \arrow[r] \arrow[d, "\cong"] & H^p(X|K \cup L) \arrow[d, "\cong"] \\
H_c^p(U \cap V|K \cap L) \arrow[r, "(i_\sharp^U, i_\sharp^V)"] & H_c^p(U|K) \oplus H_c^p(V|L) \arrow[r, "j_\sharp^U - j_\sharp^V"] & H^p(X|K \cup L)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^{p+1}(X|K \cap L) \arrow[r] \arrow[d, "\cong"] & H^{p+1}(X|K) \oplus H^{p+1}(X|L) \arrow[d, "\cong"] \\
{} \arrow[r, "\delta"] & H^{p+1}(U \cap V|K \cap L) \arrow[r] & H^{p+1}(U|K) \oplus H^{p+1}(V|L)
\end{tikzcd}$$

The upper row is an instance of a Mayer–Vietoris sequence, and therefore exact.

We are left with showing that the square involving the boundary map δ commutes. To see this, we abbreviate $t := s_{K\cup L}$ and $t_\cap := j^{U\cap V,\sharp}(s)_{K\cap L}$ and consider the following diagram:

$$\begin{tikzcd}
H^p(X|K \cup L) \arrow[r, "g^*"] \arrow[dd, "\cap t"'] & H^p(X \setminus L, X \setminus K \cup L) \arrow[d, "\cap t'"] & H^p(X \setminus K \cap L, X \setminus K) \arrow[l, "\cong"'] \arrow[r, "\delta"] & H^{p+1}(X|K \cap L) \arrow[d] \\
 & H_{n-p}(X \setminus L, V \setminus L) \arrow[d, "g_*"] & & H^{p+1}(U \cap V, U \cap V \setminus K \cap L) \arrow[d, "\cap t_\cap"] \\
H_{n-p}(X) \arrow[r, "f_*"] & H_{n-p}(X, V) & H_{n-p}(U, U \cap V) \arrow[l, "\cong"'] \arrow[r, "\partial"] & H_{n-p-1}(U \cap V)
\end{tikzcd}$$

Here we are using the maps

$$(X, X \setminus K \cup L, \varnothing) \xrightarrow{f} (X, X \setminus K \cup L, V) \xleftarrow{g} (X \setminus L, X \setminus K \cup L, V \setminus L)$$

and define $t' \in H_n(X \setminus L, (X \setminus K \cup L) \cup (V \setminus L))$ as the image of t under the composition

$$H_n(X|K \cup L) \xrightarrow{f_*} H_n(X, (X \setminus K \cup L) \cup V) \underset{\cong}{\xleftarrow{g_*}} H_n(X \setminus L, (X \setminus K \cup L) \cup (V \setminus L)),$$

using that $(X \setminus K \cup L) \cup V$ and $X \setminus L$ form an open cover of X.

By unwinding definitions, one finds that the composition of the horizontal arrows along the top and bottom of the diagram are the boundary maps of the respective Mayer–Vietoris sequences. Note that the map δ comes from the long exact sequence of a triple of spaces.

It now suffices to show that both parts of this diagram commute. For the left part, we use Lemma 4.3.5 to see that

$$g_*(g^*\alpha \cap t') = \alpha \cap g_* t' = \alpha \cap f_* t = f_*(f^*\alpha \cap t) = f_*(\alpha \cap t),$$

where the final identity stems from the fact that $f^* \colon H^p(X|K \cup L) \to H^p(X|K \cup L)$ is the identity morphism.

The right part of the diagram can be expanded into the slightly larger diagram

$$\begin{tikzcd}
H^p(X \setminus L, X \setminus K \cup L) \arrow[d, "\cap t'"] \arrow[dr, "\cong"] \arrow[dr, phantom, "(\star)"'] & H^p(X \setminus K \cap L, X \setminus K) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "\cong"] & H^p(X \setminus K \cap L) \arrow[r, "\delta"] & H^{p+1}(X|K \cap L) \arrow[dd] \\
 & H^p(U \setminus L, U \setminus K \cup L) \arrow[d, "\cap t''"] & & \\
H_{n-p}(X \setminus L, V \setminus L) \arrow[d, "g_*"'] & H_{n-p}(U \setminus L, U \cap V \setminus L) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d] & H_{n-p-1}(U \cap V \setminus L) \arrow[dr] & H^{p+1}(U \cap V, U \cap V \setminus K \cap L) \arrow[d, "\cap t_\cap"] \\
H_{n-p}(X, V) & H_{n-p}(U, U \cap V) \arrow[l, "\cong"'] \arrow[rr, "\partial"] & & H_{n-p-1}(U \cap V)
\end{tikzcd}$$

All occurrences of δ and ∂ are now boundary maps of long exact sequences of pairs, and all unlabelled arrows are induced by inclusion maps. All isomorphisms arise from excision. The class t'' is defined as the preimage of t' under the excision isomorphism

$$H_n(U \setminus L, (U \setminus K \cup L) \cup (U \cap V \setminus L)) \xrightarrow{\cong} H_n(X \setminus L, (X \setminus K \cup L) \cup (V \setminus L)).$$

The part of the diagram labelled $(\star)$ commutes by Lemma 4.3.5, and the two other parts on the left of the diagram commute already on the space level.

For the remaining part of the diagram, we make some explicit calculations on the chain level. Since $U \setminus L$, $U \cap V$ and $V \setminus K$ form an open cover of X, the class t can be represented by a singular chain

$$z = z_{U \setminus L} + z_{U \cap V} + z_{V \setminus K}$$

such that each z_A is a singular chain on the subset A and $d(z)$ is a cycle on $X \setminus K \cap L$. Then $f_*[z] = [z_{U \setminus L}]$. Since $d(z_{U \setminus L}) = d(z) - d(z_{U \cap V}) - d(z_{V \setminus K})$, this is a cycle on

$$(U \setminus L) \cap \big((X \setminus K \cup L) \cup (U \cap V) \cup (V \setminus K)\big) = (U \setminus K \cup L) \cup (U \cap V \setminus L).$$

Consequently, $z_{U \setminus L}$ represents the class t''.

Similarly, $t_\cap$ is the image of t under the composite

$$H_n(X, X \setminus K \cup L) \to H_n(X, X \setminus K \cap L) \xleftarrow{\cong} H_n(U \cap V, U \cap V \setminus K \cap L).$$

Since $z_{U \setminus L} + z_{V \setminus K}$ is a chain on $(U \setminus L) \cup (V \setminus K) = X \setminus K \cap L$, the equality $d(z_{U \cap V}) = d(z) - d(z_{U \setminus L}) - d(z_{V \setminus K})$ shows that $z_{U \cap V}$ and z represent the same class in $H_n(X, X \setminus K \cap L)$. Since $z_{U \cap V}$ is a singular chain on $U \cap V$ such that $d(z_{U \cap V})$ is a cycle on

$$(U \cap V) \cap \big((X \setminus K \cup L) \cup (U \setminus L) \cup (V \setminus K)\big) = U \cap V \setminus K \cap L,$$

the class $t_\cap$ is represented by $z_{U \cap V}$.

Now represent an arbitrary class in $H^p(X \setminus K \cap L, X \setminus K)$ by a homomorphism $\alpha \colon C_p^{\mathrm{sing}}(X \setminus K) \to M$ such that $\alpha|_{X \setminus K} = 0$ and $\delta\alpha = 0$; here and in the rest of the proof, we will write $\alpha|_A$ for the restriction of α to the subgroup of singular chains lying in a given subset A.

We first look for a cycle representative of $[\alpha|_{U \setminus L}] \cap t''$. Letting $\iota \colon (U \setminus L, \varnothing, \varnothing) \to (U \setminus L, U \setminus K \cup L, U \cap V \setminus L)$ denote the map induced by the identity, the chain level statement of Lemma 4.3.5 shows that this class is represented by

$$\iota_*(\alpha|_{U \setminus L} \cap z_{U \setminus L}),$$

using the absolute cap product on $U \setminus L$. By the explicit description of the boundary map in homology, the class $\partial([\alpha|_{U\setminus L}] \cap t'')$ is therefore represented by

$$\begin{aligned} d(\alpha|_{U\setminus L} \cap z_{U\setminus L}) &= \delta\alpha|_{U\setminus L} \cap z_{U\setminus L} + (-1)^p \alpha|_{U\setminus L} \cap d(z_{U\setminus L}) \\ &= (-1)^p \alpha|_{U\setminus L} \cap d(z_{U\setminus L}), \end{aligned}$$

where we have used that the cap product is a chain map and that $\delta\alpha = 0$. Now observe that $d(z_{U\cap V})$ is a cycle on $U \cap V \setminus K \cap L$, so the equation

$$d(z_{U\setminus L}) + d(z_{U\cap V}) = d(z - z_{V\setminus K})$$

shows that this element is a cycle on $U \setminus K \cup L \subseteq U \setminus L$. Moreover, $d(z_{U\cap V})$ is then also a cycle on $U \setminus L$. Applying Lemma 4.3.5 to the map $(U \setminus K \cup L, \varnothing, \varnothing) \to (U \setminus L, \varnothing, \varnothing)$, we obtain the equality (!)

$$\alpha|_{U\setminus L} \cap d(z - z_{V\setminus K}) = \alpha|_{U\setminus K\cup L} \cap d(z - z_{V\setminus K}),$$

where the second cap product is taken over $U \setminus K \cup L$. Since $\alpha|_{X\setminus K} = 0$, it follows that this cycle is zero. Consequently, we obtain

$$(-1)^p \alpha|_{U\setminus L} \cap d(z_{U\setminus L}) = (-1)^{p+1} \alpha|_{U\setminus L} \cap d(z_{U\cap V}).$$

This element represents the image of α under the composition along the bottom left corner. In particular, it is an $(n-p-1)$-cycle on $U \cap V \setminus L$.

In order to determine the image of $[\alpha]$ under the composition along the top right, extend α to a map $\overline{\alpha}\colon C_p^{\mathrm{sing}}(X) \to M$. Using the explicit description of the boundary δ, the image of $[\alpha]$ is then represented by

$$\delta\overline{\alpha}|_{U\cap V} \cap z_{U\cap V}.$$

Applying Lemma 4.3.5 to the map $(U \cap V, \varnothing, \varnothing) \to (U \cap V, U \cap V \setminus K \cap L, \varnothing)$, we see that this representative may be computed using the absolute cap product over $U \cap V$. Since the cap product is a chain map, we have

$$\delta\overline{\alpha}|_{U\cap V} \cap z_{U\cap V} = (-1)^{p+1} \overline{\alpha}|_{U\cap V} \cap d(z_{U\cap V}) + d(\overline{\alpha}|_{U\cap V} \cap z_{U\cap V}),$$

so $(-1)^{p+1}\overline{\alpha}|_{U\cap V} \cap d(z_{U\cap V})$ is a cycle representative of the image.

Now we can apply Lemma 4.3.5 to the inclusions $U \setminus L \to U \leftarrow U \cap V$ and use that $d(z_{U\cap V})$ is a cycle on $U \cap V \setminus L$ to obtain the following chain of identities in $C^{\mathrm{sing}}(U)$:

$$\begin{aligned} \alpha|_{U\setminus L} \cap d(z_{U\cap V}) &= \overline{\alpha}|_{U\setminus L} \cap d(z_{U\cap V}) \\ &= \overline{\alpha}|_{U} \cap d(z_{U\cap V}) \\ &= \overline{\alpha}|_{U\cap V} \cap d(z_{U\cap V}). \end{aligned}$$

Since $C^{\mathrm{sing}}(U \cap V) \to C^{\mathrm{sing}}(U)$ is injective, this shows that these are the same cycle in $C^{\mathrm{sing}}(U \cap V)$, which finishes the proof. □

Proof. The strategy of the proof is to show that the theorem holds for individual charts, and then to successively patch these together to obtain the global statement.

We make the following claims:

(1) The theorem holds for open subsets of $\mathbb{R}^n$.
(2) If U and V are open subsets of M such that the theorem holds for U, V and $U \cap V$, then the theorem holds for $U \cup V$.
(3) If I is a linearly ordered and $\{U_i\}_{i\in I}$ is a family of open subsets such that $U_i \subseteq U_j$ for $i \leq j$ and each U_i satisfies the theorem, then $\bigcup_{i\in I} U_i$ satisfies the theorem.

The first and second claims imply that the theorem holds for finite unions of coordinate charts. Writing X as an ascending union of such, the third claim shows that the theorem holds for X.

Observe that, if $i\colon U \to X$ is the inclusion of an open subset, $i^\sharp(o_X)$ is an R-orientation of U. In particular, the second claim is an immediate consequence of Lemma 4.3.13.

Let us first show that the theorem holds for $\mathbb{R}^n$. The collection of closed balls around 0 defines a cofinal subposet of the poset of all compact subsets in $\mathbb{R}^n$, so we have

$$H^p_{\mathrm{c}}(\mathbb{R}^n; M) \cong \operatorname*{colim}_R H^p(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B_R}(0); M) \cong H^p(\mathbb{R}^n, \mathbb{R}^n \setminus 0; M),$$

the second isomorphism coming from the fact that all maps in the colimit system are isomorphisms. In particular, $H^p_{\mathrm{c}}(\mathbb{R}^n; M) = 0$ for $p \neq n$, and the theorem holds trivially in this case. For $p = n$, the duality map is isomorphic to the map

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; M) \xrightarrow{\cap o_{\mathbb{R}^n,0}} H_0(\mathbb{R}^n) \cong \mathbb{Z}.$$

Since $o_{\mathbb{R}^n,0}$ is a generator, it follows from Lemma 4.3.4 and the universal coefficient theorem that this map is an isomorphism.

Next, we prove the third claim. Set $U := \operatorname{colim}_i U_i$ and let o_{U_i} be the image of o_X under $H^{\mathrm{lf}}_n(X; R) \to H^{\mathrm{lf}}_n(U_i; R)$ (and analogously for U). For $i \leq j$, the square

$$\begin{tikzcd}
H^p_{\mathrm{c}}(U_i; M) \arrow[r, "\cap o_{U_i}"] \arrow[d, "\mathrm{inc}_\sharp"'] & H_{n-p}(U_i; M) \arrow[d, "\mathrm{inc}_*"] \\
H^p_{\mathrm{c}}(U_j; M) \arrow[r, "\cap o_{U_j}"] & H_{n-p}(U_j; M)
\end{tikzcd}$$

commutes by Lemma 4.3.13 because $o_{U_i} = \mathrm{inc}^\sharp\, o_{U_j}$. The same holds if we replace U_j by U. Hence we obtain a commutative square

$$\begin{tikzcd}
\operatorname{colim}_i H^p_{\mathrm{c}}(U_i; M) \arrow[r] \arrow[d] & \operatorname{colim}_i H_{n-p}(U_i; M) \arrow[d] \\
H^p_{\mathrm{c}}(U; M) \arrow[r, "\cap o_U"] & H_{n-p}(U; M)
\end{tikzcd}$$

in which the upper horizontal arrow is an isomorphism. The right vertical map is an isomorphism by Lemma 4.1.8. For the left vertical map, observe that the map of posets

$$\{(i, K) \mid i \in I, K \subseteq U_i \text{ compact}\} \to \mathcal{K}_U, \quad (i, K) \mapsto K$$

is cofinal, so

$$\operatorname*{colim}_i \operatorname*{colim}_{K \subseteq U_i \text{ compact}} H^p(U_i, U_i \setminus K; M) \to \operatorname*{colim}_{K \in \mathcal{K}_U} H^p(U, U \setminus K; M)$$

is an isomorphism. It follows that the theorem holds for U.

This can now be put to use to prove the first claim: an arbitrary open subset U of $\mathbb{R}^n$ can be written as a countable union

$$U = \bigcup_{i=1}^{\infty} B_{\varepsilon_i}(x_i)$$

of open balls (pick a countable dense set in U, and consider all rational radii such that the corresponding balls are contained in U). Then

$$U = \bigcup_{i=1}^{\infty} \bigcup_{j<i} B_{\varepsilon_i}(x_i)$$

exhibits U as a countable increasing union over finite unions of convex open subsets of $\mathbb{R}^n$. Since we have already proved the third claim, it suffices to show that the theorem holds for finite unions of convex open subsets of $\mathbb{R}^n$. This can be proven by induction on the number of open convex subsets. For a single convex open subset U, the theorem holds since U is homeomorphic to $\mathbb{R}^n$. If U is a union of n convex open subsets, U satisfies the theorem by induction. For V a convex open subset, $U \cap V$ is also a union of n convex open subsets, so the second claim and the inductive hypothesis imply that the theorem holds for unions of $n+1$ convex open subsets. This finishes the proof. □

4.4. **Consequences and applications of Poincaré duality.** Poincaré duality imposes some restrictions on the Euler characteristics of closed manifolds.

4.4.1. **Proposition.** *Let X be a closed odd-dimensional manifold. Then $\chi(X) = 0$.*

Proof. Since every manifold is $\mathbb{F}_2$-oriented, Poincaré duality and the universal coefficient theorem imply that

$$b_{n-p}(X; \mathbb{F}_2) = \dim_{\mathbb{F}_2} H^p(X; \mathbb{F}_2) = \dim_{\mathbb{F}_2} H_p(X; \mathbb{F}_2) = b_p(X; \mathbb{F}_2).$$

It follows that

$$\begin{aligned}
\chi(X) &= \sum_{p=0}^{n} (-1)^p b_p(X; \mathbb{F}_2) \\
&= \sum_{p=0}^{\frac{n-1}{2}} (-1)^p b_p(X; \mathbb{F}_2) + \sum_{p=\frac{n-1}{2}+1}^{n} (-1)^p b_p(X; \mathbb{F}_2) \\
&= \sum_{p=0}^{\frac{n-1}{2}} (-1)^p b_{n-p}(X; \mathbb{F}_2) + \sum_{p=\frac{n-1}{2}+1}^{n} (-1)^p b_p(X; \mathbb{F}_2) \\
&= \sum_{p=\frac{n-1}{2}+1}^{n} (-1)^{n-p} b_p(X; \mathbb{F}_2) + \sum_{p=\frac{n-1}{2}+1}^{n} (-1)^p b_n(X; \mathbb{F}_2) \\
&= 0
\end{aligned}$$

because p and $n-p$ have different parities for odd n. □

Next, we want to explain how Poincaré duality can be used to compute cup products. For an R-module M, define

$$M^{\mathrm{tors}} := \{m \in M \mid \exists\, r \in R \colon rm = 0\}$$

and

$$M^{\mathrm{tf}} := M/M^{\mathrm{tors}}.$$

Then M^{tf} is the initial torsionfree module under M: any R-linear map $M \to N$ to a torsionfree R-module N factors uniquely over M^{tf}.

