Summer term 2025 | 8. Mai 2025

Algebraic K-theory

Sheet 0

Exercise 1. Let R be a ring and $x \in R$ an element. Show that the following are equivalent.

(1) $x \in \mathrm{Jac}(R)$,

(2) for all $y \in R$, we have that $1 - xy$ is right-invertible,

(3) $Mx = 0$ for all simple R-modules M.

Solution. (1)⇒(2): If $1-xy$ is not right-invertible, then $(1-xy)R$ is a proper right ideal of R, hence by Zorn's lemma $1-xy$ is contained in a maximal right ideal $\mathfrak{m}$. Since $1 \notin \mathfrak{m}$, we deduce $xy \notin \mathfrak{m}$ and hence $x \notin \mathrm{Jac}(R) \subseteq \mathfrak{m}$. (2)⇒(3): Assume $Mx \neq 0$ and take $m \in M$ with $mx \neq 0$. Then $0 \neq mxR \subseteq M$ so $mxR = M$ since M is simple. Hence, there exists $y \in R$ such that $mxy = m$ or equivalently, $m(1-xy) = 0$. Since $(1-xy)$ is right-invertible, this implies $m = 0$; contradiction. (3)⇒(1): Let $\mathfrak{m}$ be a maximal ideal of R. Then $R/\mathfrak{m}$ is simple, and hence $(R/\mathfrak{m})x = 0$ and hence $x \in \mathfrak{m}$. □

Exercise 2. For an R-module M, define $\mathrm{Ann}(M) = \{r \in R \mid Mr = 0\}$. Show that

$$\mathrm{Jac}(R) = \bigcap_{M \text{ simple}} \mathrm{Ann}(M).$$

Deduce that $\mathrm{Jac}(R)$ is a 2-sided ideal of R.

Solution. The equality is a direct consequence of the equivalence of (1) and (3) above. For the second, we need to show that if $r \in R$ and $x \in \mathrm{Jac}(R)$, then $rx \in \mathrm{Jac}(R)$. But if M is simple, then $Mrx \subseteq Mx = 0$, so $rx \in \mathrm{Jac}(R)$ again by characterization (3) above. □

Exercise 3. Let R be a ring and $J \subseteq \mathrm{Jac}(R)$. Show that $\mathrm{Proj}^{\mathrm{fg}}(R) \to \mathrm{Proj}^{\mathrm{fg}}(R/J)$ is conservative.

Proof. Suppose $f\colon P \to Q$ is such that $f \otimes_R R/J = g$. We show that f is an isomorphism. Note that $\mathrm{coker}(f)$ is finitely generated and $\mathrm{coker}(f) \otimes_R R/J = \mathrm{coker}(g) = 0$; hence Nakayama implies that $\mathrm{coker}(f) = 0$, so f is surjective. In particular, $P \cong Q \oplus \ker(f)$, so that $\ker(f)$ is again finitely generated (and projective) and $\ker(f) \otimes_R R/J = \ker(g) = 0$ since the functor $- \otimes_R R/J$ preserves split exact sequences. Another application of Nakayama gives $\ker(f) = 0$ and hence f is an isomorphism. □

Exercise 4. Let I be an ideal of a ring R such that R is I-adically complete. Show that idempotent elements of R/I can be lifted to idempotent elements of R.

Solution. First we treat the case where $I^n = 0$ is nilpotent. So let $\bar{e}$ be an idempotent in R/I. Let $x \in R$ be a lift of $\bar{e}$ and set $y = 1 - x$ so that $1 = x + y$. Then x and y commute and $yx = xy = x - x^2 \in I$. Since $I^n = 0$ we find $(xy)^n = 0$. Then we have

$$1 = (x+y)^{2n} = \sum_{k=0}^{n} \binom{2n}{k} x^{2n-k} y^k + \sum_{k=n+1}^{2n} \binom{2n}{k} x^{2n-k} y^k.$$

Let the first summand be e, so the second summand is $1 - e$. Since $(xy)^n = 0$, we have $e(1-e) = 0$, i.e. $e^2 = e$ so e is an idempotent. Moreover,

$$e = x^{2n} + 2nx^{2n-1}y + \cdots + \binom{2n}{n-1} x^{n+1} y^{n-1} \in x^{2n} + I$$

so e lifts $\bar{e}$, as x^{2n} lifts $\bar{e}^{2n} = \bar{e}$.

This construction provides compatible idempotents in R/I^n, and hence an idempotent in $\lim_n R/I^n \cong R$, by completeness. $\square$

This sheet was discussed on May 8.