TOPOLOGIE IV – EXERCISE SHEET 1

MARCUS NICOLAS

Notation. Given an object x in a $\mathcal{C}$ an ∞-category, define $\mathrm{BAut}_{\mathcal{C}}(x)$ to be the subcategory of $\mathcal{C}$ on objects equivalent to x and equivalences. The notation is justified by the fact that $\mathrm{BAut}_{\mathcal{C}}(x)$ is by definition a connected groupoid, such that $\Omega_x \mathrm{BAut}_{\mathcal{C}}(x) \simeq \mathrm{Aut}_{\mathcal{C}}(x)$. Any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a canonical morphism $\mathrm{BAut}_{\mathcal{C}}(x) \to \mathrm{BAut}_{\mathcal{D}}(Fx)$.

When $p\colon E \to B$ is a functor between groupoids with *small fibers* (meaning that the pullback of p along any map $B' \to B$ whose domain is a *small* groupoid is small), then the naturality of the Grothendieck construction gives a cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}_* \arrow[d] \\ B \arrow[r] & \mathrm{Gpd} \end{tikzcd}$$

where $\mathrm{Gpd}_* \to \mathrm{Gpd}$ is the left fibration induced by $\mathrm{id}_{\mathrm{Gpd}}$, also known as the universal left fibration (with small fibers). When p has typical fiber F, then the straightening $B \to \mathrm{Gpd}$ factorises through $\mathrm{BAut}(F)$. Define $\mathrm{BAut}_*(F)$ by the following pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{BAut}_*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gpd}_* \arrow[d] \\ B \arrow[r] & \mathrm{BAut}(F) \arrow[r] & \mathrm{Gpd} \end{tikzcd}$$

Observe that $\mathrm{BAut}_*(F)$ is connected iff F is, so that $\mathrm{BAut}_*(F)$ is not stricto sensu the delooping of a group $\mathrm{Aut}_*(F)$, even though $\Omega_{(F,*)}\mathrm{BAut}_*(F) \simeq \mathrm{Aut}_*(F)$.

Existence and computation of (co)limits in Gpd**.** Given a small category I, remember that we have canonical adjunctions

$$\begin{tikzcd} \mathrm{Cat}_{/I} \arrow[r, bend left, "p_!"] \arrow[r, bend right, "p_*"'] & \mathrm{Cat} \arrow[l, "p^*"' description] \arrow[r, bend left, "\Pi_\infty"] \arrow[r, phantom, "\perp"] & \mathrm{Gpd} \arrow[l, hook'] \end{tikzcd}$$

where p^* is the pullback along $p\colon I \to *$, or in other words the functor $I \times (-)\colon \mathrm{Cat} \to \mathrm{Cat}_{/I}$. The left adjoint $p_!$ is given by composition with p, and the right adjoint is given by taking sections. More explicitely $p_* \simeq \{\mathrm{id}_I\} \times_{\mathrm{Fun}(I,I)} \mathrm{Fun}(I, p_!(-))$.

Observe now that these adjunctions induce adjunctions on the full subcategories of left fibrations on both sides:

$$\begin{tikzcd} \mathrm{LFib}(I) \arrow[r, bend left, "\Pi_\infty\, p_!"] \arrow[r, bend right, "p_*"'] & \mathrm{LFib}(*) \simeq \mathrm{Gpd} \arrow[l, "p^*"' description] \end{tikzcd}$$

where p^* and p_* are the restrictions of the previous functors.

For instance, to show that p_* sends left fibrations to left fibrations, one uses the two following facts:

(i) the functor $\mathrm{Fun}(I, -)$ preserves left fibrations
(ii) left fibrations are stable under base change

Now, the naturality of the Grothendieck construction makes the following diagram commute

$$\begin{tikzcd}
\mathrm{Fun}(I, \mathrm{Gpd}) \arrow[d, equal] & \mathrm{Fun}(*, \mathrm{Gpd}) \arrow[l, "p^*"'] \arrow[d, equal] \\
\mathrm{LFib}(I) & \mathrm{LFib}(*) \arrow[l, "p^*"]
\end{tikzcd}$$

and the two vertical maps are equivalences. The constant diagram functor $p^*\colon \mathrm{Gpd} \to \mathrm{Fun}(I, \mathrm{Gpd})$ thus has a left adjoint $\mathrm{colim}_I(-)$ and a right adjoint $\lim_I(-)$, given by the formulas

$$\operatorname*{colim}_I(-) \simeq \Pi_\infty\, p_! \circ \mathrm{Un} \quad \text{and} \quad \lim_I(-) \simeq p_* \circ \mathrm{Un}$$

where Un denotes the canonical identification $\mathrm{Fun}(I, \mathrm{Gpd}) \simeq \mathrm{LFib}(I)$.