4.4.2. **Definition.** Let M and N be finitely generated R-modules over an integral domain R. A bilinear form $\varphi \colon M \otimes_R N \to R$ is *non-degenerate* if both maps

$$M^{\mathrm{tf}} \to \mathrm{Hom}_R(N^{\mathrm{tf}}, R) \quad \text{and} \quad N^{\mathrm{tf}} \to \mathrm{Hom}_R(M^{\mathrm{tf}}, R)$$

induced by the adjoints of φ are isomorphisms.

4.4.3. **Proposition.** *Let R be a hereditary integral domain (eg a principal ideal domain) and let X be an R-oriented, closed and connected n-manifold. Then the cup product pairing*

$$\langle\ ,\ \rangle \colon H^p(X; R) \otimes_R H^{n-p}(X; R) \xrightarrow{\cup} H^n(X; R) \underset{\cong}{\xrightarrow{\cap o_X}} H_0(X; R) \cong R.$$

is non-degenerate.

Proof. Lemma 4.3.6 shows that the diagram

$$\begin{tikzcd} H^p(X;R) \otimes_R H^{n-p}(X;R) \arrow[r, "\cup"] \arrow[d, "\mathrm{id} \otimes (- \cap o_X)"', "\cong"] & H^n(X;R) \arrow[d, "\cong"', "- \cap o_X"] \\ H^p(X;R) \otimes_R H_{n-p}(X;R) \arrow[r, "\cap"] & H_0(X;R) \cong R \end{tikzcd}$$

commutes. The vertical maps are isomorphisms by virtue of Poincaré duality. Using Lemma 4.3.4, this identifies the adjoint of $\langle\ ,\ \rangle$ with the evaluation map of the universal coefficient theorem. Since the homology groups of X are finitely generated by Proposition 4.1.12, it follows from the universal coefficient theorems that the map $H^{n-p}(X;R)^{\mathrm{tf}} \to \mathrm{Hom}_R(H_{n-p}(X;R)^{\mathrm{tf}}, R)$ induced by evaluation is an isomorphism.

Since the cup product is graded-commutative, it follows that $\langle\ ,\ \rangle$ is non-degenerate. □

For illustration, we determine the cup product structure on the cohomology of $\mathbb{C}\mathrm{P}^n$; similar arguments apply to real and quaternionic projective spaces. In particular, this provides an alternative proof for Theorem 3.3.22.

4.4.4. **Corollary.**

$$H^*(\mathbb{C}\mathrm{P}^n) \cong P_{\mathbb{Z}}(c)$$

with $|c| = 2$.

Proof. We know that

$$H^k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n \text{ even}, \\ 0 & \text{else} \end{cases}$$

as a consequence of the universal coefficient theorem. Since the inclusion $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ induces an isomorphism in degrees $\leq 2k$, it suffices to show by induction that generators in degree $2p$ and $2(n-p)$ multiply to give a generator of $H^{2n}(\mathbb{C}\mathrm{P}^n)$. Proposition 4.4.3 implies that $c^p \cup c^{n-p}$ is a generator of $H^{2n}(\mathbb{C}\mathrm{P}^n)$. □

There are also versions of Poincaré duality for manifolds with boundary which we prove next.

4.4.5. **Definition.** An $(n+1)$-*manifold with boundary* W is a second countable Hausdorff space such that every point $x \in W$ has a neighbourhood which is homeomorphic to $\mathbb{R}^{n+1}$ or to the closed half-space $\mathbb{R}^n \times [0, \infty)$.

4.4.6. **Remark.** Suppose that $x \in W$ has a neighbourhood which is homeomorphic to $\mathbb{R}^n \times [0,\infty)$. If x corresponds to a point whose last coordinate is positive, then x also has a neighbourhood homeomorphic to $\mathbb{R}^{n+1}$. Otherwise,

$$H_n(W|x) \cong H_n(\mathbb{R}^n \times [0,\infty), \mathbb{R}^n \times [0,\infty) \setminus \{0\}) = 0.$$

Hence the points of W which correspond to points in $\mathbb{R}^n \times \{0\}$ are a well-defined subset called the *boundary* of W and denoted ∂W. Evidently, the boundary of W is an n-manifold.

The subspace of non-boundary points W is called the *interior* of W. We will denote it by $\mathrm{int}\, W$ and hope that this does not cause too much confusion. Note that $\mathrm{int}\, W$ is an open subset of W, so ∂W is closed.

4.4.7. **Example.**

(1) If V is k-manifold with boundary and W is an l-manifolds with boundary, then $V \times W$ is an $(k+l)$-manifold with boundary satisfying

$$\partial(V \times W) \cong (\partial V \times W) \cup (V \times \partial W).$$

(2) The disc D^{n+1} is an $(n+1)$-manifold with boundary S^n.
(3) If X is a closed n-manifold, then $X \times [0,1]$ is a compact $(n+1)$-manifold with boundary satisfying
$$\partial(X \times [0,1]) \cong X \times \{0,1\}.$$
(4) The solid torus $D^2 \times S^1$ is a 3-manifold with boundary satisfying
$$\partial(D^2 \times S^1) \cong T^2.$$

It is relatively easy to prove a version of Poincaré duality for manifolds with boundary if one accepts the following.

4.4.8. **Proposition** (Existence of collar neighbourhoods)**.** *Let W be an $(n+1)$-manifold with boundary. Then the inclusion map $\partial W \to W$ extends to a homeomorphism $W \times [0,1) \to V$ onto an open neighbourhood of ∂W in W.*

Proof. See eg [Hat02, Proposition 3.42]. □

4.4.9. **Corollary.** *Let W be an $(n+1)$-manifold with boundary. Then the inclusion $\operatorname{int} W \to W$ is a homotopy equivalence.*

4.4.10. **Definition.** Let W be an $(n+1)$-manifold with boundary. An *R-orientation of W* is an R-orientation of $\operatorname{int} W$.

4.4.11. **Construction.** By definition, an R-orientation of a n $(n+1)$-manifold W with boundary is an element $o_W \in H^{\mathrm{lf}}_{n+1}(\operatorname{int} W)$. If W is compact, we can translate this to a class in ordinary homology through the following observation. The collection of subsets
$$\mathcal{C}_W := \{W \setminus C \mid C \text{ is an open collar neighbourhood of } \partial W\}$$
is a cofinal sub-poset of $\mathcal{K}_{\operatorname{int} W}$. By Lemma 4.3.9, we have
$$H^{\mathrm{lf}}_p(\operatorname{int} W) \xrightarrow{\cong} \lim_{K \in \mathcal{C}_W} H_p(\operatorname{int} W | K).$$
Since each $W \setminus K$ is an open collar neighbourhood of ∂W for $K \in \mathcal{C}_W$, we also have
$$\lim_{K \in \mathcal{C}_W} H_p(\operatorname{int} W | K) \xrightarrow{\cong} \lim_{K \in \mathcal{C}_W} H_p(W, W \setminus K) \xleftarrow{\cong} H_p(W, \partial W)$$
by excision. This provides an identification
$$H^{\mathrm{lf}}_p(\operatorname{int} W) \cong H_p(W, \partial W),$$
so we may equivalently view an orientation of W as a class in $H_{n+1}(W, \partial W)$.

A completely analogous argument provides an isomorphism
$$H^p(W, \partial W) \xleftarrow{\cong} \operatorname*{colim}_{K \in \mathcal{C}_W} H^p(W|K) \xrightarrow{\cong} \operatorname*{colim}_{K \in \mathcal{C}_W} H^p(\operatorname{int} W | K) \xrightarrow{\cong} H^p_c(\operatorname{int} W).$$

4.4.12. **Corollary.** *Let W be an R-oriented, compact $(n+1)$-manifold with boundary. Denote by o_W the image of the orientation class of W under the isomorphism $H^{\mathrm{lf}}_{n+1}(\operatorname{int} W; R) \xrightarrow{\cong} H_{n+1}(W, \partial W; R)$.*

Then
$$H^p(W, \partial W; M) \xrightarrow{- \cap o_W} H_{n-p}(W; M)$$
is an isomorphism for every R-module M.

Proof. Consider the diagram
$$\begin{array}{ccc} H^p(W, \partial W; M) & \xrightarrow{\cong} & H^p_c(\operatorname{int} W; M) \\ \downarrow{\scriptstyle \cap o_W} & & \downarrow{\scriptstyle \cap o_{\operatorname{int} W}} \\ H_{n+1-p}(W; M) & \xleftarrow[\cong]{} & H_{n+1-p}(\operatorname{int} W; M) \end{array}$$

whose top horizontal maps are given by Construction 4.4.11. This diagram commutes as a consequence of Lemma 4.3.5, and the left vertical map is an isomorphism by Poincaré duality. ☐

There is also a version of Poincaré duality for compact manifolds with boundary which compares the cohomology to the relative homology. This will follow from the next lemmas.

4.4.13. **Lemma.** *Let W be a compact $(n+1)$-manifold with boundary.*

(1) *If o_W is an R-orientation of W, then $\partial o_W \in H_n(\partial W; R)$ is an R-orientation of ∂W.*
(2) *If $s \in H_{n+1}(W, \partial W; R)$ is a class such that $\partial s \in H_n(\partial W; R)$ is an R-orientation of ∂W, then s is an R-orientation for every component of W with non-empty boundary.*

Proof. Let $x \in \partial W$. Choose a collar for ∂W such that there exists an open neighbourhood U of x in ∂W such that $U \times [0,1]$ is contained in a coordinate chart around x. Letting w denote the point corresponding to $(x,1)$ and suppressing coefficients everywhere, we obtain the following diagram:

$$\begin{tikzcd}
H^{\mathrm{lf}}_{n+1}(\operatorname{int} W) \arrow[d, "\cong"'] \arrow[dr, "\rho_w"] & & \\
\lim\limits_{K \in \mathcal{C}_W} H_{n+1}(\operatorname{int} W, \operatorname{int} W \setminus K) \arrow[r] \arrow[d, "\cong"] & H_{n+1}(\operatorname{int} W, \operatorname{int} W \setminus w) \arrow[d, "\cong"] & \\
\lim\limits_{K \in \mathcal{C}_W} H_{n+1}(W, W \setminus K) \arrow[r] & H_{n+1}(W, W \setminus w) \arrow[r, "\partial"] & H_n(W \setminus w, W \setminus L) \\
H_{n+1}(W, \partial W) \arrow[u, "\cong"] \arrow[r, "\cong"] \arrow[dr, "\partial"'] & H_{n+1}(W, \partial W \times [0,1)) \arrow[u] \arrow[r, "\partial"] & H_n(\partial W \times [0,1), (\partial W \setminus x) \times [0,1)) \arrow[u, "\cong"'] \\
& H_n(\partial W) \arrow[r, "\rho_x"'] & H_n(\partial W, \partial W \setminus x) \arrow[u, "\cong"']
\end{tikzcd}$$

Here, we have identified $\partial W \times [0,1]$ with its image, and L denotes the line segment $\{x\} \times [0,1]$. The diagram commutes by the definition of the limit and naturality of the boundary map—note that the horizontal boundary maps are those associated to a triple of spaces.

The boundary map $\partial\colon H_{n+1}(W, W \setminus w) \to H_n(W \setminus w, W \setminus L)$ is also an isomorphism because excision yields an identification

$$\begin{tikzcd}
H_{n+1}(W, W \setminus w) \arrow[r, "\partial"] & H_n(W \setminus w, W \setminus L) \\
H_{n+1}(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathbb{R}^n \times \mathbb{R}_{\geq 0} \setminus (x,t)) \arrow[u, "\cong"] \arrow[r, "\partial"] & H_n(\mathbb{R}^n \times \mathbb{R}_{\geq 0} \setminus (x,1), \mathbb{R}^n \times \mathbb{R}_{\geq 0} \setminus \{x\} \times [0,1]) \arrow[u, "\cong"']
\end{tikzcd}$$

and the lower map is an isomorphism since $\mathbb{R}^n \times \mathbb{R}_{\geq 0} \setminus \{x\} \times [0,t] \to \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is a homotopy equivalence.

The lemma follows now by tracing the respective classes through the above diagram. The second assertion also uses that ρ_w and $\rho_{w'}$ are isomorphic maps whenever w and w' lie in the same path component. ☐

4.4.14. **Lemma.** *Let (X, A, B) be an excisive triad. For a class $s \in H_n(X, A \cup B)$, define $t \in H_n(B, A \cap B)$ as the image of s under*

$$H_n(X, A \cup B) \xrightarrow{\partial} H_{n-1}(A \cup B, A) \xleftarrow{\cong} H_{n-1}(B, A \cap B).$$

Then the diagram

$$\begin{tikzcd} H^p(X,A) \arrow[r, "(-1)^p"] \arrow[d, "\cap s"'] & H^p(B, A\cap B) \arrow[d, "\cap t"] \\ H_{n-p}(X,B) \arrow[r, "\partial"] & H_{n-p-1}(B) \end{tikzcd}$$

commutes.

Proof. Let $z \in C^{\mathrm{sing}}(X)$ represent the class s. In particular, $dz \in C^{\mathrm{sing}}(A \cup B)$ and $\partial(s) = [dz] \in H_{n-1}(A\cup B)$. Using excision, we can write

$$dz = a + b + dc$$

for some $a \in C^{\mathrm{sing}}_{n-1}(A)$, $b \in C^{\mathrm{sing}}_{n-1}(B)$ and $c \in C^{\mathrm{sing}}_n(A\cup B)$. Replacing z by $z - c$, we obtain a representative z of s satisfying $dz = a + b$ for some $a \in C^{\mathrm{sing}}_{n-1}(A)$ and $b \in C^{\mathrm{sing}}_{n-1}(B)$.

Consider $\alpha \in C_{\mathrm{sing}}(X,B)$ with $\delta\alpha = 0$. Letting $j\colon (X,\varnothing) \to (X,B)$ denote the inclusion map and using Lemma 4.3.5, the class $\partial([\alpha]\cap s)$ is represented by

$$\begin{aligned} d(j^*\alpha \cap z) &= \delta j^*\alpha \cap z + (-1)^{|\alpha|} j^*\alpha \cap dz \\ &= (-1)^{|\alpha|} j^*\alpha \cap a + b) \\ &= (-1)^{|\alpha|} \left((i_A)_*(i_A^* j^*\alpha \cap a) + (i_B)_*(i_B^* j^*\alpha \cap b)\right) \\ &= (-1)^{|\alpha|} (i_B)_*(i_B^* j^*\alpha \cap b) \end{aligned}$$

because α vanishes on A by assumption. Hence $\partial([a] \cap s)$ is represented by $(-1)^{|\alpha|} i_B^* j^*\alpha \cap b$. By construction, $t = [b]$ and the lemma follows. □

4.4.15. **Lemma.** *Let (X,A) be a pair of spaces and let $s \in H_n(X,A)$. Then*

$$\begin{tikzcd} H^p(A) \arrow[r, "\delta"] \arrow[d, "\cap \partial s"'] & H^{p+1}(X,A) \arrow[d, "\cap s"] \\ H_{n-p-1}(A) \arrow[r, "(-1)^{p+1} i_*"] & H_{n-p-1}(X) \end{tikzcd}$$

commutes.

Proof. Let $\alpha \in C_{\mathrm{sing}}(A)_{-p}$ with $\delta\alpha = 0$ and let $z \in C^{\mathrm{sing}}_n(X)$ represent the class s. Then $\delta[\alpha]$ is represented by $\delta\overline{\alpha}$, where $\overline{\alpha}$ is some extension of α to $C^{\mathrm{sing}}(X)$. Using Lemma 4.3.5, we obtain

$$\begin{aligned} \delta[\alpha] \cap s &= [\delta\overline{\alpha} \cap z] \\ &= [d(\overline{\alpha} \cap z) - (-1)^{|\alpha|}\overline{\alpha} \cap dz] \\ &= (-1)^{p+1}[\overline{\alpha} \cap dz] \\ &= (-1)^{p+1} i_*[\alpha \cap dz] \\ &= (-1)^{p+1} i_*([\alpha] \cap \partial s) \end{aligned}$$

because $dz \in C^{\mathrm{sing}}(A)$. □

4.4.16. **Proposition.** *Let W be a compact, oriented $(n+1)$-manifold. Denote by o_W the image of the orientation class in $H_{n+1}(W, \partial W)$, and let $i\colon \partial W \to W$ be the inclusion map. Then the diagram*

$$\begin{tikzcd} H^p(W,\partial W) \arrow[r] \arrow[d, "\cap o_W"] & H^p(W) \arrow[r, "(-1)^p i^*"] \arrow[d, "\cap o_W"] & H^p(\partial W) \arrow[r, "\delta"] \arrow[d, "\cap \partial o_W"] & H^{p+1}(W,\partial W) \arrow[d, "\cap o_W"] \\ H_{n+1-p}(W) \arrow[r] & H_{n+1-p}(W,\partial W) \arrow[r, "\partial"] & H_{n-p}(\partial W) \arrow[r, "(-1)^{p+1} i_*"] & H_{n-p}(W) \end{tikzcd}$$

commutes.

Proof. By applying Lemma 4.3.5 to the maps of triads

$$(W, \partial W, \varnothing) \to (W, \partial W, \partial W) \leftarrow (W, \varnothing, \partial W),$$

we see that the left square commutes. Applying Lemma 4.4.14 to the excisive triad $(W, \varnothing, \partial W)$ shows that the middle square commutes, and Lemma 4.4.15 implies that the right square commutes. □

4.4.17. **Corollary.** *Let W be a compact, oriented $(n+1)$-manifold and denote by o_W the image of the orientation class in $H_{n+1}(W, \partial W)$. Then both*

$$- \cap o_W : H^p(W, \partial W) \to H_{n+1-p}(W)$$

and

$$- \cap o_W : H^p(W) \to H_{n+1-p}(W, \partial W)$$

are isomorphisms for all p.

Proof. The first statement is Corollary 4.4.12. The other follows from Proposition 4.4.16 and Lemma 4.4.13 from Poincaré duality for ∂W; note that the signs in 4.4.16 do not affect the exactness of the two rows. □

One can ask which closed manifold can appear as the boundaries of compact manifolds with boundary. The following shows that the Euler characteristic of a closed manifold provides an obstruction.

