Summer term 2025 7. Mai 2025

Algebraic K-theory

Sheet 2

Exercise 1. Show that the canonical map $K_0(R \times S) \to K_0(R) \times K_0(S)$ is an isomorphism.

Exercise 2. Let K be a field and V a countably infinite dimensional K-vector space. Let $R = \mathrm{End}_K(V)$. Show that $K_0(R) = 0$.

Exercise 3. Let R be a ring. Show that $K_0(R) \cong K_0(M_n(R))$. Hint: Show that R^n is an $(R, M_n(R))$-bimodule which implements an equivalence of categories $\mathrm{Proj}(R) \simeq \mathrm{Proj}(M_n(R))$.

Exercise 4. Let R be a ring and consider the canonical ring homomorphism $R \to M_n(R)$. Compute the composite

$$K_0(R) \to K_0(M_n(R)) \cong K_0(R)$$

obtained in Exercise 2.

Exercise 5. Show that if $I \ni i \mapsto R_i$ is a filtered diagram of rings with $\mathrm{colim}_i R_i = R$, then

$$\mathrm{colim}_i K_0(R_i) \to K_0(R)$$

is an isomorphism. Construct a ring R with $K_0(R) \cong \mathbb{Q}$. Can such a ring be commutative? Are there commutative rings with $K_0(R) = \mathbb{Z}/n$?

Exercise 6. Let TS^2 be the tangent bundle of S^2. Show that $\Gamma(TS^2; S^2)$ is a stably free $C(S^2; \mathbb{C})$-module, but it is not free.

This sheet will be discussed in the week of 23 October 2023.