On objects, this means that for a diagram $X\colon I \to \mathrm{Gpd}$ whose corresponding left fibration is denoted $\mathrm{Un}(X) \to I$, one has

$$\operatorname*{colim}_I X \simeq \Pi_\infty \mathrm{Un}(X) \quad \text{and} \quad \lim_I X \simeq \{\mathrm{id}_I\} \times_{\mathrm{Fun}(I,I)} \mathrm{Fun}(I, \mathrm{Un}(X))$$

As a slogan, colimits and limits of diagrams of groupoids are computed by localizing or by taking sections of the Grothendieck construction. We give two exercises on this thema for the interested student.

(1) If I is a small category, show the map induced by restriction along the unit $\ell\colon I \to \Pi_\infty(I)$ induces a natural isomorphism
$$\lim_I \ell^*(-) \simeq \lim_{\Pi_\infty(I)}(-)$$
between functors $\mathrm{Fun}(\Pi_\infty(I), \mathrm{Gpd}) \to \mathrm{Gpd}$, and deduce that (co)limits of constant diagrams indexed by I in any ∞-category only depend on the fundamental groupoid $\Pi_\infty(I)$. Hint: left fibrations are conservative.

(2) Adapt the above discussion in order to compute (co)limits in Cat.

Morphisms in arrow categories. Let $\mathcal{C}$ be an ∞-category, and denote $\mathrm{Mor}(\mathcal{C}) :\simeq \mathrm{Fun}([1], \mathcal{C})$. Given two objects $f\colon x \to y$ and $g\colon z \to w$ of $\mathrm{Mor}(\mathcal{C})$, then uncurrying yields a pullback square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1]\times[1], \mathcal{C}) \arrow[d] \\
* \arrow[r] & \mathrm{Fun}\big([(0,0) < (0,1)], \mathcal{C}\big) \times \mathrm{Fun}\big([(1,0) < (1,1)], \mathcal{C}\big)
\end{tikzcd}$$

Since the right vertical map sits as the red map inside the cube

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Fun}\big([(0,0) < (0,1)], \mathcal{C}\big) \times \mathrm{Fun}\big([(1,0) < (1,1)], \mathcal{C}\big) \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}\big([(1,0) < (1,1)], \mathcal{C}\big) \arrow[dd] \\
\mathrm{Fun}([1]\times[1], \mathcal{C}) \arrow[ur, red] \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}\big([(0,0) < (1,0) < (1,1)], \mathcal{C}\big) \arrow[ur, blue] \arrow[dd] & \\
 & \mathrm{Fun}\big([(0,0) < (0,1)], \mathcal{C}\big) \arrow[rr] & & \mathrm{Fun}(\varnothing, \mathcal{C}) \\
\mathrm{Fun}\big([(0,0) < (0,1) < (1,1)], \mathcal{C}\big) \arrow[ur, blue] \arrow[rr] & & \mathrm{Fun}\big([(0,0) < (1,1)], \mathcal{C}\big) \arrow[ur, blue] &
\end{tikzcd}$$

whose front and back faces are cartesian, its fibers are computed by taking the fiber product of the fibers of the blue maps. At (f, g), this shows that outer square of the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mor}(\mathcal{C})}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{C}}(x,z) \arrow[r] \arrow[d, "g_*"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}_{/z} \arrow[d, "g_*"] \\
\mathrm{Hom}_{\mathcal{C}}(y,w) \arrow[r, "f^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{C}}(x,w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}_{/w} \arrow[d] \\
\mathcal{C}_{y/} \arrow[r, "f^*"] & \mathcal{C}_{x/} \arrow[r] & \mathrm{Mor}(\mathcal{C})
\end{tikzcd}$$

is cartesian. The inner squares are filled by pasting, for example the fact that the lower left square is cartesian can be seen on the following diagram

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{C}}(x,w) \arrow[rr] \arrow[dd] & & \mathcal{C}_{x/} \arrow[dd] \\
\mathrm{Hom}_{\mathcal{C}}(y,w) \arrow[ur, "f^*"] \arrow[rr] \arrow[dd] & & \mathcal{C}_{y/} \arrow[ur, "f^*"] & \\
& * \arrow[rr, "w"] \arrow[dl, equal] & & \mathcal{C} \arrow[dl, equal] \\
* \arrow[rr, "w"] & & \mathcal{C} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

since the lower face is cartesian. Subsequently, we will use that this formula can be made natural in f and g, even though we only have established it pointwise.