4.4.18. **Corollary.** *Let X be a closed n-manifold. If there exists a compact $(n+1)$-manifold W with $\partial W \cong X$, then the Euler characteristic $\chi(X)$ is even.*

Proof. By Proposition 4.4.1, we only need to consider the case $n = 2k$. Suppose that W is such an $(n+1)$-manifold. Then the product $W \times I$ is a compact $(n+2)$-manifold with boundary

$$\partial(W \times I) = W \times \{0\} \cup \partial W \times I \cup W \times \{1\}.$$

Then $\{\partial(W \times I) \setminus W \times \{i\}\}_{i=0,1}$ is an open cover of $\partial(W \times I)$ such that both members of the cover are homotopy equivalent to W. Since $\partial(W \times I)$ is $(2k+1)$-dimensional, we obtain from Proposition 4.4.1 that

$$0 = \chi(\partial(W \times I)) = \chi(W) + \chi(W) - \chi(X),$$

so $\chi(X) = 2\chi(W)$. □

4.4.19. **Corollary.** *Neither $\mathbb{R}\mathrm{P}^{2n}$ nor $\mathbb{C}\mathrm{P}^{2n}$ are boundaries of compact manifolds.*

Proof. We have $\chi(\mathbb{R}\mathrm{P}^{2n}) = 1$ and $\chi(\mathbb{C}\mathrm{P}^n) = 2n+1$, both of which are odd. □

One can extract further numerical invariants from the middle dimensional homology of closed manifolds.

4.4.20. **Definition.** Let X be a closed, connected and oriented $2n$-manifold. Define the *intersection form* of X as the bilinear form

$$I : H_n(X) \otimes H_n(X) \to H_0(X) \cong \mathbb{Z}, \quad x \otimes y \mapsto \mathrm{PD}^{-1}(x) \cap y,$$

where $\mathrm{PD} : H^n(X) \to H_n(X)$ denotes the Poincaré duality isomorphism.

4.4.21. **Lemma.** *Let X be a closed, connected and oriented $2n$-manifold.*

1. *The intersection form is non-degenerate.*
2. *If n is even, the intersection form is symmetric.*
3. *If n is odd, the intersection form is skew-symmetric.*

Proof. Denote the orientation class of X by o_X. By Lemma 4.3.6, we have

$$\mathrm{PD}^{-1}(x) \cap y = \mathrm{PD}^{-1}(x) \cap \left(\mathrm{PD}^{-1}(y) \cap o_X\right) = \left(\mathrm{PD}^{-1}(x) \cup \mathrm{PD}^{-1}(y)\right) \cap o_X,$$

so the intersection form is non-degenerate is non-degenerate because the cup product is non-degenerate. The same formula shows that

$$\begin{aligned} I(x,y) &= \left(\mathrm{PD}^{-1}(x) \cup \mathrm{PD}^{-1}(y)\right) \cap o_X \\ &= (-1)^{n^2} \left(\mathrm{PD}^{-1}(y) \cup \mathrm{PD}^{-1}(x)\right) \cap o_X \\ &= (-1)^{n^2} I(y,x). \end{aligned}$$

If n is even, so is n^2. Otherwise, n^2 is odd. This shows that the intersection form is symmetric or skew-symmetric, depending on the parity of n. □

4.4.22. **Corollary.** *Let X be a closed, orientable $(4n+2)$-manifold. Then the Euler characteristic $\chi(X)$ is even.*

Proof. By Poincaré duality, we again have $b_p(X;\mathbb{Q}) = b_{4n+2-p}(X;\mathbb{Q})$. Since $4n+2$ is even, p and $4n+2-p$ have the same parity, so

$$\chi(X) = \sum_{i=0}^{4n+2} (-1)^i b_p(X;\mathbb{Q}) = -b_{2n+1}(X;\mathbb{Q}) + \sum_{i=0}^{2n} (-1)^i 2 b_p(X;\mathbb{Q}).$$

Thus, it suffices to show that $b := b_{2n+1}(X;\mathbb{Q})$ is even.

The rationalised intersection form $I \otimes \mathbb{Q}$ is a non-degenerate, skew symmetric bilinear form on $H_{2n+1}(X;\mathbb{Q})$ by Lemma 4.4.21 (and the universal coefficient theorem for homology). Choosing a $\mathbb{Q}$-basis of $H_{2n+1}(X;\mathbb{Q})$, the intersection form $I \otimes \mathbb{Q}$ is represented by an invertible quadratic matrix $A \in M_b(\mathbb{Q})$. Then

$$I(x,y) = -I(y,x) = -y^t A x = -x^t A^t y,$$

so $A = -A^t$. It follows that

$$\det(A) = \det(-A^t) = (-1)^b \det(A).$$

Since $\det(A) \neq 0$, it follows that b is even. □

4.4.23. **Definition.** Let X be a closed, connected and oriented $4n$-manifold. Then Lemma 4.4.21 and Sylvester's law of inertia imply that

$$\sigma(X) := \#\text{positive eigenvalues of } I \otimes \mathbb{R} - \#\text{negative eigenvalues of } I \otimes \mathbb{R} \in \mathbb{Z}$$

is a well-defined integer, called the *signature* of X.

4.4.24. **Example.**

(1) Since its middle homology vanishes, the signature of S^{4n} is zero.
(2) Consider $\mathbb{C}\mathrm{P}^{2n}$ and let $c \in H^2(\mathbb{C}\mathrm{P}^{2n};\mathbb{R})$ be a generator. Let $o_{\mathbb{C}\mathrm{P}^{2n}} \in H_{4n}(\mathbb{C}\mathrm{P}^2;\mathbb{R})$ be the element whose dual corresponds to c^{2n} under the isomorphism

$$H^{4n}(\mathbb{C}\mathrm{P}^{2n};\mathbb{R}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{R}}(H_{4n}(\mathbb{C}\mathrm{P}^{2n};\mathbb{R}),\mathbb{R})$$

of the universal coefficient theorem (note that this choice of orientation is independent of the choice of c). Then $\mathrm{PD}(c^n)$ is a generator of $H_{2n}(\mathbb{C}\mathrm{P}^2 n;\mathbb{R})$, and we obtain

$$I(\mathrm{PD}(c^n),\mathrm{PD}(c^n)) = c^n \cap \mathrm{PD}(c^n) \overset{\textit{Lemma } 4.3.6}{=} c^{2n} \cap o_{\mathbb{C}\mathrm{P}^{2n}} = 1.$$

We conclude that $\sigma(\mathbb{C}\mathrm{P}^{2n}) = 1$.

4.4.25. **Lemma.** *Let $\varphi\colon V \otimes_{\mathbb{R}} V \to \mathbb{R}$ be a non-degenerate, symmetric bilinear form over the finite-dimensional real vector space V. If φ admits a* Lagrangian, *ie there exists a subspace $L \subseteq V$ such that $2\dim_{\mathbb{R}} L = \dim_{\mathbb{R}} V$ and $\varphi|_{L\otimes L} = 0$, then the signature of φ is zero.*

Proof. Let n be the dimension of L. It suffices to show that φ has at least n positive eigenvalues: by considering $-\varphi$, it then follows that there are also at least n negative eigenvalues, and consequently the signature of φ is zero.

We argue by induction. Let $x_1, \dots, x_n \in L$ be a basis and let $x_1^*, \dots, x_n^*$ be the dual basis in $\mathrm{Hom}_{\mathbb{R}}(L, \mathbb{R})$. Let $\alpha_i \colon V \to \mathbb{R}$ be some extension of x_i^*. Now define $y_1 \in V$ by $\varphi(y_1, -) = \alpha_1$, and set

$$a_1 := \frac{1 - \varphi(y_1, y_1)}{2}.$$

Then

$$\varphi(a_1 x_1 + y_1, a_1 x_1 + y_1) = a_1^2 \varphi(x_1, x_1) + 2a_1 \varphi(x_1, y_1) + \varphi(y_1, y_1) = \varphi(x_1, y_1) = 1,$$

so $a_1 x_1 + y_1$ is an eigenvector with eigenvalue 1. The homomorphism

$$\psi := (\varphi(x_1, -), \varphi(y_1, -)) \colon V \to \mathbb{R}^2$$

is surjective because $\psi(x_1) = (0, 1)$ and $\psi(y_1) = (1, \varphi(y_1, y_1))$. Since φ is a non-degenerate, symmetric bilinear form on $\ker \psi$ and $(x_2, \dots, x_n)$ spans an $(n-1)$-dimensional subspace of $\ker \psi$ on which $\varphi|_{\ker \psi \otimes \ker \psi}$ vanishes, the inductive assumption applies. □

4.4.26. **Proposition.** *Let W be a compact, oriented $(4n+1)$-manifold. Then*

$$\sigma(\partial W) = 0.$$

Proof. Let $i \colon \partial W \to W$ be the inclusion map. By Lemma 4.4.25, it suffices to show that

$$\ker \left(i_* \colon H_{2n}(\partial W; \mathbb{R}) \to H_{2n}(W; \mathbb{R})\right)$$

is a Lagrangian of the intersection form.

From Proposition 4.4.16, we conclude that the Poincaré duality isomorphism induces isomorphisms

$$\mathrm{img}(i_*) \cong \mathrm{coker}(i^*) \quad \text{and} \quad \ker(i_*) \cong \mathrm{img}(i^*).$$

The universal coefficient theorem implies that the dual of $\ker(i^*)$ is isomorphic to $\mathrm{coker}(i^*)$. Since $H_{2n}(\partial W)$ is finite-dimensional,

$$\dim_{\mathbb{R}} H_{2n}(\partial W; \mathbb{R}) = \dim_{\mathbb{R}} \ker(i_*) + \dim_{\mathbb{R}} \mathrm{img}(i_*) = 2 \dim_{\mathbb{R}} \ker(i_*).$$

Moreover, we have for $x, y \in \ker(i_*)$ that

$$i_* I(x, y) = i_*(\mathrm{PD}^{-1}(x) \cap y)) = i_*(i^* \alpha \cap y) = \alpha \cap i_* y = 0.$$

Since $i_* \colon H_0(X) \to H_0(W)$ is injective for every connected component X of ∂W, it follows that I vanishes on $\ker(i_*) \otimes \ker(i_*)$. □

4.4.27. **Definition.** Let X_0 and X_1 be closed and oriented n-manifolds. An *oriented bordism* from X_0 to X_1 is a compact, oriented $(n+1)$-manifold W with boundary together with a homeomorphism $h \colon X_0 \sqcup X_1 \xrightarrow{\cong} \partial W$ such that h is orientation-preserving on X_0 and orientation-reversing on X_1.

4.4.28. **Lemma.** *Oriented bordism is an equivalence relation on closed and oriented n-manifolds.*

Proof. For reflexivity, consider for an oriented and closed n-manifold X the compact $(n+1)$-manifold with boundary $X \times [0, 1]$. From the exact sequence

$$0 = H_{n+1}(X \times [0,1]) \to H_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\partial} H_n(X \times \{0,1\}) \to H_n(X \times [0,1]),$$

we conclude that the element $(o_X, -o_X) \in H_n(X \times \{0, 1\})$ has a unique lift $o_{X \times [0,1]}$ under the boundary map. Lemma 4.4.13 implies that $o_{X \times [0,1]}$ is an orientation of $X \times [0, 1]$.

For symmetry, observe that an oriented bordism (W, o_W, h) from X_0 to X_1 produces an oriented bordism $(W, -o_W, h)$ from X_1 to X_0.

For transitivity, let (W, o_W, h) be an oriented bordism from X_0 to X_1 and let $(W', o_{W'}, h')$ be an oriented bordism from X_1 to X_2. Consider the pushout

$$\begin{tikzcd} X_1 \arrow[r, "h|_{X_1}"] \arrow[d, "h'|_{X_1}"'] & W \arrow[d] \\ W' \arrow[r] & V \end{tikzcd}$$

For points in X_1, we obtain charts by gluing a collar of X_1 in W to a collar of X_1 in W', so V is an $(n+1)$-manifold. Now consider the commutative diagram

$$\begin{tikzcd} & & H_{n+1}(W, \partial W) \oplus H_{n+1}(W', \partial W') \arrow[r, "\partial"] \arrow[d, "i_*+j_*"] & H_n(\partial W) \oplus H_n(\partial W') \arrow[d] \\ H_{n+1}(\partial V \sqcup X_1, \partial V) \arrow[r] & H_{n+1}(V, \partial V) \arrow[r] & H_{n+1}(V, \partial V \sqcup X_1) \arrow[r, "\partial"] & H_n(\partial V \sqcup X_1, \partial V) \end{tikzcd}$$

Note $H_k(\partial V \sqcup X_1, \partial V) \cong H_k(X_1)$. It follows that $\partial(i_* o_W + j_* o_{W'}) = 0$, so $i_* o_W + j_* o_{W'}$ has a unique lift to an element $o_V \in H_{n+1}(V, \partial V)$ since $H_{n+1}(X_1) = 0$ by Proposition 4.2.5. Replacing the pair $(\partial V \sqcup X_1, \partial V)$ by the pair $(\partial V \sqcup X_1, X_1)$ in the above diagram shows that o_V induces the same orientations on X_0 and X_2 as o_W and $o_{W'}$. It follows from Lemma 4.4.13 that o_V is an orientation of V. $\square$

4.4.29. **Corollary.** *The disjoint union operation induces an abelian group structure on the set of bordism classes of closed, oriented n-manifolds.*

Proof. The empty manifold (which is not only a closed manifold of any dimension, but also carries a unique orientation!) represents the neutral element. Associativity and commutativity are clear. Since oriented bordism is reflexive, $[X, -o_X]$ is the inverse of $[X, o_X]$. $\square$

4.4.30. **Definition.** Denote by Ω_n^{STOP} the abelian group of bordism classes of closed and oriented n-manifolds.

4.4.31. **Corollary.** *The signature defines a homomorphism*

$$\sigma \colon \Omega_{4n}^{\mathrm{STOP}} \to \mathbb{Z}.$$

4.4.32. **Remark.** One can refine the definition of Ω_k^{STOP} to define an abelian group $\Omega_k^{\mathrm{STOP}}(Z)$ for topological spaces Z: elements are represented by closed, oriented k-manifolds X with a reference map $X \to Z$, and the bordism relation asks for an extension of the maps at the two ends of the bordism to a map defined on the entire bordism.

It is possible to show that the assignment $Z \mapsto \Omega_k^{\mathrm{STOP}}$ defines a homology theory (at least for sufficiently nice topological spaces) in the sense that it is homotopy invariant, admits a long exact sequence for pairs of spaces (by associating to a pair of spaces (X, A) the abelian group $\Omega_k^{\mathrm{STOP}}(X \cup_A \mathrm{C}(A))$) and satisfies a form of the excision theorem—this becomes quite a bit easier, but remains non-trivial, if one restricts to smooth manifolds everywhere, which is also the case typically discussed in the literature. The corresponding groups are then denoted by $\Omega_k^{\mathrm{SO}}(Z)$.

The second part of Example 4.4.24 now asserts that $\mathbb{C}\mathrm{P}^{2n}$ defines a non-trivial element in $\Omega_{4n}^{\mathrm{STOP}} \cong \Omega_{4n}^{\mathrm{STOP}}(*)$, and the same is true for $\Omega_{4n}^{\mathrm{SO}}$. This indicates that this homology theory is significantly different from singular homology, even though it enjoys similar formal properties.

As a final application of Poincaré duality, we will obtain some information about the homology of compact Euclidean neighbourhood retracts (ie compact subsets

$K \subseteq \mathbb{R}^n$ satisfying the equivalent conditions of Proposition 4.1.10). Our goal is to prove the following.

4.4.33. **Theorem.** *Let X be a closed, connected and orientable n-manifold and let $K \subseteq X$ be a compact subset. Then there exists an isomorphism*

$$\operatorname*{colim}_{\substack{U\,\text{open nbhd.}\\ \text{of } K \text{ in } X}} H^p(U) \cong H_{n-p}(X, X \setminus K).$$

If K is in addition weakly locally contractible, then the comparison map

$$\operatorname*{colim}_{\substack{U\,\text{open nbhd.}\\ \text{of } K \text{ in } X}} H^p(U) \to H^p(K)$$

is an isomorphism.

Proof. We may assume without loss of generality that K is a proper subset, otherwise this is just the Poincaré duality theorem. Let $\mathcal{U}$ denote the poset consisting of open neighbourhoods U of K in X, ordered by inclusion. Then the map

$$\mathcal{U}^{\mathrm{op}} \to \mathcal{K}_{X \setminus K}, \qquad U \mapsto X \setminus U$$

is an isomorphism of posets: the inverse map sends L to $X \setminus L$. This allows us to express

$$H^p_c(X \setminus K) \cong \operatorname*{colim}_{U \in \mathcal{U}^{\mathrm{op}}} H^p(X \setminus K, U \setminus K)$$

and

$$H^{\mathrm{lf}}_n(X \setminus K) \cong \lim_{U \in \mathcal{U}} H^p(X \setminus K, U \setminus K).$$

Choose an orientation $o_X \in H_n(X)$. Then o_X induces an orientation on $o_{X \setminus K} = (o_{X \setminus K, U})_{U \in \mathcal{U}} \in H^{\mathrm{lf}}_n(X \setminus K)$.

For a fixed open neighbourhood U of K, consider the following diagram:

$$\begin{tikzcd}
H^p(X,U) \arrow[r] \arrow[d, "(-1)^{p-1}"', "\cong"] & H^p(X) \arrow[r] \arrow[dd, "(-1)^{p+1}(-\cap o_X)"'] & H^p(U) \arrow[r, "\delta"] \arrow[d, "\cap o_{U,K}"] & H^{p+1}(X,U) \arrow[d, "(-1)^p"', "\cong"] \\
H^p(X \setminus K, U \setminus K) \arrow[d, "\cap o_{X \setminus K, U}"'] & & H_{n-p}(U, U \setminus K) \arrow[d, "(-1)^{p+1}"', "\cong"] & H^{p+1}(X \setminus K, U \setminus K) \arrow[d, "\cap o_{X \setminus K, U}"] \\
H_{n-p}(X \setminus K) \arrow[r] & H_{n-p}(X) \arrow[r] & H_{n-p}(X, X \setminus K) \arrow[r, "\partial"] & H_{n-p-1}(X \setminus K)
\end{tikzcd}$$

The rows come the corresponding sequences of pairs, and all isomorphisms come from excision. Since U is an open subset of X, it is an n-manifold, and U inherits an orientation from X via the excision isomorphisms $H_n(U, U \setminus L) \xrightarrow{\cong} H_n(X, X \setminus L)$, which in particular defines the class $o_{U,K} \in H_n(U, U \setminus K)$. We claim that this diagram commutes.

