LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATICHES INSTITUT

Summer term 2025 | 12. Mai 2025

# Algebraic $K$-theory

Sheet 2

**Exercise 1.** Let $R$ be a ring. Show that the elementary matrices $E_{i,j}(r) \in \mathrm{GL}(R)$ satisfy the following relations:

(1) $E_{i,j}(r)E_{i,j}(r') = E_{i,k}(r + r')$,

(2) $[E_{i,j}(r), E_{j,k}(r')] = E_{i,k}(rr')$, if $i \neq k$ and

(3) $[E_{i,j}(r), E_{k,l}(r')] = 1$ if $i \neq l$ and $j \neq k$.

**Exercise 2.** Let $R$ be a ring. Show that the center $C(E(R))$ of the group of elementary matrices is trivial. Hint: Show that if $A \in \mathrm{GL}_n(R)$ commutes with all elements of $E_n(R)$, then $A$ is a homothetie, i.e. a diagonal matrix $(r, \ldots, r)$ with $r \in C(R)$. Deduce that no element of $E_{n-1}(R)$ lies in the center of $E_n(R)$ and let $n$ go to infinity.

**Exercise 3.** Let $R$ be a ring and $C \in \mathrm{GL}_n(R)$. Show that the following matrices are elementary:

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -C^{-1} & 1 \end{pmatrix}$$

**Exercise 4.** Show that the map $R^{n-1} \to \mathrm{St}(R)$ given by

$$(r_1, \ldots, r_{n-1}) \mapsto e_{1,n}(r_1) \cdot e_{2,n}(r_2) \cdots e_{n-1,n}(r_{n-1})$$

is an injective group homomorphism.

**Exercise 5.** Use Matsumoto's theorem to show that $K_2(\mathbb{F}_q) = 0$ where $\mathbb{F}_q$ is a finite field.

This sheet will be discussed in the week of 23 October 2023.