Recall that for an object x the slice categories over and under $\mathcal{C}$ are defined by the following cartesian squares

$$\begin{tikzcd}
\mathcal{C}_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Mor}(\mathcal{C}) \arrow[d, "\mathrm{ev}_1"] \\
* \arrow[r, "x"] & \mathcal{C}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{C}_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Mor}(\mathcal{C}) \arrow[d, "\mathrm{ev}_0"] \\
* \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}$$

From the above, we obtain natural identifications

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{/x}}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{C}}(a,b) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(a,x)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{x/}}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{C}}(a,b) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(x,b)
\end{tikzcd}$$

for objects a and b living either above or below x.

Colimits in underslices. Let $\mathcal{C}$ be an ∞-category together with a choice of object x. By the above, the hom groupoid from a to b between two objects over x sits inside a natural cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{x/}}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{C}}(a,b) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(x,b)
\end{tikzcd}$$

Therefore the colimit of a diagram $F\colon I \to \mathcal{C}_{x/}$ can be computed as the bottom map in the following pushout of $\mathcal{C}$

$$\begin{tikzcd}
\operatorname{colim} \underline{x} \arrow[r] \arrow[d] & \operatorname{colim} F \arrow[d] \\
x \arrow[r] \arrow[ur, phantom, "\ulcorner"] & q
\end{tikzcd}$$

as soon as all these colimits exists in $\mathcal{C}$. Indeed, for any object b under x one has the chain of identifications:

$$\mathrm{Hom}_{\mathcal{C}_{x/}}(q,b) \simeq \mathrm{Hom}_{\mathcal{C}}(q,b) \times_{\mathrm{Hom}_{\mathcal{C}}(x,b)} *$$

$$
\begin{aligned}
&\simeq \Big(\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} F, b) \times_{\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} \underline{x}, b)} \mathrm{Hom}_{\mathcal{C}}(x, b)\Big) \times_{\mathrm{Hom}_{\mathcal{C}}(x,b)} * \\
&\simeq \lim \mathrm{Hom}_{\mathcal{C}}(F(-), b) \times_{\mathrm{Hom}_{\mathcal{C}}(x,b)} * \\
&\simeq \lim \mathrm{Hom}_{\mathcal{C}_{x/}}(F(-), b)
\end{aligned}
$$

natural in b.

A direct consequence of this fact is that the forgetful functor $\mathcal{C}_{x/} \to \mathcal{C}$ preserves colimits indexed by weakly contractible categories, for instance pushouts. Since $\mathrm{Gpd}_* :\simeq \mathrm{Gpd}_{*/}$ is an underslice category, we derive for instance from this computation that the forgetful functor $\mathrm{Gpd}_* \to \mathrm{Gpd}$ commutes with suspension.

Exercise 1. All small colimits exist in Gpd_* by what was done above. More explicitely, the colimit of a diagram $X\colon I \to \mathrm{Gpd}_*$ is computed in Gpd by the following pushout

$$
\begin{tikzcd}
\Pi_\infty(I) \arrow[r, "\Pi_\infty(s)"] \arrow[d, equal] & \Pi_\infty \mathrm{Un}(X) \arrow[d, equal] \\
\operatorname{colim} \underline{*} \arrow[r] \arrow[d] & \operatorname{colim} \pi X \arrow[d] \\
* \arrow[r] & \operatorname{colim} X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

where $\pi\colon \mathrm{Gpd}_* \to \mathrm{Gpd}$ is the forgetful functor and s is the section of $\mathrm{Un}(X) \to I$ obtained by unstraightening the natural transformation $\underline{*} \to X$.