Noting that $(-1)^{p-1} = (-1)^{p+1}$, one sees that the left square commutes by applying Lemma 4.3.5 the maps of excisive triads

$$(X \setminus K, U \setminus K, \varnothing) \to (X, U, \varnothing) \leftarrow (X, \varnothing, \varnothing).$$

For the middle square, commutativity follows from Lemma 4.3.5 applied to the maps of excisive triads

$$(U, \varnothing, U \setminus K) \to (X, \varnothing, X \setminus K) \leftarrow (X, \varnothing, \varnothing).$$

The right square can be expanded into the diagram

$$
\begin{tikzcd}
 & & H^{p+1}(X,U) \arrow[d, "(-1)^p"] \\
H^p(U) \arrow[r, "(-1)^p"'] \arrow[urr, "\delta"] \arrow[d, "\cap o_{U,K}"'] & H^p(U \setminus K) \arrow[r, "\delta"] \arrow[d, "\partial o_{U,K}"] & H^{p+1}(X \setminus K, U \setminus K) \arrow[d, "o_{X \setminus K, U}"] \\
H_{n-p}(U, U \setminus K) \arrow[r, "\partial"] \arrow[d, "(-1)^{p+1}"] & H_{n-p-1}(U \setminus K) \arrow[r, "(-1)^{p+1}"] & H_{n-p-1}(X \setminus K) \\
H_{n-p}(X, X \setminus K) \arrow[urr, "\partial"'] & &
\end{tikzcd}
$$

The two triangles commute by naturality of the boundary map. The left square commutes as a consequence of Lemma 4.4.14, and the right square commutes as a consequence of Lemma 4.4.15.

In sum, the large ladder diagram above commutes. Passing to the colimit over U, the outermost vertical maps become isomorphisms by Poincaré duality for $X \setminus K$. Since X also satisfies Poincaré duality, it follows that the map

$$\operatorname*{colim}_{U \in \mathfrak{U}^{\mathrm{op}}} H^p(U) \to H_{n-p}(X, X \setminus K)$$

induced by the cap products with $(o_{U,K})_{U \in \mathfrak{U}}$ is an isomorphism.

We are left with showing that the domain of this map is isomorphic to $H^p(K)$ if K is weakly locally contractible. We know that X can be embedded into some Euclidean space $\mathbb{R}^N$. This also embeds K into $\mathbb{R}^X$, and since K is weakly locally contractible, Proposition 4.1.10 shows that there exists an open neighbourhood V of K in $\mathbb{R}^N$ which retracts onto K. Setting $U := X \cap V$, we obtain an open neighbourhood of K in X which retracts onto K. In particular, every class in $H^p(K)$ admits a preimage via pullback along the retraction $U \to K$.

For injectivity of the comparison map, let U be an arbitrary neighbourhood of K in X and let $u \in H^p(U)$ represent an element in the colimit which maps to zero in $H^p(K)$. Since K admits an open neighbourhood which retracts onto K, assume without loss generality that there exists a retraction $r \colon U \to K$. Then the map $U \xrightarrow{r} K \subseteq \mathbb{R}^N$ is homotopic to the inclusion map $U \to \mathbb{R}^N$ since $\mathbb{R}^N$ is contractible. Let $h \colon U \times [0,1] \to \mathbb{R}^N$ be such a homotopy.

By Proposition 4.1.10, X admits an open neighbourhood V in $\mathbb{R}^N$ and a retraction $r_X \colon V \to X$ to the inclusion map. Then $h^{-1}(V)$ is an open neighbourhood of $K \times [0,1]$ in $U \times [0,1]$; by compactness, there exists an open neighbourhood $U' \subseteq U$ of K such that $h|_{U' \times [0,1]}$ is a homotopy in V. By composing this restricted homotopy with r_X, we obtain a homotopy

$$h' \colon U' \times [0,1] \to X$$

which is constant on K, which is the inclusion map at one end, and which is given by a composition $U' \xrightarrow{r|_{U'}} K \hookrightarrow X$.

The same type of argument applied to $(h')^{-1}(U)$ yields a homotopy

$$h'' \colon U'' \times [0,1] \to U$$

between the inclusion map and the composition $U'' \xrightarrow{r|_{U''}} K \hookrightarrow U$. Hence $H^p(U) \to H^p(K) \xrightarrow{r|_{U''}^*} H^p(U'')$ agrees with the map induced by the inclusion $U'' \subseteq U$, so $u \in H^p(U)$ represents zero in the colimit. $\square$

4.4.34. **Corollary** (Alexander duality). *Let K be a non-empty, proper, compact subset of S^n. Then there exists an isomorphism*

$$\operatorname*{colim}_{\substack{U \text{ open nbhd.} \\ \text{of } K \text{ in } S^n}} \widetilde{H}^p(U) \cong \widetilde{H}_{n-p-1}(S^n \setminus K).$$

If K is in addition weakly locally contractible, this induces an isomorphism

$$\widetilde{H}^p(K) \cong \widetilde{H}_{n-p-1}(S^n \setminus K).$$

Proof. For $n = 0$, all terms involved are zero, so assume $n > 0$. From the long exact sequence of the pair $(S^n, S^n \setminus K)$, we find that

$$H_{n-p}(S^n, S^n \setminus K) \underset{\cong}{\overset{\partial}{\to}} H_{n-p-1}(S^n \setminus K)$$

is an isomorphism for $p \neq 0$ (the case $p = n$ uses that K is a proper subset). In the case $p = 0$, choose a point $x \in K$. Then an inspection of the proof of Theorem 4.4.33 yields a commutative diagram

$$\begin{tikzcd}
H^0(K) \arrow[d] & \operatorname{colim}_{K\subseteq U} H^0(U) \arrow[l] \arrow[r, "\cong"] \arrow[d] & H_n(S^n, S^n \setminus K) \arrow[d] \\
H^0(*) & \operatorname{colim}_{*\in U} H^0(U) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(S^n, S^n \setminus x)
\end{tikzcd}$$

The upper map pointing left is an isomorphism if K is weakly locally contractible.

Since the composite

$$H_n(S^n) \to H_n(S^n, S^n \setminus K) \to H_n(S^n, S^n \setminus x)$$

is an isomorphism, the right vertical map splits, and this yields the desired identification of reduced (co)homology groups. $\square$

4.4.35. **Remark.**

(1) If K is as in Corollary 4.4.34, then Corollary 4.4.34 implies in particular that the homology of $S^n \setminus K$ is concentrated below degree n.

(2) The invariant

$$\operatorname*{colim}_{\substack{U \text{open nbhd.} \\ \text{of } K \text{ in } S^n}} H^p(U)$$

is called the *Čech cohomology of K*. Theorem 4.4.33 asserts in particular that the Čech cohomology of a compact space $K \subseteq \mathbb{R}^N$ coincides with its singular cohomology if K is weakly locally contractible.

(3) Suppose that $f\colon S^{n-1} \to S^n$ is a topological embedding. Then Corollary 4.4.34 implies that

$$\widetilde{H}_0(S^n \setminus f(S^{n-1})) \cong \widetilde{H}^{n-1}(S^{n-1}) \cong \mathbb{Z},$$

so the complement of $f(S^{n-1})$ has exactly two path components.

For $n = 2$, this is also a consequence of the Jordan curve theorem, which additionally asserts that both components of the complement are homeomorphic to discs. This is not true in higher dimensions: for $n = 3$, the "Alexander horned sphere" gives a counterexample. If one assumes in addition that the embedding of S^{n-1} extends to an embedding of $S^{n-1} \times [0, 1]$, then one can show without restrictions on the dimension that both components are discs.

5. Homotopy theory

One of the heuristics to describe what singular homology is doing says that singular homology detects "holes" in topological spaces. For example, the calculation $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ is compatible with our intuition that the n-sphere has exactly one n-dimensional "hole". The homotopy groups of a space constitue an alternative formalisation of this idea. We can think of continuous maps $S^n \to X$ as tentative "holes" in the space X, and we can ask whether this can be filled (extending the map to D^{n+1}) or not. By now, we have grown used to the fact that good invariants are homotopy invariant, so we consider such maps up to homotopy. Generalising the construction of the fundamental group, we want to equip these sets of homotopy classes of pointed maps with a composition operation. Geometrically, this operation is induced by the pinching map $S^n \to S^n \vee S^n$. However, the definition is easier to write down if we identify $S^n \cong I^n/\partial I^n$, where $I^n := [0,1]^n$ denotes the n-cube.

5.1. Homotopy groups.

The contents of this section are mostly a recap of notions and statements discussed in Topology I.

In order to construct homotopies, it is convenient to know that we can obtain some homotopies "for free" as encoded by the homotopy extension property.

5.1.1. **Definition.** A map $i\colon A \to X$ is a *cofibration* if it has the homotopy extension property: every map

$$(A \times [0,1]) \cup_{A\times\{0\}} (X \times \{0\}) \to Y$$

extends to a map $X \times [0,1] \to Y$.

5.1.2. **Remark.** By applying the homotopy extension property to the identity map, it follows that i is a cofibration if and only if $(A \times [0,1]) \cup_{A\times\{0\}} (X \times \{0\})$ is a retract of $X \times [0,1]$.

5.1.3. **Example.**

(1) The inclusion map $S^n \to D^{n+1}$ is a cofibration.
(2) If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout and i is a cofibration, then j is also a cofibration.
(3) If $(i_\alpha\colon A_\alpha \to X_\alpha)_\alpha$ is a family of cofibrations, then $\coprod_\alpha i_\alpha\colon \coprod A_\alpha \to \coprod_\alpha X_\alpha$ is also a cofibration.
(4) The previous two examples imply that the inclusion $A \to X$ of a subcomplex into a CW-complex is a cofibration.

5.1.4. **Lemma.**

(1) *Consider a commutative diagram*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

in which i and j are cofibrations. If f is a homotopy equivalence, then f is also a homotopy equivalence relative A.
(2) *If $i\colon A \to X$ is a cofibration and a homotopy equivalence, then A is a deformation retract of X.*

Proof. See [May99, Chapter 6.5]. □

We will make use of the following notation:

(1) $I^n := [0,1]^n$ denotes the n-cube;
(2) $\partial I^n := \{(x_1,\dots,x_n) \in I^n \mid \exists\, i\colon x_i = 0\}$ is the boundary of the n-cube.

Note that the pair $(I^n, \partial I^n)$ is homeomorphic to the pair (D^n, S^{n-1}).

5.1.5. **Definition.** Let (X,x) be a pointed space. Define

$$\pi_n(X,x) := \{\text{homotopy classes of maps } (I^n,\partial I^n) \to (X,x)\}.$$

5.1.6. **Definition.** Let $f,g\colon (I^n,\partial I^n) \to (X,x)$ be maps. Define

$$\begin{aligned} f *_1 g\colon I^n &\to X \\ (t_1,\dots,t_n) &\mapsto \begin{cases} f(2t_1,t_2,\dots,t_n) & 0 \le t_1 \le \frac{1}{2} \\ g(2t_1-1,t_2,\dots,t_n) & \frac{1}{2} \le t_1 \le 1 \end{cases}\end{aligned}$$

The resulting map is a map of pairs $f *_1 g\colon (I^n,\partial I^n) \to (X,x)$.

Let $\tau_{1,i}\colon I^n \to I^n$ be the automorphism which switches the first and i-th coordinate. Then define

$$f *_i g := ((f \circ \tau_{1,i}) *_1 (g \circ \tau_{1,i})) \circ \tau_{1,i}\colon (I^n,\partial I^n) \to (X,x).$$

The first key point is that, despite appearances, we have defined a single composition operation on $\pi_n(X,x)$. This rests on the following observation.

5.1.7. **Lemma** (Eckmann–Hilton argument)**.** *Let M be a set with two unital binary operations $\cdot$ and $\star$. If*

$$(v \cdot w) \star (x \cdot y) = (v \star x) \cdot (w \star y)$$

for all $v,w,x,y \in M$, then $\cdot = \star$, and this operation is commutative.

Proof. Let e be the neutral element for $\cdot$, and let $e_\star$ be the neutral element for $\star$. Then

$$e_\star = e_\star \star e_\star = (e \cdot e_\star) \star (e_\star \cdot e) = (e \star e_\star) \cdot (e_\star \star e) = e \cdot e = e,$$

so the neutral elements coincide. Now it follows that

$$x \cdot y = (e \star x) \cdot (y \star e) = (e \cdot y) \star (x \cdot e) = y \star x$$

and

$$x \cdot y = (x \star e) \cdot (e \cdot y) = (x \cdot e) \star (e \cdot y) = x \star y,$$

so both operations conincide and are commutative. □

Remark. One way to remember the equation featuring in Lemma 5.1.7 is to say that $\cdot$ is a homomorphism with respect to $\star$ (or the other way around). With this observation, one can upgrade the Eckmann–Hilton argument to the assertion that the category of monoid objects in monoids is equivalent to the category of commutative monoids.

5.1.8. **Proposition.** *Each operation $*_i$ induces the same group structure on $\pi_n(X,x)$, and this group structure is commutative for $n \geq 2$.*

Proof. The proofs for well-definedness, associativity and unitality work essentially as in the case $n = 1$, where the given formulas define the fundamental group. Inverses for $*_i$ are obtained by applying the inversion $[0,1] \to [0,1]$, $t \mapsto 1-t$, to the i-th coordinate.

By inspection, one finds that $(f *_i g) *_j (h *_i k) = (f *_j h) *_i (g *_j k)$, so the Eckmann–Hilton argument shows that all operations $*_i$ coincide and are commutative if $n \geq 2$. □

5.1.9. **Corollary.** *The preceding constructions define homotopy invariant functors*

$$\begin{aligned}\pi_0 &\colon \mathrm{Top}_* \to \mathrm{Set}_*,\\ \pi_1 &\colon \mathrm{Top}_* \to \mathrm{Grp},\\ \pi_n &\colon \mathrm{Top}_* \to \mathrm{Ab}, \quad n \geq 2.\end{aligned}$$

5.1.10. **Remark.** Note that Corollary 5.1.9 asserts that maps which are homotopic relative to the given basepoints induce the same maps on π_n, and therefore pointed spaces which are pointed homotopy equivalent have isomorphic homotopy groups. This restriction is not overly problematic due to Lemma 5.1.4: if we consider *well-pointed* spaces, ie spaces X for which the inclusion $\{x\} \to X$ of the base point is a cofibration, then a pointed map which is a homotopy equivalence is also a pointed homotopy equivalence.

5.1.11. **Remark.** Observing that

$$I^n/\partial I^n \cong S^n,$$

we can identify $\pi_n(X, x)$ with the set of pointed homotopy classes of pointed maps $(S^n, s) \to (X, x)$ for some basepoint $s \in S$.

After fixing such an identification, the composition operation $*_i$ is induced by the pinch map $S^n \to S^n \vee S^n$ which collapses an equatorial $(n-1)$-sphere, the index i designating the hyperplane which contains the equator.

5.1.12. **Definition.** A map $f\colon X \to Y$ is a *weak homotopy equivalence* if the induced map $f_*\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection for all $x \in X$ and all $k \geq 0$.

5.2. **Fibrations and the Whitehead theorem.** A key concept that will allow us to organise homotopy theoretic information is that of a *Serre fibration*, which is dual to the notion of a cofibration. Among other things, it will allow us to establish long exact sequences of homotopy groups for arbitrary maps of spaces, and it will allow us to show that weak homotopy equivalences between CW-complexes are homotopy equivalences.

5.2.1. **Definition.** A map $p\colon X \to Y$ is a *Serre fibration* if, for all $n \geq 0$, every lifting problem

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution: given any commutative square as depicted by the solid arrows, there exists a dotted arrow making the resulting diagram commute.

Let us first make sure that we have a reasonable supply of Serre fibrations.

5.2.2. **Lemma.** *Every fibre bundle is a Serre fibration.*

Proof sketch. For the purpose of this proof, identify $D^n \cong I^n$. Let $p\colon E \to B$ be a fibre bundle and consider a lifting problem

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g_0"] \arrow[d, "\mathrm{inc}"'] & E \arrow[d, "p"] \\ I^n \times [0,1] \arrow[r, "f"] \arrow[ur, dotted] & B \end{tikzcd}$$

Pick an open cover $\mathcal{U} = (U_\alpha)_\alpha$ of B and trivialisations

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr, "p"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}"] \\ & U_\alpha & \end{tikzcd}$$

for each α. Then $f^{-1}\mathcal{U}$ is an open cover of $I^n \times I$. Pick a subdivision $0 = t_0 < \ldots < t_r = 1$ of the interval such that each cube $\prod_{k=1}^{n+1}[t_{i_k}, t_{i_k+1}]$ is contained in an element of $f^{-1}\mathcal{U}$. Proceeding along the lexicographic ordering of $\{0, \ldots, r-1\}^{n+1}$, one constructs a lift using the local trivialisations φ_α. The important observation is that in each step of the construction, one needs to extend a map from a union of faces of the $(n+1)$-cube which is properly contained in the boundary to the entire cube; this can be achieved by composing with a suitable retraction of the cube onto the subcomplex of the boundary. □

5.2.3. **Example.**

(1) The unique map $X \to *$ is a Serre fibration for every topological space X.
(2) The projection map $X \times F \to X$ is a Serre fibration for any two topological spaces X and F.
(3) Every covering is a Serre fibration since it is a fibre bundle with discrete fibres.
(4) The projection maps $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ are fibre bundles with fibres homeomorphic to S^1, so they are also Serre fibrations. This includes in particular the Hopf map $\eta\colon S^3 \to S^2$ (ie the attaching map of the 4-cell in $\mathbb{C}\mathrm{P}^2$).

5.2.4. **Lemma.**

(1) *If $p\colon X \to Y$ and $q\colon Y \to Z$ are Serre fibrations, then $q \circ p$ is also a Serre fibration.*
(2) *If $p\colon X \to Y$ is a Serre fibration and*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a pullback, then p' is also a Serre fibration.
(3) *If $(p_\alpha\colon X_\alpha \to Y_\alpha)_\alpha$ is a family of Serre fibrations, then their product $\prod_\alpha p_\alpha\colon \prod_\alpha X_\alpha \to \prod_\alpha Y_\alpha$ is also a Serre fibration.*

Proof. For (1), the lifting problem

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[dd, "\mathrm{inc}"'] & X \arrow[d, "p"] \\ & Y \arrow[d, "q"] \\ D^n \times [0,1] \arrow[r] \arrow[uur, dotted] & Z \end{tikzcd}$$

can be solved by successively choosing lifts along q, and then along p.

For (2), note that a lifting problem for p' gives rise to a diagram

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, "\mathrm{inc}"'] & X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dotted] & Y' \arrow[r] & Y \end{tikzcd}$$

Since p is a Serre fibration, there exists a map $D^n \times [0,1] \to X$ which solves the lifting problem given by the outer corners. Since X' is a pullback, this induces the desired solution to the original lifting problem.

For (3), a lifting problem

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r] \arrow[d, "\mathrm{inc}"'] & \prod_\alpha X_\alpha \arrow[d, "p_\alpha"] \\
D^n \times [0,1] \arrow[r] \arrow[ur, dotted] & Y_\alpha
\end{tikzcd}$$

can be solved by choosing a solution in each component separately. □

The homotopy lifting property which defines Serre fibrations holds for a much larger class of spaces than just discs.

