ALGEBRAIC K-THEORY

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Abstract. These are lecture notes for my lecture "Algebraic K-theory" which I taught in the summer term 2025 at LMU Munich.

Contents

1. Organization

There will be **no lectures** on 23.06. and 25.06. There will be weekly exercises, starting on May 8. I will upload exercise sheets and this script to the homepage of the course

Course Webpage

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weakly and after each lecture, respectively. The examination will be an oral exam at the end of the term.

2. Introduction and several motivations

2.1. **History.** We begin with some historical remarks. In its simplest form, algebraic K-theory can be viewed as a sequence of functors

$$K_n(-)\colon \text{Rings} \to \text{Ab}, \quad n \in \mathbb{Z}$$

where $K_0(R)$ is the *group completion* of the abelian monoid of isomorphism classes of finitely generated projective R-modules $\text{Proj}(R)/\text{iso}$ under direct sum. Concretely:

$$K_0(R) = \mathbb{Z}[\text{Proj}(R)/\text{iso}]/\langle [P] + [Q] = [P \oplus Q]\rangle.$$

This group was introduced by Grothendieck in 1957 (in fact in greater generality as we indicate below). At the same time, Bott proved his famous periodicity theorem for the homotopy groups of the stable unitary group U, and hence also for the classifying space for stable vector bundles BU, and Atiyah and Hirzebruch defined the topological K-groups $K^*(X)$ in 1959. In 1964, Bass defined $K_1(R) := \text{GL}(R)^{\text{ab}}$ and proved what is called the fundamental theorem of algebraic K-theory: There is an exact sequence of abelian groups

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t^{\pm 1}]) \xrightarrow{\partial} K_0(R) \to 0$$

and the map ∂ is split by the map induced by sending P to $\cdot t\colon P \otimes_R R[t^{\pm 1}] \to P \otimes_R R[t^{\pm 1}]$.[1] Bass used this to define negative K-groups inductively: For $n < 0$, he sets

$$K_n(R) = \text{coker}(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t^{\pm 1}])).$$

Moreover, Bass, Milnor, and Murthy established an *excision exact sequence* in K-groups starting with $K_1(-)$ and lowering degree: For a map $f\colon A \to B$ of rings carrying an ideal $I \subseteq A$ isomorphically to an ideal $J \subseteq B$, there is a long exact sequence:

$$K_1(A) \to K_1(A/I) \oplus K_1(B) \to K_1(B/J) \xrightarrow{\partial} K_0(A) \to K_0(A/I) \oplus K_0(B) \to K_0(B/J)$$

which in fact can be contined indefinitely to the right using Bass' definition of negative K-groups. Swan proved that there is no functorial way to extend this sequence to putative higher K-groups to the left. Nevertheless, in 1967, Milnor defines $K_2(R)$ and computes $K_2(\mathbb{Z})$. It is slightly more involved to describe $K_2(R)$ than $K_1(R)$, but it goes as follows. One defines the *Steinberg group* $\text{St}(R)$ of a ring R as the group generated by symbols $e_{i,j}(r)$, where $i \neq j$ are natural numbers and $r \in R$, subject to the standard relations that the elementary matrices $E_{i,j}(r) \in \text{GL}(R)$ satisfy.[2] One obtains a group homomorphism $\text{St}(R) \to E(R) \subseteq \text{GL}(R)$ and Milnor defines $K_2(R) = \ker(\text{St}(R) \to \text{GL}(R))$; one can show that this agrees with the center $C(\text{St}(R))$ of $\text{St}(R)$, and in particular $K_2(R)$ is indeed abelian. Since $E(R) = [\text{GL}(R), \text{GL}(R)]$ by Whiteheads lemma, there is an exact sequence

$$0 \to K_2(R) \to \text{St}(R) \to \text{GL}(R) \to K_1(R) \to 0.$$

As described, both $K_1(R)$ and $K_2(R)$ are purely algebraic definitions, and there was good reason to believe that these are "the correct definitions" – mostly, because they participate in certain long exact sequences for quotients by a two-sided ideal. In his thesis in 1968,

[1]Here, we need to note that an automorphism of a finitely generated projective can be extended to an automorphism of a finitely generated free module; this can be represented by a matrix and then represents an element in $K_1(-)$.