Given a pointed spherical fibration $p\colon E \to B$ straightened to $\xi\colon B \to \mathrm{Gpd}_*$ we obtain in particular

$$
\begin{tikzcd}
B \arrow[r, "s"] \arrow[d] & E \arrow[d] \\
* \arrow[r] & \operatorname{colim} \xi \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

where $s\colon B \to E$ is the section induced by the natural transformation $\underline{*} \to \xi$. This furnishes a canonical equivalence $\mathrm{Th}_*(p) \simeq \operatorname{colim} \xi$.

Exercise 2. Since $ps \simeq \mathrm{id}$, pasting cocartesian squares gives

$$
\begin{tikzcd}
B \arrow[r, "s"] \arrow[d] & E \arrow[r, "p"] \arrow[d] & B \arrow[d] \\
* \arrow[r] & \mathrm{Th}_*(p) \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] & * \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
& * \arrow[r] & \mathrm{Th}(p) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

and in particular yields a canonical equivalence $\mathrm{Th}(p) \simeq \Sigma \mathrm{Th}_*(p)$.

Exercise 3. Fix an integer $d \geq 0$.

1) The fiber sequence

$$
\begin{tikzcd}
S^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{BF}(d) \arrow[d] \\
* \arrow[r] & \mathrm{BG}(d+1)
\end{tikzcd}
$$

shows that the fiber of the forgetful map $\mathrm{BF}(d) \to \mathrm{BG}(d+1)$ is $(d-1)$-connected, so that the map itself is d-connected.

2) Given a pair of composable adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \arrow[r, bend left, "L'"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "R'"] \end{tikzcd}$$

remember that the unit η'' of the composite $L'L \dashv RR'$ is defined as the composite

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{\eta} RL \xrightarrow{R\eta' L} RR'L'L$$

where η and η' are the respective units of the adjunctions $L \dashv R$ and $L' \dashv R'$. Using this observation, remark that the following diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(L(-),-) \arrow[r, "L'"] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{E}}(L'L(-),L'(-)) \arrow[d, equal] \\ \mathrm{Hom}_{\mathcal{C}}(-,R(-)) \arrow[r, "(R\eta')_*"] & \mathrm{Hom}_{\mathcal{C}}(-,RR'L'(-)) \end{tikzcd}$$

canonically commutes.

Applying this observation the endoadjunctions $\Sigma^d \dashv \Omega^d$ and $\Sigma \dashv \Omega$ of Gpd_* yields

$$\begin{tikzcd} \mathrm{Hom}_*(\Sigma^d(-),-) \arrow[r, "\Sigma"] \arrow[d, equal] & \mathrm{Hom}_*(\Sigma^{d+1}(-),\Sigma(-)) \arrow[d, equal] \\ \mathrm{Hom}_*(-,\Omega^d(-)) \arrow[r, "(\Omega^d\eta)_*"] & \mathrm{Hom}_*(-,\Omega^{d+1}\Sigma(-)) \end{tikzcd}$$

where $\eta\colon \mathrm{id}_{\mathrm{Gpd}_*} \to \Omega\Sigma$ is the unit. Evaluating the left variable at S^0 gives

$$\begin{tikzcd} \mathrm{Hom}_*(S^d,-) \arrow[r, "\Sigma"] \arrow[d, equal] & \mathrm{Hom}_*(S^{d+1},\Sigma(-)) \arrow[d, equal] \\ \Omega^d \arrow[r, "\Omega^d\eta"] & \Omega^{d+1}\Sigma \end{tikzcd}$$

and in particular

$$\begin{tikzcd} \mathrm{End}_*(S^d) \arrow[r, "\Sigma"] \arrow[d, equal] & \mathrm{End}_*(S^{d+1}) \arrow[d, equal] \\ \Omega^d S^d \arrow[r, "\Omega^d\eta"] & \Omega^{d+1}S^{d+1} \end{tikzcd}$$

Recall that Freudental theorem states that the unit $\eta\colon S^d \to \Omega S^{d+1}$ is $(2d-1)$-connected, and therefore $\Sigma\colon \mathrm{End}_*(S^d) \to \mathrm{End}_*(S^{d+1})$ is $(d-1)$-connected. Since $\Sigma\colon \mathrm{F}(d) \to \mathrm{F}(d+1)$ is obtained by restricting this map along the same connected components of maps of degree ± 1, it must be $(d-1)$-connected as well. Finally, $\Sigma\colon \mathrm{BF}(d) \to \mathrm{BF}(d+1)$ is d-connected.