Summer term 2025 May 9, 2025

Topology IV

Sheet 2

Exercise 1. Show that there is an equivalence of functors $\Sigma(-) \simeq S^1 \wedge -\colon \mathrm{An}_* \to \mathrm{An}_*$.

The following exercises go towards proving the James splitting - since we have discussed a number of its ingredients in class already.

Exercise 2. Let $X, Y \in \mathrm{An}_*$ and suppose X is n-connected and Y is m-connected. Show that $X \wedge Y$ is $(n+m+1)$-connected.

Exercise 3. Using the natural equivalence $\mathrm{An}_{/X} \simeq \mathrm{Fun}(X, \mathrm{An})$, show the following. Consider the two squares

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \qquad\qquad \begin{array}{ccc} A \times_D X & \longrightarrow & B \times_D X \\ \downarrow & & \downarrow \\ C \times_D X & \longrightarrow & X \end{array}$$

in An where the right square is obtained from the left by pullback along a map $X \to D$. Show that if the left square is a pushout, then so is the right square. (This is called Mather's second cube lemma). Deduce that for every $X \in \mathrm{An}_*$, there is a canonical pushout square

$$\begin{array}{ccc} X \times \Omega\Sigma X & \xrightarrow{\mathrm{pr}_2} & \Omega\Sigma X \\ {\scriptstyle a_X}\downarrow & & \downarrow \\ \Omega\Sigma X & \longrightarrow & * \end{array}$$

Can the map a_X be the projection?

Exercise 4. Show that for $X \in \mathrm{An}_*$, there is a canonical equivalence

$$\Sigma(\Omega\Sigma X) \simeq \Sigma X \vee \Sigma(X \wedge \Omega\Sigma X).$$

Deduce that when X is connected, there is a canonical equivalence

$$\Sigma(\Omega\Sigma X) \simeq \bigvee_{n \geq 0} \Sigma(X^{\wedge n}).$$

(This is called the James splitting).

Exercise 5. Let $\gamma_{\mathbb{K}}$ be the tautological $\mathbb{K}$-line bundle over $\mathbb{KP}^n$; recall that it has an underlying spherical fibration (with fibre type the sphere $S(\mathbb{K})$ inside $\mathbb{K}$) – here $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Determine $\mathrm{Th}(\gamma_{\mathbb{K}})$ in terms of more familiar symbols.

This sheet will be discussed on May 16.