[2]We will make this more explicit later in the course.

Matsumoto gave an explicit presentation for $K_2(-)$ for fields, leading Milnor to define a form of higher K-groups for fields now known as Milnor K-theory, but it seems to have been clear that his is not even the "correct" definition of higher K-groups for fields, let alone for more general rings. In particular, it was not at all clear at the tine how to correctly define higher K-groups. This was eventually solved by Quillen in 1971. In modern language (and to the best of my knowledge largely inspired by insights of Segal) he observed that it is better to consider $\iota\mathrm{Proj}(R)$ as a symmetric monoidal groupoid and not take its isomorphism classes (which is then an abelian monoid). Symmetric monoidal groupoids are then examples of commutative monoids in anima (aka spaces, ∞-groupoids, etc..). The collection of such form an ∞-category $\mathrm{CMon}(\mathrm{An})$ which contains a full subcategory $\mathrm{CGrp}(\mathrm{An})$ of grouplike monoids, i.e. those, where every point admits an inverse (or equivalently $\pi_0(-)$ forms an ordinary abelian group). Just like in the case of abelian monoids and groups in sets, the inclusion $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ admits a left adjoint, the *group completion* $(-)^{\mathrm{gp}}$.[3] Quillen came up with an ad hoc construction, the *Q-construction*, which implements this group completion, and defines the K-theory space:

$$K(R) := (\iota\mathrm{Proj}(R))^{\mathrm{gp}}.$$

By comparing universal properties, one finds $\pi_0(K(R)) = K_0(R)$. It is less obvious that $\pi_1(K(R)) = K_1(R)$ and $\pi_2(K(R)) = K_2(R)$, these rely on the *group completion theorem* of Segal and McDuff as we will prove in this course. The following are among the most important first computations about K-theory. It is fair to say, that there is not a single "simple" computation of $K(R)$; all computations really invoke or establish deep mathematics.

(1) In 1971, when defining $K(R)$ in general and setting up a number of influential basic results [Qui73b], Quillen also computed $K(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field [Qui72]; the result is that for $n > 0$, $K_{2n}(\mathbb{F}_q) = 0$, and $K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/q^n - 1$. In fact, Quillen constructs a map of spaces

$$K(\mathbb{F}_q) \to \mathrm{fib}(\mathrm{BU} \xrightarrow{\psi^q - 1} \mathrm{BU})$$

where ψ^q is an Adams operation, and shows that this map induces an isomorphism on positive homotopy groups. Moreover, Quillen showed that $K_n(\mathcal{O}_F)$ is a finitely generated abelian group for all $n \geq 0$, where F is a number field with ring of integers $\mathcal{O}_F$ [Qui73a].

(2) In 1974, Borel then computed $K(\mathcal{O}_F) \otimes \mathbb{Q}$: These groups are trivial in even (positive) degrees, and have rank $r_1 + r_2 - 1$ in degree 1, and for degrees larger than 1, the ranks are given by $r_1 + r_2$ in degrees 1 mod 4 and r_2 in degrees 3 mod 4. Here, r_1 and r_2 are the numbers of real and pairs of complex conjugate complex embeddings of F, respectively. Borel's proof again uses crucially the group completion theorem to reduce the computation of *homotopy groups* to a computation of *homology groups*, in the case of interest of certain arithmetic groups.