5.2.5. **Lemma.** *Let $p\colon X \to Y$ be a Serre fibration. If (B,A) is a relative CW-complex and $i\colon A \to B$ is the inclusion map, then every lifting problem*

$$\begin{tikzcd}
A \times [0,1] \cup_{A\times\{0\}} B \times \{0\} \arrow[r] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "p"] \\
B \times [0,1] \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

has a solution.

Proof. Observe first that there is an isomorphism

$$\begin{tikzcd}
S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} \arrow[r, "\mathrm{inc}"] \arrow[d, "\cong"] & D^n \times [0,1] \arrow[d, "\cong"] \\
D^n \times \{0\} \arrow[r, "\mathrm{inc}"] & D^n \times [0,1]
\end{tikzcd}$$

Hence every lifting problem for the relative CW-complex (D^n, S^{n-1}) has a solution. It follows that every lifting problem for the relative CW-complex $(\coprod_\alpha D^n, \coprod_\alpha S^{n-1})$ has a solution.

Suppose now that B is obtained by attaching n-cells to A, ie that there exists a pushout

$$\begin{tikzcd}
\coprod_\alpha S^{n-1} \arrow[r] \arrow[d] & A \arrow[d, "i"] \\
\coprod_\alpha D^n \arrow[r] & B
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
& \coprod_\alpha S^{n-1} \times \{0\} \arrow[rr] \arrow[dl] \arrow[dd] & & A \times \{0\} \arrow[dl] \arrow[dd] \\
\coprod_\alpha S^{n-1} \times [0,1] \arrow[rr] \arrow[dd] & & A \times [0,1] \arrow[dd] & \\
& \coprod_\alpha D^n \times \{0\} \arrow[rr] \arrow[dl] & & B \times \{0\} \arrow[dl] \\
\coprod_\alpha S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} \arrow[rr] \arrow[d] & & A \times [0,1] \cup_{A\times\{0\}} B \times \{0\} \arrow[d] & \\
\coprod_\alpha D^n \times [0,1] \arrow[rr] & & B \times [0,1] &
\end{tikzcd}$$

The back face of the upper cube is a pushout by assumption, and so are the left and right faces of that cube. By the pasting lemma for pushout squares, the front face of the cube is a pushout. Since crossing with $[0,1]$ preserves pushouts, the four outer corners at the front also form a pushout square, so another application of the pasting lemma implies that the square at the bottom is a pushout. Given a lifting problem for (B,A), this pushout fits now into a diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1}\times[0,1]\cup_{S^{n-1}\times\{0\}} D^n\times\{0\} \arrow[r] \arrow[d] & A\times[0,1]\cup_{A\times\{0\}} B\times\{0\} \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ \coprod_\alpha D^n\times[0,1] \arrow[r] & B\times[0,1] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

The lifting problem given by the outer four corners has a solution, so the universal property of the pushout yields a solution to the original lifting problem.

For an arbitrary relative CW-complex (B,A), we have $B\cong \operatorname{colim}_n B^{(n)}$ with $B^{(-1)}=A$. Since $[0,1]$ is compact and colimits commute with each other, we have an identification

$$\begin{tikzcd} \operatorname{colim}_n\big(A\times[0,1]\cup_{A\times\{0\}} B^{(n)}\times\{0\}\big) \arrow[r, "\cong"] \arrow[d] & A\times[0,1]\cup_{A\times\{0\}} B\times\{0\} \arrow[d] \\ \operatorname{colim}_n\big(B^{(n)}\times[0,1]\big) \arrow[r, "\cong"] & B\times[0,1] \end{tikzcd}$$

Therefore, we can start by solving the lifting problem

$$\begin{tikzcd} A\times[0,1]\cup_{A\times\{0\}} B^{(0)}\times\{0\} \arrow[r] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "p"] \\ B^{(0)}\times[0,1] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

which is possible since B arises from A by attaching 0-cells. A choice of solution induces a map

$$B^{(0)}\times[0,1]\cup_{B^{(0)}\times\{0\}} B^{(1)}\times\{0\}\to X,$$

and now we can proceed by induction find to compatible solutions on each n-skeleton. Passing to the colimit, we obtain the desired solution to the original lifting problem. □

5.2.6. **Corollary.** *Let (B,A) be a relative CW-complex and let $p\colon X\to Y$ be a Serre fibration. If the inclusion map $i\colon A\to B$ is a homotopy equivalence, then every lifting problem*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution, and any two such solutions are homotopic through a homotopy relative A which projects to the constant homotopy in Y.

Proof. Since i is a cofibration and a homotopy equivalence, it is a deformation retraction. Let $r\colon B\to A$ be a retraction and let $h\colon B\times[0,1]\to B$ be a homotopy $ir\simeq \mathrm{id}_B$. The map $\alpha r\colon B\to X$ satisfies $\alpha r i=\alpha$ and $p\alpha r=\beta i r\simeq\beta$ via βh. By Lemma 5.2.5, the lifting problem

$$\begin{tikzcd} A\times[0,1]\cup_{A\times\{0\}} B\times\{0\} \arrow[r, "c_\alpha\cup\alpha r"] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "p"] \\ B\times[0,1] \arrow[r, "\beta h"] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution $\gamma\colon B \times [0,1] \to X$. Then $\gamma_1\colon B \to X$ is the desired solution to the original lifting problem.

In the composition $A \times [0,1] \to A \times [0,1] \cup_{A\times\{0,1\}} B \times \{0,1\} \to B \times [0,1]$, the first map and the composite map are homotopy equivalences, so the second map is also a homotopy equivalence. Since it is also the inclusion of a subcomplex, the previous paragraph applies to the lifting problem

$$\begin{tikzcd} A \times [0,1] \cup_{A\times\{0\}} B \times \{0,1\} \arrow[r, "c_\alpha \cup \gamma_1 \cup \gamma_1'"] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "p"] \\ B \times [0,1] \arrow[r, "c_\beta"] \arrow[ur, dotted] & Y \end{tikzcd}$$

arising from two such lifts, which shows that $\gamma_1 \simeq \gamma_1'$ relative A as needed. □

5.2.7. **Proposition.** *Let $p\colon E \to B$ be a Serre fibration. Let $b \in B$ be a basepoint, and let $e \in p^{-1}(b) =: F$. Then there exist maps $\partial\colon \pi_{n+1}(B,b) \to \pi_n(F,e)$ such that the sequences*

$$\pi_{n+1}(F,e) \to \pi_{n+1}(E,e) \to \pi_{n+1}(B,b) \xrightarrow{\partial} \pi_n(F,e) \to \pi_n(E,e) \to \pi_n(B,b)$$

are exact for $n \geq 0$; for $n = 0$, exactness means the following:

1. *there exists an action of $\pi_1(B,b)$ on $\pi_0(F)$ such that $\partial([\alpha]) = [\alpha]\cdot[e]$;*
2. *two classes $[a]$ and $[b]$ in $\pi_0(F)$ map to the same element in $\pi_0(E)$ if and only if they lie in the same orbit under the $\pi_1(B,b)$-action;*
3. *an element $[a] \in \pi_0(E)$ maps to the class of the basepoint $[b] \in \pi_0(B)$ if and only if it lies in the image of the map $\pi_0(F) \to \pi_0(E)$.*

Proof. We begin by constructing the boundary maps ∂. Any pair of maps

$$\beta\colon (I^{n+1}, \partial I^{n+1}) \to (B,b) \quad \text{and} \quad \varphi\colon (I^n, \partial I^n) \to (F,e)$$

gives rise to the lifting problem

$$\begin{tikzcd} \partial I^n \times [0,1] \cup_{\partial I^n \times\{0\}} I^n \times \{0\} \arrow[r, "c_e \cup \varphi"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times [0,1] \arrow[r, "\beta"] \arrow[ur, dotted] & B \end{tikzcd}$$

Applying Corollary 5.2.6, a lift $\overline{\beta}$ exists, whose endpoint $\overline{\beta}_1\colon (I^n, \partial I^n) \to (F,e)$ is well-defined up to homotopy relative ∂I^n. To see that this is independent of the choices of β and φ, let $h\colon \beta \simeq \gamma$ and $k\colon \varphi \simeq \psi$ be homotopies relative ∂I^{n+1} and ∂I^n, respectively. Then we have a lifting problem

$$\begin{tikzcd} \partial I^n \times [0,1] \times [0,1] \cup_{\partial I^n \times\{0\}\times[0,1]} I^n \times \{0\} \times [0,1] \arrow[r, "c_e \cup k"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times [0,1] \times [0,1] \arrow[r, "h"] \arrow[ur, dotted] & B \end{tikzcd}$$

which admits a solution. The restriction of the resulting lift to $I^n \times \{1\} \times [0,1]$ provides a homotopy relative ∂I^n between $\overline{\beta}_1$ and $\overline{\gamma}_1$ Hence we obtain a well-defined element

$$[\beta]\cdot[\varphi] := [\overline{\beta}_1] \in \pi_n(F,e).$$

It also follows from Corollary 5.2.6 that this construction satisfies $([\beta']\cdot[\beta])\cdot[\varphi] = [\beta']\cdot([\beta]\cdot[\varphi])$, so this defines an action of $\pi_{n+1}(B,b)$ on $\pi_n(F,e)$.

By stacking elements in $\pi_{n+1}(B,b)$ using a coordinate in I^n instead of the last coordinate, we also find that $([\beta]\cdot[\beta'])\cdot[\varphi] = ([\beta]\cdot[\varphi])\cdot(\beta'\cdot[\varphi])$.

The boundary maps $\partial\colon \pi_{n+1}(B,b) \to \pi_n(F,e)$ are now defined by $\partial[\beta] := [\beta]\cdot[c_e]$, and the preceding consideration implies that ∂ is a homomorphism.

We need to check exactness. The low-dimensional cases are left as an exercise, so let $n \geq 1$. The composite $\pi_n(F,e) \to \pi_n(B,b)$ is clearly trivial. If $\lambda\colon (I^n, \partial I^n) \to (E,e)$ maps to the trivial element in $\pi_n(B,b)$, we choose a nullhomotopy h relative ∂I^n in B and solve the lifting problem

$$\begin{tikzcd} \partial I^n \times [0,1] \cup_{\partial I^n \times \{0\}} I^n \times \{0\} \arrow[r, "c_e \cup \lambda"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times [0,1] \arrow[r, "h"] \arrow[ur, dotted] & B \end{tikzcd}$$

and obtain a preimage of $[\lambda]$ in $\pi_n(F,e)$ by evaluating the lift at the endpoint.

For $[\lambda] \in \pi_{n+1}(E,e)$, then λ itself solves the lifting problem which defines the action of $p_*[\lambda]$ on the trivial element in $\pi_n(F,e)$. This proves that $\partial p_*[\lambda] = 1$. Suppose that the solution to the problem defining $\partial[\beta]$ ends at a map which is in F homotopic relative ∂I^n to the constant map at e. Then the concatenation of the lift with this homotopy lifts $\beta * c_b$ along p, which provides a preimage of $[\beta]$ under p_*.

By construction of $\partial[\beta]$, its image in $\pi_n(E,e)$ is trivial. If an element φ in $\pi_n(F,e)$ maps to the trivial element in $\pi_n(E,e)$, this corresponds to a homotopy relative ∂I^n from φ to the constant map at e. By construction of ∂, this means that the image of this homotopy in B yields a preimage of φ under the boundary map. □

5.2.8. **Corollary.** *Let $p\colon X \to Y$ be a covering map. Then*

$$p_*\colon \pi_k(X,x) \to \pi_k(Y,p(x))$$

is an isomorphism for $k \geq 2$.

5.2.9. **Corollary.** *For $k \geq 3$, the Hopf map $\eta\colon S^3 \to S^2$ induces isomorphisms*

$$\pi_k(S^3) \cong \pi_k(S^2).$$

Proof. Since $\mathbb{R}$ is the universal covering of S^1, the group $\pi_k(S^1)$ is trivial for $k \geq 2$. Hence the corollary follows from Proposition 5.2.7. □

We can obtain a long exact sequence associated to any map since maps can be replaced by Serre fibrations. The construction is dual to the one of the mapping cylinder.

5.2.10. **Construction.** Let $f\colon X \to Y$ be a map. Define

$$PY := \mathrm{Map}([0,1], Y)$$

and consider the pullback

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "{(q(f),p(f))}"'] & PY \arrow[d, "\mathrm{ev}"] \\ X \times Y \arrow[r, "f \times \mathrm{id}_Y"] & Y \times Y \end{tikzcd}$$

The map on $\mathrm{id}_X \times f\colon X \to X \times Y$ and the map $X \to PY$ sending x to the constant path at $f(x)$ induce a map $i(f)\colon X \to E_f$ such that $\mathrm{id}_X = q(f) \circ i(f)$. Since

$$E_f \times [0,1] \to E_f, \quad ((x,\gamma),t) \mapsto (x, \gamma(t \cdot -))$$

defines a homotopy $i(f) \circ q(f) \simeq \mathrm{id}_{E_f}$, these maps are mutually inverse homotopy equivalences.

Since $(p(f), q(f))$ is a Serre fibration by Lemma 5.2.4, the map $p(f)$ is a Serre fibration satisfying $p(f) \circ i = f$.

5.2.11. **Example.** Let Y be a space and let $y \in Y$. Replacing the map $y\colon * \to Y$ by a Serre fibration, one obtains the endpoint projection $P_yY \to Y$ from the pointed path space. In particular, we observe that $P_yY \simeq *$.

5.2.12. **Definition.** Let $f\colon X \to Y$ be a map and let $y \in Y$. Define the *homotopy fibre of f over y* as the pullback

$$\begin{tikzcd} \mathrm{hofib}_y(f) \arrow[r] \arrow[d] & P_yY \arrow[d, "\mathrm{ev}_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

5.2.13. **Remark.** By construction, the homotopy fibre of $f\colon X \to Y$ over $y \in Y$ has the following universal property: any map $a\colon A \to X$ together with a nullhomotopy $h\colon c_y \simeq fa$ induces a map $A \to \mathrm{hofib}_y(f)$ by interpreting the nullhomotopy h as a map $A \to P_yY$. This observation gives reason to define a *homotopy fibre sequence* as a sequence of composable maps $F \xrightarrow{i} X \xrightarrow{f} Y$ together with a nullhomotopy $c_y \simeq fi$ such that the induced map $F \to \mathrm{hofib}_y(Y)$ is a weak homotopy equivalence.

From Proposition 5.2.7, it follows that every homotopy fibre sequence induces a long exact sequence of homotopy groups.

There is a very useful analogue of Corollary 5.2.6 in which we require the Serre fibration to be a weak equivalence (instead of the inclusion of the relative CW-complex being a homotopy equivalence). To prepare for the proof, we formulate an easy lemma.

5.2.14. **Lemma.** *Let $\alpha\colon S^n \to X$ be a map and let $s \in S^n$. Then α represents the trivial element in $\pi_n(X, \alpha(s))$ if and only if there exists a commutative diagram*

$$\begin{tikzcd} S^n \arrow[r, "\alpha"] \arrow[d, "\mathrm{inc}"'] & X \\ D^{n+1} \arrow[ur] & \end{tikzcd}$$

Proof. If α represents the trivial element, there exists a pointed nullhomotopy $S^n \times [0,1] \to X$. Since $S^n \times [0,1]/S^n \times \{1\} \cong D^{n+1}$, the nullhomotopy induces a diagram of the required form.

Conversely, such a diagram induces a commutative diagram

$$\begin{tikzcd} \pi_n(S^n, s) \arrow[r, "\alpha_*"] \arrow[d, "\mathrm{inc}_*"'] & \pi_n(X, \alpha(s)) \\ \pi_n(D^{n+1}, s) \arrow[ur] & \end{tikzcd}$$

Note that $\alpha_*[\mathrm{id}_{S^n}] = [\alpha]$. Since D^{n+1} is convex, it deformation retracts onto any given basepoint, so $\pi_n(D^{n+1}, s) = 1$. □

5.2.15. **Proposition.** *Let $D \subseteq \mathbb{N}$ be a set of natural numbers. Let (B, A) be a relative CW-complex which contains only d-cells for $d \in D$. Suppose that $p\colon X \to Y$ is a Serre fibration which induces*

(1) *an injection $\pi_{d-1}(X, x) \to \pi_{d-1}(Y, p(x))$ for all $d \in D$ and all $x \in X$;*
(2) *a surjection $\pi_d(X, x) \to \pi_d(Y, p(x))$ for all $d \in D$ and all $x \in X$.*

Then every lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"] \arrow[ur, dotted] & Y \end{tikzcd}$$

has a solution.

Remark. By Proposition 5.2.7, the assumptions of Proposition 5.2.15 are equivalent to requiring that $\pi_d(p^{-1}(p(x)), x)$ is trivial for all $d \in D$.

Proof of Proposition 5.2.15. We begin by considering the relative CW-complex $(B, A) = (D^d, S^{d-1})$ with $d \in D$, which we may replace by the homeomorphic pair $(I^d, \partial I^d)$. After a choice of basepoint s, the map α represents an element in $\pi_{d-1}(X, \alpha(s))$. Then Lemma 5.2.14 implies that $p_*[\alpha] \in \pi_{d-1}(Y, p\alpha(s))$ is trivial. By assumption, $[\alpha] \in \pi_{d-1}(X, \alpha(s))$ is then also trivial, and we obtain a map $\overline{\alpha}\colon I^d \cong D^d \to X$ extending α.

We begin by considering the relative CW-complex $(B, A) = (D^d, S^{d-1})$ with $d \in D$. Pick a basepoint $s \in S^{d-1}$. Then Lemma 5.2.14 implies that $p_*[\alpha] \in \pi_{d-1}(Y, p\alpha(s))$ is trivial. By assumption, $[\alpha] \in \pi_{d-1}(X, \alpha(s))$ is then also trivial, and we obtain a map $\overline{\alpha}\colon D^d \to X$ extending α.

Then $\overline{\alpha}$ induces a map

$$S^d \cong D^d_+ \cup_{S^{d-1}} D^d_- \xrightarrow{p\overline{\alpha} \cup \beta} Y,$$

which defines an element $[p\overline{\alpha} \cup \beta] \in \pi_d(Y, p\alpha(s))$. By assumption, there exists a map $\gamma\colon (S^d, s) \to (X, \alpha(s))$ such that $p\gamma$ is pointed homotopic to $p\overline{\alpha} \cup \beta$.

We claim that we may assume without loss of generality that $\gamma|_{D^d_+} = \overline{\alpha}$. To see this, note that $\overline{\alpha}$ and $\gamma|_{D^d_+}$ are pointed homotopic through a homotopy h since their domain D^d_+ deformation retracts onto the basepoint s. Since $D^d_+ \to S^d$ is a cofibration, we can apply the homotopy extension property to solve the extension problem

$$\begin{array}{ccc} D^d_+ \times [0,1] \cup_{D^d \times \{0\}} S^d \times \{0\} & \xrightarrow{h \cup \gamma} & X \\ \downarrow & \nearrow & \\ S^d \times [0,1] & & \end{array}$$

The endpoint of this pointed homotopy is a map $(S^d, s) \to (X, \alpha(s))$ which represents the element $[\gamma] \in \pi_d(X, \alpha(s))$ and is given by $\overline{\alpha}$ on D^d_+.

