Summer term 2025 8. Mai 2025

Algebraic K-theory

Sheet 1

Exercise 1. Show that the canonical map $K_0(R \times S) \to K_0(R) \times K_0(S)$ is an isomorphism.

Solution. We claim that the additive functor $\mathrm{Proj}(R \times S) \to \mathrm{Proj}(R) \times \mathrm{Proj}(S)$ given by sending V to the pair $(V \otimes_{R\times S} R, V \otimes_{R \times S} S)$ admits an inverse given by $(P, Q) \mapsto P \times Q$. Indeed, there are natural isomorphisms $(P \times Q) \otimes_{R\times S} R \cong P$ and $(P \times Q) \otimes_{R\times S} S = Q$, showing that the one composite is equivalent to the identity. Conversely, if V is a projective $R \times S$ module, we need to show that V is isomorphic to $V \otimes_{R\times S} R \times V \otimes_{R\times S} S$. To that end, note that the elements $e = (1, 0)$ and $1 - e = (0, 1)$ in $R \times S$ are complementary idempotents. Hence for an $R \times S$ module V, we have $V \otimes_{R\times S} R = Ve$ and $V \otimes_{R\times S} S = V(1 - e)$, so we obtain $V = Ve \oplus V(1 - e)$. The result then follows by passing to isomorphism classes and group completing. □

Exercise 2. Let K be a field and V a countably infinite dimensional K-vector space. Let $R = \mathrm{End}_K(V)$. Show that $K_0(R) = 0$.

Solution. Denote by M the K-vector space $\mathrm{Hom}_K(V, \bigoplus_{\mathbb{N}} V)$. This is canonically a right R-module via precomposition, and a left R-module via postcomposition with the induced endomorphism of $\bigoplus_{\mathbb{N}} V$ – since these two module structures commute, M is an (R, R)-bimodule. Note that any choice of isomorphism $V \cong \bigoplus_{\mathbb{N}} V$ shows that the underlying right R-module of M is isomorphic to R itself. Now consider the evident isomorphism

$$V \oplus \bigoplus_{\mathbb{N}} \to \bigoplus_{\mathbb{N}} V, \quad (v, (v_0, v_1, v_2, \dots)) \mapsto (v, v_0, v_1, \dots)$$

and note that this map is one of left R-modules; again, the left R-module structure is just componentwise. Therefore, we obtain an induced isomorphism

$$R \oplus M = \mathrm{Hom}_K(V, V) \oplus \mathrm{Hom}_K(V, \bigoplus_{\mathbb{N}} V) \cong \mathrm{Hom}_K(V, V \oplus \bigoplus_{\mathbb{N}} V) \xrightarrow{\cong} \mathrm{Hom}_K(V, \bigoplus_{\mathbb{N}} V)$$

and this is an isomorphism of (R, R)-bimodules. We conclude that for every finitely generated projective R-module P, we obtain an isomorphism

$$P \otimes_R M \cong P \oplus P \otimes_R M;$$

here we use that the right module underlying M is isomorphic to R, and hence in particular finite projective; This ensures that $- \otimes_R M$ is indeed a functor $\mathrm{Proj}(R) \to \mathrm{Proj}(R)$. It follows that any monoid homomorphism $f\colon \tau_{\leq 0}\mathrm{Proj}(R) \to A$ sends P to zero since in the group A, we can cancel the term $f(P \otimes_R M)$ from the above equation. Hence 0 satisfies the universal property of $[\tau_{\leq 0}\mathrm{Proj}(R)]^{\mathrm{gp}}$ as needed.

Remark: Note that the place where we have used that V is infinite dimensional is in the argument that M is finite projective over R as right R-module; the same is not true for V finite dimensional as the next exercise shows. $\square$

Exercise 3. Let R be a ring. Show that $K_0(R) \cong K_0(M_n(R))$. Hint: Show that R^n is an $(R, M_n(R))$-bimodule which implements an equivalence of categories $\mathrm{Proj}(R) \simeq \mathrm{Proj}(M_n(R))$.