(3) In 1984, Suslin computed $K(k)/n$, where k is an algebraically (or separably) closed field and $n \in k^\times$. In fact, he proves a *rigidity theorem*, that whenever $k \subseteq k'$ is an inclusion of algebraically closed fields and $n \in k^\times$, then the inclusion induced map $K(k)/n \to K(k')/n$ is an equivalence, and the common term is in turn equivalent to

[3] Warning: This is not quite as simple as the one for sets described above. Consider for instance the symmetric monoidal groupoid of finite sets with bijections under disjoint union. Its group completion is then the (anima underlying the) sphere spectrum, whose homotopy groups are the famously notoriously mysterious and hard to compute stable homotopy groups of spheres.

ku$/n$; here ku denotes the (connective) complex K-theory spectrum – to see this, by rigidity, it suffices to study the cases $K(\overline{\mathbb{F}}_p)$ and $K(\mathbb{C})$; the former essentially then follows from Quillen's computation, and the latter is another result of Suslin from 1984, confirming a conjecture of Milnor's about the relation of the group homologies of $\mathrm{GL}_n(\mathbb{C})^\delta$ and $\mathrm{GL}_n(\mathbb{C})$ – once with the discrete and once with its analytic topology.

(4) In 1983, Gabber gave a talk explaining the following theorem (his result was then published in 1989 [**?**]): If (A, I) is a henselian pair[4] and $n \in A^\times$, then the induced map $K(A)/n \to K(A/I)/n$ is an equivalence. Examples of henselian pairs include the case where A is I-adically complete (in particular if I is nilpotent), and the case where I is locally nilpotent.

Let us now turn to some results from different fields which aim to convey the slogan: K-theory is everywhere and everywhere interesting.

2.2. **Algebraic geometry.** We begin with the origin of K-theory: Grothendieck's goal to understand (and vastly generalize in his typical manner) the theorem of Riemann and Roch. Let us recall the theorem of Riemann–Roch from the 1850's:

So let Σ be a compact Riemann surface of genus $g(\Sigma)$. Let $D \in \mathbb{Z}[\Sigma]$ be a divisor on Σ, that is, a formal finite linear combination of points in Σ with coefficients in $\mathbb{Z}$. The degree $\deg(D)$ of a divisor D is the sum of its coefficients. Divisors can be added and form an abelian group $\mathrm{Div}(\Sigma)$. For $f\colon \Sigma \to \mathbb{C} \in \mathcal{M}(\Sigma;\mathbb{C})$ a meromorphic function, one can consider its associated principal divisor $D(f)$ whose coefficient $D(f)_x$ at $x \in \Sigma$ is given by

$$D(f)_x = \begin{cases} n & \text{if } x \text{ is a zero of order } n \\ -n & \text{if } x \text{ is a pole of order } n \\ 0 & \text{otherwise} \end{cases}.$$

In the theory of Riemann surfaces, one is then interested in the $\mathcal{C}$-vector spaces

$$\mathcal{M}(D) = \{f \in \mathcal{M}(\Sigma;\mathbb{C}) \mid D(f)_x \geq -D_x\}$$

and in particular, one would like to compute the dimension of $\mathcal{M}(D)$. Riemann proved the inequality

$$\dim \mathcal{M}(D) \geq \deg(D) + 1 - g(\Sigma).$$

This in particular implies that $\mathcal{M}(D)$ is non-empty if $\deg(D) + 1 - g(\Sigma) \geq 0$. Riemann's inequality was then improved by his student Roch as follows. First, note that $\dim \mathcal{M}(D) = \dim \mathcal{M}(D + D(f))$ and $\deg(D) = \deg(D + D(f))$; it follows that $D \mapsto \dim_{\mathbb{C}} \mathcal{M}(D)$ can be thought of as a function on the divisor class group $\mathrm{Cl}(\Sigma) = \mathrm{Div}(\Sigma)/\mathrm{pDiv}(\Sigma)$. This group is isomorphic to the Picard group $\mathrm{Pic}(\Sigma)$ consisting of holomorphic line bundles (under tensor product) on Σ. Let K_Σ be the canonical bundle on Σ (i.e. the holomorphic cotangent bundle) and $D_\Sigma \in \mathrm{Cl}(\Sigma)$ be its associated divisor (up to principal divisors). For any $D \in \mathrm{Cl}(\Sigma)$, set $D^\vee := D_\Sigma - D$. Then the Riemann–Roch (RR) theorem states:

$$\dim \mathcal{M}(D) - \dim \mathcal{M}(D^\vee) = \deg(D) + 1 - g(\Sigma).$$

Let us now go towards Grothendieck's generalization of the Riemann–Roch theorem. We aim to reinterprete several players involved in the above formula. Firstly, when L is the line bundle with associated divisor D, then $\mathcal{M}(D)$ canonically identifies with $H^0_{\mathrm{sh}}(\Sigma; L) = \Gamma(\Sigma; L)$,

[4] That is, A is a commutative ring, $I \subseteq \mathrm{Jac}(A)$ is contained in the Jacobson radical and for every monic polynomial $f \in A[X]$ with factorization $\bar{f} = \bar{g}\bar{h}$ with $\bar{g}, \bar{h} \in A/I[X]$ monic and generating the unit ideal, there exists a lifted factorization $f = gh$ with $g, h \in A[X]$ monic.

i.e. the holomorphic global sections of the sheaf on Σ represented by L. Therefore, the left hand side of RR becomes

$$\dim H^0_{\mathrm{sh}}(\Sigma; L) - \dim H^0_{\mathrm{sh}}(\Sigma; L^{-1} \otimes K_\Sigma)$$

which by Serre duality (and the fact that Riemann surfaces are complex curves, i.e. 1-dimensional) is equal to

$$\dim H^0_{\mathrm{sh}}(\Sigma; L) - \dim H^1_{\mathrm{sh}}(\Sigma; L)^\vee$$

which is the Euler characteristic $\chi(\Sigma; L)$ of Σ with coefficients in L (as all cohomology groups here are finite dimensional $\mathbb{C}$-vector spaces). In 1954, Hirzebruch then found the following generalization of RR, the Hirzebruch–Riemann–Roch theorem: For $E \to X$ a holomorphic vector bundle over a compact complex d-dimensional manifold X, there is the formula

$$\chi(X; E) = \langle \mathrm{ch}(E) \cdot \mathrm{td}(TX), [X] \rangle$$

where $\mathrm{ch}(E)$ is the Chern character of E and $\mathrm{td}(TX)$ is the Todd genus (another characteristic class describable in terms of Chern classes) of the tangent bundle TX. Specialized to $L \to \Sigma$ with Σ a Riemann surface and L a holomorphic line bundle, the right hand side of the above equality can be computed to be $\deg(D)+1-g(\Sigma)$, so Hirzebruch really generalizes the classical Riemann–Roch theorem to higher dimensional compact complex manifolds.

Grothendieck, among other things, wanted to generalize the above result to a relative setting, where one considers a proper morphism $f\colon X \to Y$ where Y need not be a point. In this situation, how could one generalise left and right hand sides of the equation? Recalling that

$$\chi(X; E) = \sum_{i \geq 0} (-1)^i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{sh}}(X; E)$$

and noting that $H^i(X; E) = R^i p_*(E)$, where $R^i p_*$ is the ith right derived functor of the functor $\Gamma(-; E)\colon \mathrm{Sh}(X; \mathrm{Ab}) \to \mathrm{Ab}$, here $p\colon X \to *$ is the map to the point. For a morphism $f\colon X \to Y$, one can still consider the values $R^i f_*(E)$ of the right derived functors $R^i f_*$ of $f_*\colon \mathrm{Sh}(X; \mathrm{Ab}) \to \mathrm{Sh}(Y; \mathrm{Ab})$ and one would like to form

$$\sum_{i \geq 0} (-1)^i \cdot R^i f_*(E).$$

But how are we supposed to interpret this alternating sum? You see that for the above formula for $\chi(X; E)$ to make sense, we have used $\dim_{\mathbb{C}}(-)$ to obtain natural numbers, and then know what it means to take an alternating sum. Grothendieck's insight here was to simply *define* an abelian group in which the above formula involving alternating sums of higher pushforward sheaves makes sense. Indeed, he defined[5] $K_0(X)$ to be the group completion of the abelian monoid of isomorphism classes of of coherent sheaves on X, modulo the relation $[\mathcal{F}_1] = [\mathcal{F}_0] + [\mathcal{F}_2]$ if there is a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0.$$