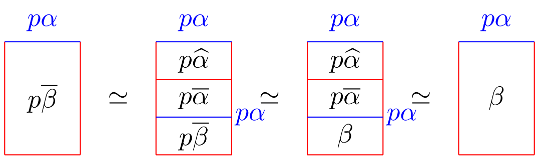
Set $\overline{\beta} := \gamma|_{D^d_-}$. Then $\overline{\beta}$ extends α, and we claim that $p\overline{\beta}$ is homotopic to β relative S^{d-1}. For this, it is convenient to regard $\overline{\beta}$ as map

$$\overline{\beta}\colon (I^d, \partial I^{d-1} \times [0,1] \cup_{\partial I^{d-1} \times \{1\}} I^{d-1} \times \{1\}) \to (X, \alpha(s))$$

so α corresponds to the restriction of this map to $I^{d-1} \times \{0\}$. The same reasoning applies to $\overline{\alpha}$ and β. Letting $\widehat{\alpha}$ denote $\overline{\alpha}$ composed with the inversion map on the last coordinate, we obtain a chain of homotopies relative ∂I^d

$$p\overline{\beta} \simeq p\widehat{\alpha} * p\overline{\alpha} * p\overline{\beta} \simeq p\widehat{\alpha} * p\overline{\alpha} * \beta \simeq \beta,$$

where the second homotopy witnesses that $p\gamma = p\overline{\alpha} * p\overline{\beta}$ is pointed homotopic to $p\overline{\alpha} \cup \beta$. Pictorially, this chain of homotopies looks as follows, red lines indicating points that map to the basepoint of X:



Passing to the quotients again, this corresponds precisely to the required homotopy $H\colon p\overline{\beta} \simeq \beta$ relative S^{d-1}.

To finish the proof, solve the lifting problem

$$\begin{tikzcd} S^{d-1}\times[0,1]\cup_{S^{d-1}\times\{0\}} D^d\times\{0\} \arrow[r, "c_\alpha\cup\overline{\beta}"] \arrow[d] & X \arrow[d, "p"] \\ D^d\times[0,1] \arrow[r, "H"] \arrow[ur, dotted] & Y \end{tikzcd}$$

and observe that the endpoint of the lift solves the original lifting problem.

Knowing this, it is immediate that the proposition holds for CW-pairs of the form $(\bigsqcup_\alpha D^d, \bigsqcup_\alpha S^{d-1})$ with $d \in D$. As in the proof of Lemma 5.2.5, this implies the proposition whenever B arises from A by attaching d-cells, and the general statement follows by induction along the skeletal filtration of B. □

5.2.16. **Corollary.** *Let $D \subseteq \mathbb{N}$ be a set of natural numbers. Let (B, A) be a relative CW-complex which contains only d-cells for $d \in D$. Suppose that $f\colon X \to Y$ is a map which induces*

(1) *an injection $\pi_{d-1}(X, x) \to \pi_{d-1}(Y, p(x))$ for all $d \in D$ and all $x \in X$;*
(2) *a surjection $\pi_d(X, x) \to \pi_d(Y, p(x))$ for all $d \in D$ and all $x \in X$.*

Then for every commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"] & Y \end{tikzcd}$$

there exists a map $\overline{\beta}\colon B \to X$ such that $\overline{\beta} i = \alpha$ and $p\overline{\beta} \simeq \beta$ relative to A.

Proof. Using Construction 5.2.10, the map f can be factored into a homotopy equivalence $j\colon X \to E_f$ followed by a Serre fibration $p\colon E_f \to Y$, and X is even a deformation retract of E_f. Since j is in particular a weak homotopy equivalence, it follows that p is a Serre fibration and has the same properties as f. By Proposition 5.2.15, the lifting problem

$$\begin{tikzcd} A \arrow[r, "j\alpha"] \arrow[d, "i"'] & E_f \arrow[d, "p(f)"] \\ B \arrow[r, "\beta"] \arrow[ur, dotted, "\gamma"] & Y \end{tikzcd}$$

has a solution. Letting $j^{-1}\colon E_f \to X$ be a homotopy inverse retraction to j, the map $\overline{\beta} := j^{-1}\gamma$ satisfies $\overline{\beta} i = j^{-1} j\alpha = \alpha$ and

$$p\overline{\beta} = pj^{-1}\gamma = p(f)jj^{-1}\gamma \simeq p(f)\gamma = \beta$$

as desired. □

5.2.17. **Definition.**

(1) A space X is d-connected if $\pi_n(X, x)$ is trivial for all $d \leq n$ and all $x \in X$.
(2) A map $f\colon X \to Y$ is *d-connected* if $\mathrm{hofib}_y(f)$ is $(d-1)$-connected for all $y \in Y$.

Remark. Equivalently, a map f is d-connected if $f_*\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection for all $n < d$ and an epimorphism for $n = d$. This follows directly from the long exact sequence for homotopy groups.

5.2.18. **Theorem** (Whitehead)**.** *Let $f\colon X \to Y$ be a d-connected map and let A be a CW-complex. Denote by*

$$f_*\colon [A, X] \to [A, Y]$$

the map on the sets of homotopy classes of maps given by postcomposition with f.

(1) *If the dimenson of A is $\leq d$, then f_* is surjective.*
(2) *If the dimenson of A is $< d$, then f_* is bijective.*

Proof. If the dimension of A is at most d, apply Corollary 5.2.16 to

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "g"] & Y \end{tikzcd}$$

to obtain a preimage of $[g]$ under f_*.

Assume that the dimension of A is $< d$. Then injectivity follows by applying Corollary 5.2.16 to the commutative square

$$\begin{tikzcd} A \times \{0,1\} \arrow[r, "ig_0 \sqcup ig_1"] \arrow[d] & X \arrow[d, "f"] \\ A \times [0,1] \arrow[r, "h"] \arrow[ur, dotted] & Y \end{tikzcd}$$

where h is a homotopy $pg_0 \simeq pg_1$; this works because $A \times [0,1]$ has dimension $\leq d$. □

5.2.19. **Corollary.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proof. Let $f\colon X \to Y$ be such a weak homotopy equivalence. By Theorem 5.2.18, there exists a map $g\colon Y \to X$ such that $fg \simeq \mathrm{id}$. By inspection, g is necessarily also a weak homotopy equivalence. Hence Theorem 5.2.18 also implies that there exists $f'\colon X \to Y$ with $gf' \simeq \mathrm{id}$. Now use the two-out-of-six property for homotopy equivalences or observe directly that

$$gf \simeq gfgf' \simeq gf' \simeq \mathrm{id}$$

to see that f is a homotopy equivalence. □

5.3. **The cellular approximation theorem.** Something that gives us some control over the homotopy groups of spaces is that there exist connectivity estimates for cell attachments. The key statements is the following lemma.

5.3.1. **Lemma.** *Let A be a topological space and suppose that X arises from A by attaching n-cells. Then the inclusion map $A \to X$ is $(n-1)$-connected.*

In particular, $\pi_k(S^n)$ is trivial for $k < n$.

Proof. Note that the statement about $\pi_k(S^n)$ is the special case in which $A = *$ and we attach a single n-cell.

The assertion that $A \to X$ is $(n-1)$-connected is equivalent to showing that for every diagram

$$\begin{tikzcd} \partial I^k \arrow[r, "f"] \arrow[d, "j"'] & A \arrow[d, "i"] \\ I^k \arrow[r, "g"] & X \end{tikzcd}$$

with $k < n$, there exists a map $\overline{g}\colon I^k \to A$ such that $\overline{g}j = g$ and $i\overline{g} \simeq g$ relative ∂I^k: by taking f to be the constant map at a basepoint of A, we find that $\pi_k(A) \to \pi_k(X)$ is surjective. If $f\colon S^{k-1} \cong \partial I^k$ represents an element in $\pi_k(A)$ which becomes trivial in $\pi_k(X)$, then Lemma 5.2.14 provides such a diagram, and the assumption together with Lemma 5.2.14 implies that f is trivial in $\pi_k(A)$. The converse follows from Corollary 5.2.16

We prove the lemma by induction on n. For $n = 1$, we have $k = 0$, and the statement is obvious.

Assume that the lemma holds for n and pick a pushout

$$\begin{tikzcd} \coprod_\alpha \partial I^{n+1} \arrow[r, "\partial\chi"] \arrow[d] & A \arrow[d] \\ \coprod_\alpha I^{n+1} \arrow[r, "\chi"] & X \end{tikzcd}$$

Let C be an open collar around A, and choose for each α an open cube $U_\alpha \subseteq I^{n+1}$ such that the open subsets $U_\alpha := \chi(U'_\alpha) \subseteq \chi_\alpha(I^{n+1} \setminus \partial I^{n+1})$ give rise to an open cover $\mathcal{U} := \{C, U_\alpha\}_\alpha$ of X. Consider the open cover $g^{-1}\mathcal{U}$ of I^k. For an appropriate subdivision $0 = t_0 < \ldots < t_r = 1$ of the interval, each cube $\prod_{i=1}^k [t_{s_i}, t_{s_i+1}]$ is contained in some element of $g^{-1}\mathcal{U}$.

Since A is a deformation retract of C, it suffices to find a homotopy relative ∂I^k between g and a map sending I^k to C. We construct such a homotopy by a sub-induction on the CW-structure of I^k which is given by our choice of subdivision. Concretely, consider maps $\sigma\colon \{1, \ldots, k\} \to (\{1, \ldots, r-1\} \times \{0, 1\}) \cup \{(0, 1)\}$ and define for each such map

$$I^k_\sigma := \prod_{i=1}^k \left[t_{\sigma_1(i)}, t_{\sigma_1(i)+\sigma_2(i)}\right],$$

where σ_1 and σ_2 denote the first and second component of σ. Letting

$$l(\sigma) := \#\sigma^{-1}(\{0, \ldots, r-1\} \times \{1\}),$$

we observe that I^k_σ is homeomorphic to the $l(\sigma)$-cube $I^{l(\sigma)}$.

The 0-cells in this CW-structure correspond to the cubes I^k_σ with $l(\sigma) = 0$. If I^k_σ gets mapped to C, choose the constant homotopy on this point. Otherwise, this point is an interior point of I^k and maps to a unique $(n+1)$-cell in X. Since $n + 1 \geq 1$, we can connect the image of this point under g with a point in C by some path. This defines a homotopy on the union of ∂I^k and the 0-skeleton of I^k. Use the homotopy extension property to find a homotopy $I^k \times [0, 1] \to X$ relative ∂I^k whose end point maps the entire 0-skeleton to C.

By induction, assume that g maps the l-skeleton of I^k to C for some $l < k$. Then we choose the constant homotopy on ∂I^k and consider all cubes I^k_σ with $l(\sigma) = l+1$. If such a cube is mapped to C, we also pick the constant homotopy on this cube. Otherwise, the choice of the open cover $\{C, U_\alpha\}_\alpha$ forces I^k_σ to map to a unique open set U_α. By assumption for the sub-induction, the boundary ∂I^k_σ gets mapped to C, so $g|_{\partial I^k_\sigma}$ defines an element in $\pi_l(C \cap U_\alpha) \cong \pi_l(S^n)$ (with respect to some choice of basepoint). The inductive assumption implies that this group is trivial, so $g|_{\partial I^k_\sigma}$ extends to a map $I^k_\sigma \to C \cap U_\alpha$. Since U_α is the homeomorphic image of a convex subset of $\mathbb{R}^k$, there exists a linear homotopy between these maps; as they agree on the boundary, this homotopy is relative to ∂I^k_σ. Now use the homotopy extension property to obtain a homotopy on the entirety of I^k. □

5.3.2. **Corollary.**

$$\pi_2(S^2) \cong \mathbb{Z}$$

Proof. Since S^3 is 2-connected by Lemma 5.3.1, the long exact sequence of homotopy groups associated to the Hopf bundle $\eta\colon S^3 \to S^2$ implies that

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$ □

5.3.3. **Corollary.** *For every CW-complex X, the inclusion $X^{(n)} \to X$ is n-connected.*

Proof. Since $X^{(k)}$ arises from $X^{(k-1)}$ by attaching k-cells, this is immediate from Lemma 5.3.1. □

5.3.4. **Corollary.** *Let I be a finite set and let $n \geq 1$. Then the inclusion map*

$$\bigvee_I S^n \to \prod_I S^n$$

is $(2n-1)$-connected.

Proof. The standard CW-structure on S^n induces a CW-structure on the product $\prod_I S^n$ whose $(2n-1)$-skeleton is $\bigvee_I S^n$. Hence the corollary follows from Corollary 5.3.3. ☐

The cellular approximation theorem states that every map between CW-complexes is homotopic to a cellular map. The hard work towards proving this theorem has already been done. Before we give the proof, we recall the following lemma, which also features in the proof that the inclusion of a subcomplex into a CW-complex is a cofibration.

5.3.5. **Lemma.** *Let (X, A) be a relative CW-complex and suppose that*

$$\{h_n \colon X \times [0,1] \to Y\}_{n \geq 1}$$

is a sequence of homotopies on X with the following properties:

(1) $h_n(-,1) = h_{n+1}(-,0)$*;*
(2) *there exists a sequence $(k_n)_n$ of natural numbers such that $\lim_n k_n = \infty$ and h_n is a homotopy relative the k_n-skeleton of X.*

Then there exists a homotopy $h \colon X \times [0,1] \to Y$ relative A such that $h(-,0) = h_1(-,0)$ and $h(x,1) = \lim_{n\to\infty} h_n(x,1)$.

Proof. Define $h \colon X \times [0,1] \to Y$ by

$$h(x,t) := h_n\left(x, n(n+1)\left(t - 1 + \frac{1}{n}\right)\right)$$

if $t \in [1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$. This is well-defined as a map of sets since the end point of h_n is the start point of h_{n+1}. To check continuity, it is enough to show that h restricts to a continuous map $X^{(k)} \times [0,1] \to Y$ on each skeleton. For each k, the assumptions imply that there exists some N such that h_n is a homotopy relative $X^{(k)}$ for $n \geq N$. Hence h is given by the concatenation of finitely many homotopies and a constant homotopy on $X^{(k)}$, and this is obviously continuous. Since each h_n is a homotopy relative A, the same is true for h. ☐

5.3.6. **Theorem** (Cellular approximation theorem)**.** *Let (B, A) and (Y, X) be relative CW-complexes. Suppose that $f \colon (B, A) \to (Y, X)$ is a map. Then there exists a cellular map $g \colon B \to Y$ such that g is homotopic to f relative A.*

Proof. By Lemma 5.3.5, it is enough to show the following: assuming that A and X are the n-skeleta of B and Y, respectively, $n \geq -1$, and that $f|_A \colon A \to B$ is cellular, there exists a homotopy relative A from f to a map $g \colon B \to Y$ such that g maps the $(n+1)$-skeleton of B to the $(n+1)$-skeleton of Y.

The inclusion map $Y^{(n+1)} \to Y$ is $(n+1)$-connected by Corollary 5.3.3. Since $B^{(n+1)}$ arises from A by attaching $(n+1)$-cells, Corollary 5.2.16 implies that $f|_{B^{(n+1)}}$ is homotopic relative A to map $g' \colon B^{(n+1)} \to Y^{(n+1)}$. As $B^{(n+1)} \to B$ is a cofibration, this homotopy extends to a homotopy $B \times [0,1] \to Y$ relative A whose end point restricts to g'. This is all we have to show. ☐

5.3.7. **Remark.** One consequence of Theorem 5.3.6 is the fact that every topological space Z admits a weak homotopy equivalence $X \to Z$ from a CW-complex. See the lecture notes for Topology I for a proof.

This gives a good excuse to restrict one's attention to CW-complexes if one wishes to avoid point-set pathologies.

5.4. **The homotopy excision theorem.** One of the key features which allows for calculations in singular (co)homology is the excision isomorphism. A prominent incarnation of of this isomorphism appears in the case that we express a CW-complex X as a union of two subcomplexes A and B, in which case the excision isomorphism on relative homology groups gives rise to the long exact Mayer–Vietoris sequence relating the homologies of $A \cap B$, A, B and X. It appears natural to ask whether a similar statement holds for homotopy groups, and the following theorem tells us to which degree the analogous statement is true.

5.4.1. **Theorem** (Blakers–Massey; Homotopy excision theorem)**.** *Let X be a space and let A and B be open subsets of X with non-empty intersection $A \cap B$. Consider the commutative diagram*

$$\begin{tikzcd} A \cap B \arrow[r, "i^A"] \arrow[d, "i^B"'] & A \arrow[d, "j^A"] \\ B \arrow[r, "j^B"] & X \end{tikzcd}$$

Let $p, q \geq 1$ be natural numbers. Suppose that

(1) *the inclusion map i^A is p-connected;*
(2) *the inclusion map i^B is q-connected.*

Then the comparison map $\mathrm{hofib}_a(i^A) \to \mathrm{hofib}_a(j^B)$ *is* $(p+q-1)$*-connected for every* $a \in A$.

Most of the techniques that go into the proof of Theorem 5.4.1 have already appeared in earlier arguments. The main additional ingredient is the following technical lemma.

5.4.2. **Lemma.** *Let $g_0 \colon I^l \to Z$ and let $Y \subseteq Z$. Define*

$$S_p^k := \{x \in I^l \mid x_i < \frac{1}{2} \textit{ for at least } p \textit{ components of } x\}$$

and

$$B_p^k := \{x \in I^l \mid x_i > \frac{1}{2} \textit{ for at least } p \textit{ components of } x\}$$

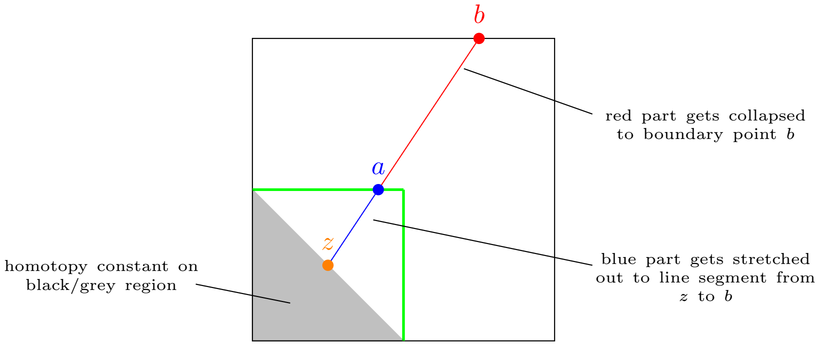
Suppose that $l \geq p$.