Solution. We define an $(R, M_n(R))$-bimodule structure on R_r^n (thought of as a row vector) by left scalar multiplication and right matrix multiplication. Likewise, define an $(M_n(R), R)$-bimodule R_r^c (thought of as column vector) by right scalar multiplication and left matrix multiplication. Then we obtain functors

$$\mathrm{Proj}(R) \to \mathrm{Proj}(M_n(R)), \quad P \mapsto P \otimes_R R_r^n$$

and

$$\mathrm{Proj}(M_n(R)) \to \mathrm{Proj}(R), \quad Q \mapsto Q \otimes_{M_n(R)} R_c^n.$$

The respective composites are therefore given by

$$P \mapsto P \otimes_R R_r^n \otimes_{M_n(R)} R_c^n$$

as well as

$$Q \mapsto Q \otimes_{M_n(R)} R_c^n \otimes_R R_r^n.$$

Hence, it suffices to show that $R_r^n \otimes_{M_n(R)} R_c^n \cong R$ as (R, R)-bimodule, as well as that $R_c^n \otimes_R R_r^n \cong M_n(R)$ as $(M_n(R), M_n(R))$-bimodule.

To that end, let $(x, y) \in R_r^n \times R_c^n$ and consider $\langle x, y\rangle = \sum_{i=1}^n x_i y_i \in R$ – i.e. the matrix multiplication of x and y. Then the associativity of matrix multiplication gives that this map factors as a map $R_r^n \otimes_{M_n(R)} R_c^n \to R$. This map is evidently an (R, R)-bimodule map, and one checks directly that it is an isomorphism.

Similarly, define a map $R_c^n \otimes_R R_r^n \to M_n(R)$ again by matrix multiplication. Then, this map is one of $(M_n(R), M_n(R))$-bimodules again by associativity of matrix multiplication, and again, it is an isomorphism by direct check. $\square$

Exercise 4. Let R be a ring and consider the canonical ring homomorphism $R \to M_n(R)$. Compute the composite

$$K_0(R) \to K_0(M_n(R)) \cong K_0(R)$$

obtained in Exercise 2.

Solution. By construction, the composite is induced by sending P to $(P \otimes_R M_n(R)) \otimes_{M_n(R)} R_c^n$, and is hence induced by the (R, R)-bimodule R^n. Consequently, the map sends P to $P^{\oplus n}$, and thus the map under investigation is the multiplication by n map. $\square$

Exercise 5. Show that if $I \ni i \mapsto R_i$ is a filtered diagram of rings with $\mathrm{colim}_i R_i = R$, then

$$\mathrm{colim}_{i\in I}\, K_0(R_i) \to K_0(R)$$

is an isomorphism. Construct a ring R with $K_0(R) \cong \mathbb{Q}$. Can such a ring be commutative? Are there commutative rings with $K_0(R) = \mathbb{Z}/n$?

Solution. As usual, we first prove that $\mathrm{colim}\,\mathrm{Proj}(R_i) \to \mathrm{Proj}(R = \mathrm{colim}\, R_i)$ is an equivalence. Then the result follows from the fact that passing to groupoid cores preserves filtered colimits, and $\tau_{\leq 0}(-)$ and $(-)^{\mathrm{gp}}$ are both left adjoint functors and hence commute with all colimits.