[5]This is in fact not generally what $K_0(X)$ is, rather what one would call $G_0(X)$. $K_0(X)$ is defined with vector bundles rather than coherent sheaves instead. The fact that the two yield the same group has to do with the smoothness of X. We write $K_0(X)$ rather than $G_0(X)$ as to not make a big fuzz about the difference at this point.

Now, if $f\colon X \to Y$ is proper, the higher pushforward functors $R^i f_*(-)$ preserve coherent sheaves and hence f induces a morphism $f_*\colon K_0(X) \to K_0(Y)$, with

$$f_*(\mathcal{F}) = \sum_{i\geq 0} (-1)^i R^i f_*(\mathcal{F}) \in K_0(Y)$$

which is now perfectly well-defined and precisely the putative candidate for the left hand side of the Grothendieck–Riemann–Roch theorem (and also the reason for working with coherent modules, rather than vector bundles: In general $R^i f_*$ does not preserve vector bundles). But then the next question is how to generalize the right hand side of the RR theorem? Hirezbruch extension already showed that terms like a Chern character and the Todd class appear at least in the holomorphic case. To see how Grothendieck treated this, let us briefly talk about algebraic cycles. For X a smooth variety over a field, let the *cycle group* be

$$Z(X) = \mathbb{Z}[\text{ irred. subvarieties of X }].$$

One would like to have an intersection product on cycles, informally taking a pair (Z_1, Z_2) to $Z_1 \cap Z_2$. This turns out to work up to rational equivalence. One therefore defines the *Chow ring*

$$A(X) = Z(X)/\text{rational equivalence}$$

which may perhaps be thought of as the algebraic analog of singular (co)homology (note that $A(X)$ is graded by codimension, and in particular trivial in degrees greater than the dimension of X). The association $X \mapsto A(X)$ is contravariantly functorial for flat maps (by taking preimages) and can be given a covariant functoriality for proper maps, essentially by taking the image of a subvariety if the dimension of the image does not drop (multiplied with the degree of the resulting extensions) or taking zero if the dimension drops. Grothendieck then proved the following result about the relation between $A(X)$ and $K_0(X)$: He constructed an explicit isomorphism of rings

$$A(X) \otimes \mathbb{Q} \cong K_0(X) \otimes \mathbb{Q}$$

and obtains a Chern character $\mathrm{ch}(-)$ as the composition

$$K_0(X) \to K_0(X) \otimes \mathbb{Q} \cong A(X) \otimes \mathbb{Q}.$$

This is very much in analogy with the situation in algebraic topology, where one can construct an isomorphism

$$\bigoplus_{n\geq 0} H^{2n}(X;\mathbb{Q}) \simeq \mathrm{KU}^0(X) \otimes \mathbb{Q}$$

inducing in the same manner the topological Chern character.

Denote now by T_f the difference $TX - f^*(TY) \in K_0(X)$; a kind of relative tangent bundle. Note that $f^*(TY)$ is even a vector bundle on X so in particular a coherent sheaf. In order to define the Chern character, Grothendieck really constructed algebraic Chern classes from which he extracts the Chern character, just like Hirzebruch did in the complex case. One can then also define a Todd class $\mathrm{td}(T_f) \in A(X)\otimes\mathbb{Q}$ via these algebraic Chern classes. Indeed, the Todd class can be defined for general coherent sheaves $\mathcal{F}$ on X, satisfies $\mathrm{td}(\mathcal{F}) = \mathrm{td}(\mathcal{F}')\cdot\mathrm{td}(\mathcal{F}'')$ for all short exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, and is of the form $1 + x$ in $A(X) \otimes \mathbb{Q}$ where x sits is positive grading (with respect to the codimension grading indicated above). Hence, $\mathrm{td}(\mathcal{F})$ is in fact invertible in the ring $A(X) \otimes \mathbb{Q}$, and so the map $\mathcal{F} \mapsto \mathrm{td}(\mathcal{F})$ extends uniquely to a group homomorphism $\mathrm{td}\colon K(X) \to [A(X) \otimes \mathbb{Q}]^\times$. If f is a smooth and proper map between smooth varieties, then T_f is in fact itself a vector bundle, the tangent bundle along the fibres of f, and $TX = T_f \oplus f^*(TY)$.