(1) *If the intersection of $g_0^{-1}(Y)$ with each face of I^l is contained in S_p^{l-1}, then there exists a homotopy relative ∂I^l to a map g_1 such that $g_1^{-1}(Y) \subseteq S_p^l$.*
(2) *If the intersection of $g_0^{-1}(Y)$ with each face of I^k is contained in B_p^{l-1}, then there exists a homotopy relative ∂I^l to a map g_1 such that $g_1^{-1}(Y) \subseteq B_p^l$.*

Proof. The second assertion follows from the first by flipping all coordinates, so we only have to show the first assertion.

Set $z := (\frac{1}{4}, \ldots, \frac{1}{4})$. Given a point $x \in I^l$, consider the ray $r(x)$ emanating from z through x. If $r(x)$ has no intersection with the complement of $[0, \frac{1}{2}]^l$, then the homotopy we wish to define is constant on all points of $r(x) \cap I^k$. Otherwise, let a be the intersection point of $r(x)$ and $\partial[0, \frac{1}{2}]^l$, and let b be the intersection point of $r(x)$ and ∂I^l. Then the homotopy we want to define collapses the line segment between a and b to the boundary point b, and stretches the line segment from z to a out onto the line segment from z to b. By performing these homotopies linearly along each ray, we obtain a homotopy $h \colon I^l \times [0,1] \to I^l$ relative ∂I^l from $h_0 = \mathrm{id}$

to some map h_1 which restricts to the identity on the boundary.



Then gh is a homotopy relative ∂I^l from g_0 to $g_1 := gh_1$. Consider $x \in I^l$ with $g_1(x) \in Y$. If $x \in S^l_l$, then there is nothing to show because $l \geq p$. Otherwise, there exists one component $x_i \geq \frac{1}{2}$. Then $h_1(x)$ lies by construction in ∂I^l, so $h_1(x) \in \partial I^l \cap g_0^{-1}(Y)$. By assumption, this means that at least p many components of $h_1(x)$ are smaller than $\frac{1}{2}$. Since $h_1(x)_j = \frac{1}{4} + \lambda(x_i - \frac{1}{4})$ for some $\lambda \geq 1$, we conclude that $x_j < \frac{1}{2}$ whenever $h_1(x)_j < \frac{1}{2}$. Hence $x \in S^l_p$. □

Proof of Theorem 5.4.1. Applying Construction 5.2.10 to both i^A and j^B, we obtain the commutative square

$$\begin{tikzcd} E_{i^A} \arrow[r, "p(i^A)"] \arrow[d, "i^B"'] & A \arrow[d, "j^A"] \\ E_{j^B} \arrow[r, "p(j^B)"] & X \end{tikzcd}$$

Note that, up to a reversal of the time coordinate, the fibres of $p(i^A)$ and $p(j^B)$ are the homotopy fibres of i^A and j^B, respectively. By the long exact sequence of homotopy groups associated to a Serre fibration (Proposition 5.2.7), the theorem will follow if we can show that the induced map

$$c\colon E_{i^A} \to E_{j^B} \times_X A$$

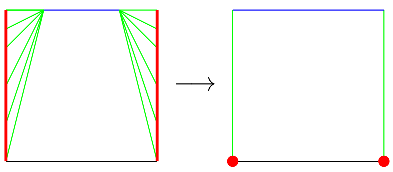
is $(p+q-1)$-connected.

As we saw at the beginning of the proof of Lemma 5.3.1, this assertion is equivalent to showing that for each commutative square

$$\begin{tikzcd} \partial I^k \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] & E_{i^A} \arrow[d, "c"] \\ I^k \arrow[r, "g"] & E_{j^B} \times_X A \end{tikzcd}$$

with $k < p+q$, there exists a map $\overline{g}\colon I^k \to E_{i^A}$ such that $\overline{g}j = f$ and $c\overline{g} \simeq g$ relative to ∂I^k.

We can actually get away with a little less. It suffices to homotope $(g,f)\colon (I^k, \partial I^k) \to (E_{j^B} \times_X A, E_{i^A})$, as a map of pairs, to some map taking values in E_{i^A}. Then we can precompose with an appropriate map $I^k \times [0,1] \to I^k \times [0,1]$ to obtain a homotopy relative ∂I^k to a map with values in E_{i^A}; the following picture indicates such a map:



It is worth noting that this works for any pair of spaces in place of $(E_{j^B} \times_X A, E_{i^A})$.

Since E_{i^A} is a subspace of $\mathrm{Map}([0,1],A)$ and $E_{j_B} \times_X A$ is a subspace of $\mathrm{Map}([0,1],X)$, we can adjoin the time coordinate to the other side and translate the above square to a commutative square

$$\begin{tikzcd} \partial I^k \times [0,1] \arrow[r, "f"] \arrow[d, "\mathrm{inc}"] & A \arrow[d, "j^A"] \\ I^k \times [0,1] \arrow[r, "g"] & X \end{tikzcd}$$

such that $g|_{I^k \times \{0\}}$ takes values in B, the restriction $f|_{\partial I^k \times \{0\}}$ takes values in $A \cap B$ and $g|_{I^k \times \{1\}}$ takes values in A. Let us call maps of pairs $(I^k \times [0,1], \partial I^k \times [0,1]) \to (X, A)$ with these properties *good*. Our goal is now to find a homotopy h through good maps from g to a map $\overline{g}$ such that $\widetilde{g}$ maps to A: such a homotopy translates precisely to a homotopy through maps of pairs $(I^k, \partial I^k) \to (E_{j^B} \times_X A, E_{i^A})$.

Main claim: There exists a homotopy $g \simeq \widetilde{g}$ through good maps such that the images of $\widetilde{g}^{-1}(X \setminus A)$ and $\widetilde{g}^{-1}(X \setminus B)$ under the projection $\mathrm{pr}\colon I^k \times [0,1] \to I^k$ are disjoint.

Assuming the claim, we can finish the proof as follows. Note that ∂I^k is contained in $\mathrm{pr}\,\widetilde{g}^{-1}(A)$, so it is disjoint from $\mathrm{pr}\,\widetilde{g}^{-1}(X \setminus A)$. Choose a function $\sigma\colon I^k \to [0,1]$ such that $\sigma|_{\partial I^k \cup \mathrm{pr}\,\widetilde{g}^{-1}(X\setminus B)} = 0$ and $\sigma|_{\mathrm{pr}\,\widetilde{g}^{-1}(X\setminus A)} = 1$; this is possible since these are disjoint closed subsets of I^k. Then

$$\begin{aligned} h\colon I^k \times [0,1] \times [0,1] &\to X \\ (x,t,s) &\mapsto \widetilde{g}(x, t + s(1-t)\sigma(x)) \end{aligned}$$

defines a homotopy through good maps from a map $\overline{g}$ to $\widetilde{g}$:

- for $x \in \partial I^k$, we have $h(x,t,s) = \widetilde{g}(x,t) \in A$ because $\sigma(x) = 1$;
- we have $h(x,0,s) = \widetilde{g}(x, s\sigma(x))$; if there exists any $u \in [0,1]$ such that $\widetilde{g}(x,u) \notin B$, then $\sigma(x) = 0$ and hence $h(x,0,s) = \widetilde{g}(x,0) \in B$; otherwise, $h(x,0,s) \in B$ as well;
- we have $h(x,1,s) = \widetilde{g}(x,t) \in A$;
- finally, $h(x,t,1) = \widetilde{g}(x, t + (1-t)\sigma(x))$; if there exists any $u \in [0,1]$ such that $\widetilde{g}(x,u) \notin A$, then $\sigma(x) = 1$ and thus $h(x,t,1) = \widetilde{g}(x,1) \in A$; otherwise, $h(x,t,1) \in A$ as well.

Therefore, it suffices to prove the claim.

Since (A,B) is an open cover of X, we can choose a subdivision $0 = t_0 < \ldots < t_r = 1$ of the interval such that each cube $\prod_{i=1}^{k+1} [t_{s_i}, t_{s_i+1}]$ maps to A or B under g. The homeomorphisms $[0,1] \xrightarrow{\cong} [t_i, t_{i+1}]$ sending s to $t_i + s(t_{i+1} - t_i)$ induces a homeomorphism between I^k and this cube which restricts to the analogous identification on each face of the cube. In particular, if C is such a cube or one of its faces with dimension l, then S^l_p and B^l_p correspond to subsets $S^l_p(C)$ and $B^l_p(C)$ of C.

We will establish the claim by showing that there is a good homotopy h between g and a map $\widetilde{g}$ such that the following holds for each face C of a cube in the subdivision:

(1) if $g(C) \subseteq A$ or $g(C) \subseteq B$, then $h(C \times [0,1]) \subseteq A$ or $h(C \times [0,1]) \subseteq B$, respectively;
(2) if $g(C) \subseteq A \cap B$, then h is constant on C;
(3) if $g(C) \subseteq A$, then $\widetilde{g}^{-1}(A \setminus A \cap B) \cap C \subseteq S^{\dim C}_{p+1}(C)$;
(4) if $g(C) \subseteq B$, then $\widetilde{g}^{-1}(B \setminus A \cap B) \cap C \subseteq B^{\dim C}_{q+1}(C)$;

Once again, we want to construct the required homotopy by an induction along the CW-structure of I^k which is given by this subdivision. Suppose by induction that g itself satisfies the conclusion of (3) and (4) for all cubes C of dimension $\leq l$ for some $l \geq -1$. Let C be a cube of dimension $l+1$. If $g(C) \subseteq A \cap B$, then we choose the constant homotopy on C. Assume $g(C) \subseteq A$. If $l \leq p$, then we use Corollary 5.2.16 to find a homotopy of $g|_C$ relative to ∂C within A to a map with image in $A \cap B$; this uses that $A \cap B \to A$ is p-connected. Then (3) holds for C. If $l \geq p+1$, we apply Lemma 5.4.2 with $Z = A$ and $Y = A \cap B$ to find a homotopy in A whose endpoint satisfies (3). The case $g(C) \subseteq B$ is handled analogously, using that $A \cap B \to B$ is q-connected.

Note that if $C \subseteq I^k \times \{0\}$, then g maps this cube to B, so the homotopy stays in B. Similarly for cubes $C \subseteq I^k \times \{1\}$, and for cubes in $\partial I^k \times \{0\}$, the chosen homotopy is even constant. This defines the required homotopy on the$(l+1)$-skeleton of $I^k \times [0,1]$. Using the homotopy extension property, we extend the homotopy cube by cube to the entirety of $I^k \times [0,1]$, making sure that the resulting homotopy is a homotopy through good maps. The endpoint of this homotopy is a good map which satisfies (3) and (4) for all cubes of dimension $\leq l+1$.

Let $\widetilde{g}$ be the final result of this homotopy and suppose that $x \in I^k$ lies in $\operatorname{pr} \widetilde{g}^{-1}(X \setminus A) \cap \operatorname{pr} \widetilde{g}^{-1}(X \setminus B)$. Then there exist t_A and t_B such that $\widetilde{g}(x, t_A) \in X \setminus A$ and $\widetilde{g}(x, t_B) \in X \setminus B$. Then (3) and (4) imply that with respect to a cube C_A containing (x, t_A), we have $(x, t_A) \in B_{p+1}^{\dim C_A}(C_A)$ ("at least $q+1$ coordinates are big"), and that for a cube C_B containing (x, t_B) we have $(x, t_B) \in S_{q+1}^{\dim C_B}(C_B)$. We can choose these cubes such that $C_A = C \times I_A$ and $C_B = C \times I_B$ for appropriate sub-intervals $I_A, I_B \subseteq [0,1]$. Hence x has at least p coordinates which are "small" in C and q coordinates which are "large" in C. But x has only $k < p+q$ coordinates. So there is no point x like this, which proves the main claim. $\square$

To avoid point-set problems in the following discussion, we will restrict our attention to CW-complexes.

5.4.3. **Definition.** Let X be a topological space. Define its suspension SX by the pushout

$$\begin{array}{ccc} X \times \{0,1\} & \longrightarrow & X \times [0,1] \\ \downarrow & & \downarrow \\ \{0,1\} & \longrightarrow & SX \end{array}$$

If X is pointed, then define the *reduced suspension* and *reduced cone* as the spaces given by

$$\Sigma X := SX/\{x\} \times [0,1] \quad \text{and} \quad CX := X \times [0,1]/X \times \{1\} \cup \{x\} \times [0,1].$$

5.4.4. **Remark.**

(1) Spelling out the universal property of the quotient space, a map $CX \to Y$ amounts to a map $X \to Y$ together with a pointed nullhomotopy.

(2) The reduced suspension is a left adjoint to the loop space functor Ω: the map

$$u\colon X \to \Omega\Sigma X, \quad x \mapsto [t \mapsto [x,t]]$$

induces a bijection $\operatorname{Hom}_{\mathrm{Top}_*}(\Sigma X, Y) \cong \operatorname{Hom}_{\mathrm{Top}_*}(X, \Omega Y)$ for all pointed spaces X and Y.

(3) ΣS^n is homeomorphic to S^{n+1} (for example, one can see this by inspecting the induced CW-structure on ΣS^n).

(4) If the inclusion of the basepoint is a cofibration, then the projection map $SX \to \Sigma X$ is a homotopy equivalence. This is easy to see if X is a CW-complex pointed by a 0-cell (since we collapse a contractible subcomplex of SX in this case). For the general case, see Corollary B.0.5.

5.4.5. **Theorem** (Freudenthal suspension theorem)**.** *Let (X, x) be a pointed space such that $\{x\} \to X$ is a cofibration. If X is p-connected, then the unit map $u_X : X \to \Omega\Sigma X$ is $(2p+1)$-connected.*

Proof. Let N and S denote the north pole and south pole of the unreduced suspension SX. Then $SX = (SX \setminus N) \cup (SX \setminus S)$ is an open cover of SX by contractible subsets, and

$$(SX \setminus N) \cup (SX \setminus S) \simeq X.$$

Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{hofib}_x(i) \arrow[r] \arrow[d] & SX \setminus \{N, S\} \arrow[r, "i"] \arrow[d] & SX \setminus N \arrow[d] \\
\mathrm{hofib}_x(j) \arrow[r] \arrow[d] & SX \setminus \{S\} \arrow[r, "j"] \arrow[d] & SX \arrow[d, "\mathrm{id}"] \\
\Omega_x SX \arrow[r] & * \arrow[r] & SX
\end{tikzcd}$$

in which all rows are homotopy fibre sequences, the bottom row using the non-trivial nullhomotopy

$$\Omega_x SX \times [0, 1] \to SX$$

which comes from evaluation. The composite map $X \to SX \setminus \{N\}$ is nullhomotopic through the contraction to the south pole, and the map $SX \setminus \{S\} \to SX$ is nullhomotopic through the contraction to the north pole. This induces the vertical maps in the left column.

Since $X \to SX \setminus \{N, S\}$ is a deformation retract, it follows that the maps $X \to \mathrm{hofib}_x(i)$ and $\mathrm{hofib}_x(j) \to \Omega_x SX$ are weak homotopy equivalences.

By assumption, the map $p : X \to *$ is $(p+1)$-connected. The homotopy excision theorem 5.4.1 therefore implies that the map

$$\mathrm{hofib}_x(i) \to \mathrm{hofib}_x(j)$$

is $(2p+1)$-connected, and therefore the composite map $X \to \Omega_x SX$ is also $(2p+1)$-connected. Tracing through the definitions, one finds that this composite is the canonical comparison map $X \to \Omega_x SX$. Composing with the homotopy equivalence $\Omega_x SX \to \Omega\Sigma X$ induced by the projection map, the corollary follows. □

5.4.6. **Construction.** Let $\alpha : (I^n, \partial I^n) \to (X, x)$ represent an element in $\pi_n(X, x)$. Then the map

$$s\alpha : I^{n+1} = I^n \times [0, 1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0, 1] \to \Sigma X$$

sends ∂I^{n+1} to the basepoint in ΣX, and therefore represents an element $\pi_{n+1}(\Sigma X, x)$. This defines the *suspension map*

$$s : \pi_n(X, x) \to \pi_{n+1}(\Sigma X, x).$$

5.4.7. **Lemma.** *The diagram*

$$\begin{tikzcd}
\pi_n(X, x) \arrow[r, "s"] \arrow[dr, "(u_X)_*"'] & \pi_{n+1}(\Sigma X, x) \arrow[d, "\partial", "\cong"'] \\
 & \pi_n(\Omega\Sigma X, x)
\end{tikzcd}$$

commutes, where ∂ is the boundary map of the Serre fibration $\mathrm{ev}_1 : P_x\Sigma X \to \Sigma X$. In particular, s is a homomorphism.

Proof. Let $\alpha\colon (I^n, \partial I^n) \to (X, x)$ represent an element in $\pi_n(X, x)$. The homotopy

$$I^n \times [0,1] \to P_x\Sigma X, \quad (t,s) \mapsto [u \mapsto [\alpha(t), su]]$$

solves the lifting problem

$$\begin{tikzcd} \partial I^n \times [0,1] \cup_{\partial I^n \times \{0\}} I^n \times \{0\} \arrow[r] \arrow[d] & P_x\Sigma X \arrow[d, "\mathrm{ev}_1"] \\ I^n \times [0,1] \arrow[r, "s\alpha"] & \Sigma X \end{tikzcd}$$

which defines $\partial s[\alpha]$, and the endpoint of this homotopy is given by $u_X \circ \alpha$.

The boundary map ∂ is an isomorphism since $P_x X$ is contractible. $\square$

5.4.8. **Corollary.**

(1) *Let $n \geq 1$. Then the suspension map $s\colon \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ is an isomorphism for $k < 2n-1$ and an epimorphism for $k = 2n-1$.*

(2) *Using a fixed identification $I^n/\partial I^n \cong S^n$, the induced map*

$$\deg\colon \pi_n(S^n) \to \mathrm{Hom}(H_n(S^n), H_n(S^n)) \cong \mathbb{Z}, \quad [\alpha] \mapsto \alpha_*$$

is an isomorphism.

(3) *$\pi_3(S^2) \cong \mathbb{Z}$, and the class of $\eta\colon S^3 \to S^2$ is a generator.*

Proof. Lemma 5.4.7 identifies the suspension map with the map

$$(u_{S^n})_*\colon \pi_k(S^n) \to \pi_k(\Omega\Sigma S^n),$$

so the first assertion follows from the Freudenthal suspension theorem 5.4.5.

We have seen already in Corollary 5.3.2 that $\pi_2(S^2) \cong \mathbb{Z}$. By the first part, the suspension map $s\colon \pi_1(S^1) \to \pi_2(S^2)$ is an epimorphism. Since this is an epimorphism from an infinite cyclic group onto an infinite cyclic group, it is an isomorphism. For $n \geq 2$, the first assertion says directly that $s\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism.