We give two arguments for the above equivalence: One computational one and one abstract: For the computational one, we note that the functor is concretely given as follows: Since the colimit is filtered, an object of the left hand side is an object P of $\mathrm{Proj}(R_i)$ for some $i \in I$; such an object is then sent to $P \otimes_{R_i} R$. Note that this implies that the finite free R-modules are in the image of this functor. So if the functor is fully faithful, then retracts of free modules are also in the image and hence the functor is also essentially surjective. Now, to see fully faithfulness, we pick objects $P \in \mathrm{Proj}(R_i)$ and $Q \in \mathrm{Proj}(R_j)$. (Since I is filtered, we could assume without loss of generality that $i = j$). Then we use that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{colim}\,\mathrm{Proj}(R_i)}(P, Q) &= \mathrm{colim}_{k\in I_{i/}\times_I I_{j/}}\, \mathrm{Hom}_{R_k}(P \otimes_{R_i} R_k, Q \otimes_{R_i} R_k)\\
&= \mathrm{colim}_{k\in I_{j/}}\, \mathrm{Hom}_{R_i}(P, Q \otimes_{R_j} R_k)\\
&= \mathrm{Hom}_{R_i}(P, \mathrm{colim}_{k\in I_{j/}}\, Q \otimes_{R_j} R_k)\\
&= \mathrm{Hom}_{R_i}(P, Q \otimes_{R_j} R)\\
&= \mathrm{Hom}_R(P \otimes_{R_i} R, Q \otimes_{R_j} R)
\end{aligned}$$

giving fully faithfulness as needed. Note that we have used in the above that P is finite projective, and hence compact, over R_i in order to pull in the colimit in line 3.

Different, less computational, approach: The functor $R \mapsto \mathrm{Proj}(R)$ is a left adjoint when taking values in pointed small additive categories $(\mathcal{A}, a)$ in which every retract splits. Indeed, there is a functor from such categories to rings given by sending $(\mathcal{A}, a)$ to $\mathrm{End}_{\mathcal{A}}(a)$, and this functor is right adjoint to $R \mapsto \mathrm{Proj}(R)$ – indeed, $R \mapsto \mathrm{Free}(R)$ is the coproduct completion of BR, the "additive category with one object and R as endomorphisms" and $\mathrm{Proj}(R)$ is then the retract closure of $\mathrm{Free}(R)$. It then remains to show that filtered colimits in pointed additive categories are computed by forgetting the specified objects, this follows from the fact that filtered diagrams are contractible. Finally, the forgetful functor from additive categories to ordinary categories also preserves filtered colimits, as does passing to the groupoid cores.

Now, to construct a ring R with $K_0(R) \cong \mathbb{Q}$, consider the filtered diagram with values $\mathbb{Z}$ and transition maps multiplication by natural numbers via the divisibility poset. Its colimit is $\mathbb{Q}$. All these maps can be realised by maps induced by ring maps on $K_0(-)$ by the above. Hence, taking the resulting colimit of rings, we are done. Such a ring cannot be commutative: For commutative rings, we have $\mathbb{Z}$ as a direct summand of $K_0(R)$, but $\mathbb{Z}$ is not a direct summand in $\mathbb{Q}$. Same applies to $K_0(R) = \mathbb{Z}/n$. $\square$

Exercise 6. Let TS^2 be the tangent bundle of S^2. Show that $\Gamma_{\mathbb{R}}(TS^2; S^2)$ is a stably free $C(S^2;\mathbb{R})$-module, but it is not free. Note that TS^2 is in fact canonically a complex vector bundle. Is $\Gamma_{\mathbb{C}}(TS^2; S^2)$ stably free as $C(S^2;\mathbb{C})$-module?

Solution. For the first, by Swan's equivalence of categories, we need to show that TS^2 is, as a real vector bundle, non-trivial but stably trivial. It is stably trivial, since the normal bundle of the canonical embedding $S^2 \subseteq \mathbb{R}^3$ is trivial, and therefore $TS^2 \oplus \epsilon = TS^3_{|S^2} = \epsilon^3$. Now, if TS^2 were trivial, we could find a non-vanishing section, but the hairy ball theorem says that there is no such non-vanishing section. The same argument in fact applies to TS^{2n} for any $n \geq 1$.

As for $\Gamma_{\mathbb{C}}(TS^2)$, we see that this is a line bundle over $C(S^2;\mathbb{C})$ - so we know from Remark 3.46 in the lecture that it is stably trivial if and only if it is trivial. But it is not trivial as a complex bundle since it is non-trivial as a real bundle as shown above. $\square$

This sheet will be discussed on May 15.