Now, Grothendieck's version of the HRR theorem, proved around 1957,[6] states that for $f\colon X \to Y$ a proper morphism between smooth varieties over a field, and any coherent sheaf $\mathcal{F}$ on X, there is the equality

$$\mathrm{ch}(f_*(\mathcal{F})) = f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(T_f)).$$

In other words, it shows $\mathrm{td}(T_f)$ is a correction term accounting for the non-commutativity of the diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f_*"'] & A(X) \otimes \mathbb{Q} \arrow[r, dashed] \arrow[d, "f_*"] & H^{2*}(X(\mathbb{C}); \mathbb{Q}) \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}"] & A(Y) \otimes \mathbb{Q} \arrow[r, dashed] & H^{2*}(Y(\mathbb{C}); \mathbb{Q}) \end{tikzcd}$$

i.e. $\mathrm{td}(T_f)$ measures the failure of the Chern character map to be compatible with the proper pushforward (here the right hand dashed maps exist if X is defined over $\mathbb{C}$).[7] Written differently, the diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{td}(TX)"] \arrow[d, "f_*"'] & A(X) \otimes \mathbb{Q} \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{td}(TY)"'] & A(Y) \otimes \mathbb{Q} \end{tikzcd}$$

commutes (this uses the usual projection formula for f_* and f^*: $f_*(a \cdot f^*(b)) = f_*(a) \cdot b$) in the Chow ring). Let us indicate that this indeed recovers the earlier results: When Y is a point, we have $K_0(Y) = \mathbb{Z}$ via the dimension, and the lower horizontal map is simply the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$. Hence the LHS of Grothendieck's version indeed becomes $\chi(X; \mathcal{F})$. Moreover, $T_f = TX$. Now, if in addition the base field is $\mathbb{C}$ and $\mathcal{F} = E$ is a holomorphic vector bundle, under the map $A(X) \otimes \mathbb{Q} \to H^{2*}(X(\mathbb{C}); \mathbb{Q})$, and the symbols $\mathrm{ch}(E)$ and $\mathrm{td}(T_f)$ give precisely the terms appearing in Hirzebruch's version of the Riemann–Roch theorem (i.e. Grothendieck's Chern character is mapped to Hirzebruch's Chern character, and similarly for the Todd class). Finally, in this situation, the map $f_*\colon A(X) \otimes \mathbb{Q} \to \mathbb{Q}$ equals the composite $A(X) \otimes \mathbb{Q} \to H^{2*}(X(\mathbb{C}); \mathbb{Q}) \to \mathbb{Q}$, where the latter map is the evaluation on the fundamental class $[X]$, so we finally arrive at Hirzebruch's formula.

There are more interesting things to say about $K_0(X) \otimes \mathbb{Q}$: It turns out that $K_0(X)$ is canonically a λ-ring and hence carries Adams operations ψ^k for all integers k. Rationally, any λ-ring Λ decomposes into "common Eigenspaces" for these Adams operations; that is, into the sum (over $i \in \mathbb{Z}$) of its subspaces $(\Lambda \otimes \mathbb{Q})_{(i)}$ where ψ^k acts via k^i for all k. Under Grothendieck's isomorphism, these recover the fact that the Chow ring is graded by codimension. Now in fact, we will see that there is a full spectrum $K(X)$ all of whose homotopy groups are interesting. Rationally, they again decompose into the common Eigenspaces for Adams operations. In 1986, Bloch developed a higher version of Chow groups and extended

[6] Grothendieck presented his version of the HRR theorem at the Arbeitstagung in Bonn in 1957 which was organized by Hirzebruch.