Applying Corollary 5.2.9, we see that η (more precisely, $\eta \circ c$, where $c\colon I^3 \to I^3/\partial^3 \cong S^3$) generates $\pi_3(S^2)$. $\square$

5.4.9. **Remark.** The connectivity estimate in the Freudenthal suspension theorem (and hence in the homotopy excision theorem) is optimal. The theorem implies that $s\colon \pi_3(S^2) \to \pi_4(S^3)$ is surjective. Using Corollary 5.4.8, we find that $\pi_4(S^3)$ is cyclic and generated by $s[\eta] = [\Sigma\eta]$. Considering $S^3 \subseteq \mathbb{C}^2$, the map η is given by $S^3 \to \mathbb{C}\mathrm{P}^1 \cong S^2$, $(z_0, z_1) \mapsto [z_0 : z_1]$. Hence η is equivariant with respect to the C_2-action given by complex conjugation on both domain and codomain. The conjugation map $c\colon S^2 \to S^2$ corresponds to reflection along the equator, so it has degree -1. On S^3, complex conjugation corresponds to reflection along two hyperplanes, and thus has degree 1. Since we have just seen that the mapping degree classifies elements in $\pi_3(S^3)$, we find that $\eta \simeq c \circ \eta$. After suspending, the degree -1 map Σc is homotopic to the reflection in the suspension coordinate, and this map evidently commutes with $\Sigma\eta$. Hence $2[\Sigma\eta] = 0 \in \pi_4(S^3)$.

This means that $\pi_4(S^3)$ is either trivial or cyclic of order two. It turns out that $\pi_4(S^3) \cong \mathbb{Z}/2$, but showing this requires more sophisticated tools.

5.4.10. **Remark.** Let us finish with an outlook on some consequences of the homotopy excision theorem. As we have seen in Remark 5.4.9, the connectivity estimates in the excision theorem are optimal in general. If we are really insistent in our desire to build some sort of "homology theory", we could try to artifically create a situation in which everything becomes arbitrarily highly connected. Since we do not want to consider only weakly contractible senses, some explanation is needed what we mean by this.

For simplicity, let us consider only CW-complexes X which are pointed by a 0-cell x in the following discussion. Note that ΣX is always connected because every point admits a path to one of the cone points of the suspension (which happen to become identified in ΣX). If X is path-connected, then the Seifert–van Kampen theorem tells us that ΣX is simply-connected. For an n-connected space X with $n \geq 1$, the Freudenthal suspension theorem (together with Lemma 5.4.7) implies that ΣX is $(n+1)$-connected. This may motivate us to define the *k-th stable homotopy group* of (X, x) as

$$\pi_k^{\mathrm{st}}(X,x) := \operatorname{colim}\left(\pi_k(X,x) \xrightarrow{s} \pi_{k+1}(\Sigma X, x) \xrightarrow{s} \pi_{k+2}(\Sigma^2 X, x) \xrightarrow{s} \dots\right).$$

Note that this colimit system is eventually constant for each X and k, again by the Freudenthal suspension theorem.

Then π_k^{st} is evidently a homotopy invariant functor with values in the category of abelian groups on the category of pointed CW-complexes. We can extend this to a functor on pairs (X, A) of CW-complexes by setting $\pi_k^{\mathrm{st}}(X, A) := \pi_k^{\mathrm{st}}(X/A)$. Since $\Sigma^{n+1}X$ and $\Sigma^{n+1}A$ are n-connected, the homotopy excision theorem can be applied to the square

$$\begin{tikzcd} \Sigma^{n+1}A \arrow[r] \arrow[d] & \Sigma^{n+1}X \arrow[d] \\ * \arrow[r] & \Sigma^{n+1}(X/A) \end{tikzcd}$$

to see that the induced map on horizontal homotopy fibres is $2n$-connected (at least for $n \geq 2$). Hence the sequence $\Sigma^{n+1}A \to \Sigma^{n+1}X \to X/A$ induces an exact sequence of homotopy groups in a range of degrees, roughly up to degree $2n$. As we are taking the colmit over all n, we find that there is an induced natural exact sequence

$$\pi_{k+1}^{\mathrm{st}}(A) \to \pi_{k+1}^{\mathrm{st}}(X) \to \pi_{k+1}^{\mathrm{st}}(X/A) \xrightarrow{\partial} \pi_k^{\mathrm{st}}(A) \to \pi_k^{\mathrm{st}}(X) \to \pi_{k+1}^{\mathrm{st}}(X/A)$$

for all k. If we want to compare this to the properties of singular homology, this sequence in some sense combines the existence of the long exact sequence of a pair with the existence of an excision isomorphism. For example, from the existence of these sequences one obtains directly that for a CW-complex X covered by subcomplexes A and B, there exists a Mayer–Vietoris type sequence comparing the stable homotopy of $A \cap B$, A, B and X (since $X/B \cong A/A \cap B$).

Moreover, if $\bigvee_{i=1}^r X_i$ is a sum of pointed CW-complexes, then ...

In this sense, the collection of functors $\{\pi_k^{\mathrm{st}}\}_k$ defines a "homology theory" on pointed CW-complexes. Note that singular homology also induces such a functor by sending a pointed CW-complex X to $H_k(X, x)$.

It is a general observation that the values of such homology theories are mainly governed by their evaluation on S^0. More precisely, assume that $\{h_k\}_k$ and $\{h'_k\}_k$ are sequences of functors on pointed CW-complexes with these properties (plus the additional data of the boundary maps required for the long exact sequences) and that $\{\tau_k \colon h_k \to h'_k\}_k$ is a sequence of natural transformations which induce transformations of long exact sequences for each pair (X, A). If $\tau_k(S^0)$ is an isomorphism for all k, then τ_k is an isomorphism for every finite CW-complex X as seen by the following argument.

By homotopy invariance, $\tau_k(D_+^n)$ is an isomorphism for all k and n. Then one argues by induction on the dimension of X, using that an $(n+1)$-dimensional pointed

CW-complex X can be written as a pushout the pushout of pointed CW-complexes

$$\begin{tikzcd}
\bigvee_\alpha S^n_+ \arrow[r] \arrow[d] & X^{(n)} \arrow[d] \\
\bigvee_\alpha D^{n+1}_+ \arrow[r] & X
\end{tikzcd}$$

and applying the inductive hypothesis to the Mayer–Vietoris sequences for $\{h_k\}_k$ and $\{h'_k\}_k$ associated to this pushout.

If the functors h_k and h'_k additionally send directed unions to directed colimits, this statement implies that τ_k is an isomorphism on all pointed CW-complexes since every CW-complex is the directed colimit over its finite subcomplexes.

The claim from Remark 5.4.9 that $\pi_4(S^3) \cong \mathbb{Z}/2$ implies that $\pi_1^{\mathrm{st}}(S^0) \cong \mathbb{Z}/2$ (by the Freudenthal suspension theorem). This shows already that stable homotopy is genuinely different from singular homology, but much more is true:

(1) $\pi_k^{\mathrm{st}}(S^0)$ is finite for $k \geq 1$;
(2) the p-torsion part of $\pi_{2p-3}^{\mathrm{st}}(S^0)$ is isomorphic to $\mathbb{Z}/p$; in particular, $\pi_k^{\mathrm{st}}(S^0)$ is non-trivial in infinitely many degrees, and there exists p-torsion in the graded abelian group $\pi^{\mathrm{st}}(S^0)$ for every p;
(3) conjecturally(!), $\pi_k^{\mathrm{st}}(S^0) = 0$ for only finitely many k;
(4) there is no general description of what $\pi_k^{\mathrm{st}}(S^0)$ looks like; for a first idea of the existing knowledge on these groups, one can start with an internet search for the keyword “stable homotopy groups of spheres”.

Appendix A. More long exact sequences

In our discussion of orientations and the Poincaré duality theorem, we had to rely on a Mayer–Vietoris type long exact sequence which is potentially not covered as part of a standard discussion. As preparation, let us prove that there exists a natural long exact sequence associated to every triple of spaces $A \subseteq B \subseteq X$.

A.0.1. **Lemma.** *Let X be a topological space and let $A \subseteq B \subseteq X$. Then the sequences*

$$H_{n+1}(X,B) \xrightarrow{\partial} H_n(B,A) \to H_n(X,A) \to H_n(X,B) \xrightarrow{\partial} H_{n-1}(B,A)$$

are exact for every n, where the unlabelled arrows are induced by inclusions and the boundary map is defined to be the composition

$$H_{n+1}(X,B) \xrightarrow{\partial} H_n(B) \to H_n(B,A),$$

with ∂ being the boundary map of the pair (X,B).

Proof. The long exact sequences associated to the pairs (X,B), (X,A) and (B,A) can be combined into the commutative braid

$$\begin{tikzcd}[column sep=small]
H_{n+1}(X,B) \arrow[rr, bend left] \arrow[dr, "\partial"] & & H_n(B,A) \arrow[rr, bend left, "\partial"] \arrow[dr] & & H_{n-1}(A) \arrow[rr, bend left] \arrow[dr] & & H_{n-1}(X) \\
& H_n(B) \arrow[ur] \arrow[dr] & & H_n(X,A) \arrow[ur, "\partial"] \arrow[dr] & & H_{n-1}(B) \arrow[ur] \arrow[dr] & \\
H_n(A) \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,B) \arrow[ur, "\partial"] \arrow[rr, bend right] & & H_{n-1}(B,A)
\end{tikzcd}$$

The claimed exact sequence is also a part of this braid, and proving exactness is a matter of diagram chasing. □

A.0.2. **Lemma.** *Let X be a Hausdorff space and let K and L be compact subsets of X. Let*

$$\begin{aligned} i^K &\colon (X, X \setminus K \cup L) \to (X, X \setminus K) \\ i^L &\colon (X, X \setminus K \cup L) \to (X, X \setminus L) \\ j^K &\colon (X, X \setminus K) \to (X, X \setminus K \cap L) \\ j^L &\colon (X, X \setminus L) \to (X, X \setminus K \cap L) \end{aligned}$$

denote the respective inclusion maps. Then

$$\begin{aligned} H_{n+1}(X|K \cap L) \xrightarrow{\partial} H_n(X|K \cup L) &\xrightarrow{(i_*^K, i_*^L)} H_n(X|K) \oplus H_n(X|L) \\ &\xrightarrow{j_*^K - j_*^L} H_n(X|K \cap L) \xrightarrow{\partial} H_{n-1}(X|K \cup L) \end{aligned}$$

is exact, where ∂ is defined to be the composition

$$H_{n+1}(X|K \cap L) \xrightarrow{\partial} H_n(X \setminus K \cap L, X \setminus L) \xleftarrow{\cong} H_n(X \setminus K, X \setminus K \cup L) \to H_n(X|K \cup L).$$

Proof. Apply Lemma A.0.1 to the triples

$$(X, X \setminus K, X \setminus K \cup L) \quad \text{and} \quad (X, X \setminus K \cap L, X \setminus L)$$

to obtain the map of exact sequences

$$\begin{tikzcd} H_{n+1}(X|K) \arrow[r, "\partial"] \arrow[d] & H_n(X \setminus K, X \setminus K \cup L) \arrow[r] \arrow[d, "\cong"] & H_n(X|K \cup L) \arrow[r] \arrow[d] & H_n(X|K) \arrow[r] \arrow[d] & H_{n-1}(X \setminus L, X \setminus K \cup L) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(X|K \cup L) \arrow[d] \\ H_{n+1}(X|K \cap L) \arrow[r, "\partial"] & H_n(X \setminus K \cap L, X \setminus L) \arrow[r] & H_n(X, X|L) \arrow[r] & H_{n-1}(X|K \cap L) \arrow[r] & H_{n-1}(X \setminus K \cap L, X \setminus L) \arrow[r] & H_{n-1}(X|L) \end{tikzcd}$$

The maps marked as isomorphisms are indeed isomorphisms by excision because $(X \setminus K, X \setminus L)$ is an open cover of $X \setminus K \cap L$ with intersection $X \setminus K \cup L$. ☐

A.0.3. **Remark.** Both Lemmas A.0.1 and A.0.2 have evident analogues for cohomology.

Appendix B. More on cofibrations

B.0.1. **Proposition.** *Let X be a topological space, let $A \subseteq X$ be a subspace, and denote the inclusion map by $i\colon A \to X$. The following are equivalent:*

1. *i is a cofibration;*
2. *the map $A \times [0,1] \cup_{A \times \{0\}} X \times \{0\} \to X \times [0,1]$ admits a retraction;*
3. *there exist a map $u\colon X \to \mathbb{R}_{\geq 0}$ and a homotopy $h\colon X \times [0,1] \to X$ relative A such that*
   (a) *$u \circ i = 0$;*
   (b) *$h|_{X \times \{0\}} = \mathrm{id}_X$;*
   (c) *$h(x,t) \in A$ for $t > u(x)$.*

Proof. If i is a cofibration, one obtains a retraction as in (2) by solving the homotopy extension problem

$$\begin{tikzcd} A \times [0,1] \cup_{A \times \{0\}} X \times \{0\} \arrow[r, "\mathrm{id}"] \arrow[d] & A \times [0,1] \cup_{A \times \{0\}} X \times \{0\} \\ X \times [0,1] \arrow[ur, dotted] & \end{tikzcd}$$

Conversely, the existence of such a retraction allows us to solve any homotopy extension problem by precomposing with the retraction.

If $r\colon X \times [0,1] \to A \times [0,1] \cup_{A \times \{0\}} X \times \{0\}$ is a retraction, define

$$u\colon X \to [0,1], \quad x \mapsto \max\{t - \mathrm{pr}_2(r(x,t)) \mid t \in [0,1]\}$$

and

$$h\colon X \times [0,1] \to X, \quad (x,t) \mapsto \mathrm{pr}_1(r(x,t)),$$

where pr_i denotes the respective projection map on $X \times [0,1]$.

The map u is continuous because $[0,1]$ is compact. Since r is a retraction, we have $r(a,t) = (a,t)$ for all $a \in A$, so h is a homotopy relative A. Similarly, we have $u(a) = 0$. The property $r(x,0) = (x,0)$ implies that $h(x,0) = x$ for all $x \in X$. If $t > u(x)$, this implies that $\mathrm{pr}_2(r(x,t)) > 0$, so $r(x,t) \in A \times [0,1]$.

Conversely, if u and h are given, we obtain a retraction by defining

$$r\colon X \times [0,1] \to A \times [0,1] \cup_{A\times\{0\}} X \times \{0\}, \quad (x,t) \mapsto \begin{cases} (h(x,t),0) & t \leq u(x), \\ (h(x,t), t-u(x)) & t > u(x). \end{cases}$$

□

B.0.2. **Remark.** Suppose that $A \subseteq X$ is a closed subspace and that the inclusion $i\colon A \to X$ is a cofibration as witnessed by maps $u\colon X \to \mathbb{R}_{\geq 0}$ and $h\colon X \times [0,1] \to X$.

Suppose that $u(x) < 1$. Then it follows for sufficiently big n that $h(x, u(x) + \frac{1}{n})$ is defined, and thus lies in A. By continuity, we conclude that

$$h(x,u(x)) = \lim_{n\to\infty} h\left(x, u(x) + \frac{1}{n}\right) \in A$$

because A is closed.

B.0.3. **Proposition.** *Let $i\colon A \to X$ be a closed cofibration and let $j\colon B \to Y$ be an arbitrary cofibration. Then the induced map*

$$i \boxtimes j\colon A \times Y \cup_{A\times B} X \times B \to X \times Y$$

is also a cofibration.

Proof. Choose $u\colon X \to \mathbb{R}_{\geq 0}$, $v\colon Y \to \mathbb{R}_{\geq 0}$, $h\colon X \times [0,1] \to X$ and $k\colon Y \times [0,1] \to Y$ as in Proposition B.0.1 (3). Then define

$$w\colon X \times Y \to \mathbb{R}_{\geq 0}, \quad (x,y) \mapsto \min(u(x), v(y))$$

and

$$H\colon X \times Y \times [0,1] \to X \times Y, \quad (x,y,t) \mapsto (h(x, \min(t, v(y)), k(y, \min(t, u(x)))\,.$$

We claim that this pair also satisfies the conditions of Proposition B.0.1 (3).

Set $C := A \times Y \cup_{A\times B} X \times B$. If (x,y) lies in C, then $x \in A$ or $y \in B$, so w vanishes on C. If $x \in A$, we have

$$H(x,y,t) = (h(x, \min(t, v(y))), k(y,0)) = (x,y)$$

because h is a homotopy relative A and $k|_{Y\times\{0\}} = \mathrm{id}_Y$. Similarly, we have $H(x,y,t) = (x,y)$ if $y \in B$. Moreover,

$$H(x,y,0) = (h(x,0), k(y,0)) = (x,y).$$

Suppose now that $t > w(x,y) = \min(u(x), v(y))$. If $v(y) \geq t > u(x)$, then $h(x, \min(t, v(y))) = h(x,t) \in A$, and similarly if $u(x) \geq t > v(y)$. The remaining case is $t > u(x)$ and $t > v(y)$. If $u(x) > v(y)$, then $k(y, \min(t, u(x)) = k(y, u(x)) \in B$. Otherwise, $v(y) \geq u(x)$. Using that $u(x) < t \leq 1$, Remark B.0.2 implies that we then have $h(x, \min(t, v(y)) = h(x, v(y)) \in A$. □

B.0.4. **Corollary.** *Let $i\colon A \to X$ be a cofibration and let Y be an arbitrary topological space. Then $i \times \mathrm{id}_Y\colon A \times Y \to X \times Y$ is a cofibration.*

Proof. This is immediate from Proposition B.0.3 since $\varnothing \to Y$ is a closed cofibration.

□

B.0.5. **Corollary.** *Let (X, x) be a pointed space such that $\{x\} \to X$ is a cofibration. Then the canonical map $SX \to \Sigma X$ is a homotopy equivalence.*

Proof. Proposition B.0.3 implies that

$$A := \{x\} \times [0,1] \cup_{\{x\}\times\{0,1\}} X \times \{0,1\} \to X \times [0,1]$$

is a cofibration. Then a choice of retraction

$$X \times [0,1] \times [0,1] \to A \times [0,1] \cup_{A\times\{0\}} X \times [0,1] \times \{0\}$$

induces a retraction

$$SX \times [0,1] \to \{x\} \times [0,1] \cup_{\{x\}\times\{0\}} SX \times \{0\},$$

so $\{x\} \times [0,1] \to SX$ is a cofibration. Since $\{x\} \times [0,1]$ is contractible, it follows that $SX \to \Sigma X$ is a homotopy equivalence. □

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