[7] This is a very interesting map: The Hodge conjecture is a conjecture about its image and assuming it, a further conjecture of Bloch and Beilinson implies that the map is an isomorphism if and only if $H^{p,q}(X) = 0$ for $p \neq q$.

Grothendieck's comparison between $A(X)$ and $K_0(X)$ to (rational) higher Chow and K-groups. These higher Chow groups define what is called (rational) *motivic cohomology*, so that we learn that rational motivic cohomology and rational algebraic K-theory determine each other: There is rational motivic cohomology $H^n_{\mathrm{mot}}(X;\mathbb{Q}(i))$ with Tate twist coefficients $\mathbb{Q}(i)$; This then identifies with $(K_{2i-n}(X)\otimes\mathbb{Q})_{(i)}$, the weight i part of K-theory, characterized by the property that for all k, the Adams operation ψ^k acts by multiplication by k^i. In particular, $K_0(X)\otimes\mathbb{Q}\simeq\oplus_i H^{2i}_{\mathrm{mot}}(X;\mathbb{Q}(i))\cong A(X)\otimes\mathbb{Q}$. If time permits, will discuss this λ-ring structure on $K_0(X)$ and possibly on $K_*(X)$ later in the course.

Another interesting interaction between algebraic K-theory and algebraic geometry is the sensitivity of K-theory to regularity or smoothness. An instance of this is the following result: Let X be a regular Noetherian scheme. Then the map $\mathbb{A}^1_X\to X$ induces an equivalence $K(X)\simeq K(\mathbb{A}^1_X)$ and $K_n(X)=0$ for $n<0$ (we will prove this later in this course). The following two prominent conjectures aim to convey that K-theory is an invariant very sensitive to singularities. Indeed, from the Bass–Milnor–Murthy excision sequence, it was already known that singular curves can have non-trivial K_{-1} (e.g. the nodal curve) but need not have non-trivial K_{-1} (e.g. the cuspidal curve). Moreover, in these cases K_{-1} is free abelian, and somewhat determined by "topology" and there are no non-trivial lower negative K-groups. Weibel then conjectured that this is generally so:

2.1. **Conjecture** (Weibel) Let X be a Noetherian scheme of Krull dimension d. Then $K_n(X)=0$ for $n<-d$ and $K_{-d}(X)$ can be described "topologically".[8]

When X is a variety over a field k with $\mathrm{char}(k)=0$, it was shown by Cortinas–Haesemeyer–Schlichting–Weibel [CHSW08] and by Geisser–Hesselholt [GH10] and Krishna [Kri09] for varieties over of field satisfying a strong form of resolution of singularities. Weibel's conjecture was fully resolved in work of Kerz–Strunk–Tamme [KST18] and has been extended to a regular schemes of valuative dimension d in the non-Noetherian situation [KST24].

2.2. **Conjecture** (Vorst) Let k be a field and A a k algebra essentially of finite type of Krull dimension n. If $K(A)\to K(A[X_1,\dots,X_{n+1}])$ is an equivalence, then A is regular.

Vorst showed this for $\dim(A)=1$ (1979), when $\mathrm{char}(k)=0$, it was shown by Cortinas–Haesemeyer–Weibel (2008) [CHW08], and for perfect fields k with $\mathrm{char}(k)=p>0$, it was shown by Geisser–Hesselholt [GH12]. A generalisation of their result, without assuming resolution of singularities was recently proven by Kerz–Strunk–Tamme [KST21].

2.3. **Number theory.** Algebraic K-theory also has a wonderful relation to number theory. For instance, to (special values of) ζ-functions. Most of what I write here is from [Kah05] which gives a very nice overview of the relations between K-theory and number theory. Recall the Riemann ζ-function

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$

[8]Precisely, $K_{-d}(X)$ is conjecturally given by $H^d_{\mathrm{cdh}}(X;\mathbb{Z})$ is given by a sheaf cohomology group, where the topology is a certain *completely decomposed* topology on schemes introduced by Voevodsky in his work on motivic cohomology.

for $s \in \mathbb{C}$. This is the special case of the arithmetic ζ-function associated to schemes X which are of finite type over $\mathrm{Spec}(\mathbb{Z})$, the Riemann ζ-function being the case of $X = \mathrm{Spec}(\mathbb{Z})$ itself:

$$\zeta_X(s) = \prod_{\substack{x \in X \\ \text{closed point}}} \frac{1}{1 - |\kappa(x)|^{-s}}$$

where $\kappa(x)$ is the residue field of X at the closed point x. The function $\zeta_X(-)$ converges for $\mathrm{Re}(s) > \dim(X)$ and is conjectured to have a meromorphic continuation to the whole complex plane; this is known at least when $\mathrm{Re}(s) > \dim(X) - \frac{1}{2}$. Soulé conjectures (in particular) the following.

2.3. **Conjecture** (Soulé) Let X be regular and of finite type over $\mathbb{Z}$ of dimension d and let $n \in \mathbb{Z}$. Then

$$\begin{aligned} \mathrm{ord}_{s=n}\zeta_X(s) &= \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}} (K_i(X) \otimes \mathbb{Q})_{(d-n)} \\ &= \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}} H^{2(d-n)-i}_{\mathrm{mot}}(X; \mathbb{Q}(d-n)). \end{aligned}$$

where the subscript denotes the weight $(d-n)$ part of the Adams operation decomposition, that is, where all Adams operations ψ^k act by multiplication by k^{d-n}, and hence the appropriate rational motivic cohomology group by what we have indicated above.[9]

In particular, this assumes that the meromorphic continuation exists, and that the dimensions appearing on the right hand side are all finite and almost surely zero. See Conjecture 2.14 below for further conjectures about finiteness of K-groups.

Soulé's conjecture is known to hold for $n > \dim(X)$ – in that case both sides vanish. For $n = \dim(X) - 1$, it implies the famous conjecture of Birch and Swinnerton-Dyer (which asserts that the rank of the K-points of an elliptic curve E agrees with the order of the zero of $L(E, s)$ at $s = 1$, where $L(E, s)$ is the Hasse-Weil L-function of E).

In order to appreciate the next conjecture, it is worthwhile to spell out Soulés conjecture for $\mathrm{Spec}(\mathcal{O}_F)$ for number fields F: Here, it turns out that the Adams Eigenspaces are explicitly known: For $i \geq 1$ we have $(K_*(\mathcal{O}_F) \otimes \mathbb{Q})_{(i)} = K_{2i-1}(\mathcal{O}_F)$; and $(K_0(\mathcal{O}_F) \otimes \mathbb{Q})_{(0)} = K_0(\mathcal{O}_F) \otimes \mathbb{Q}$. Hence we obtain for

$$\mathrm{ord}_{s=n}\zeta_F(s) = \begin{cases} 0 & \text{for } n \geq 2 \\ -1 & \text{for } n = 1 \\ \dim_{\mathbb{Q}}(K_{1-2n}(\mathcal{O}_F) \otimes \mathbb{Q}) & \text{for } n \leq 0 \end{cases}$$

As indicated above, the ranks of the K-groups have been computed by Borel, so the right hand side is known explicitly. Moreover, the ζ-function (in particular of a number field) satisfies a functional equation, relating $\zeta_F(s)$ with $\zeta_F(1-s)$ (involving so-called Γ-factors,

[9] This might actually not be what Soulé conjectures; He has a more general version for arbitrary (possibly non-regular) schemes of finite type over $\mathbb{Z}$, where one replaces $K_i(X)$ by $G_i(X)$, i.e. the K-theory of coherent sheaves, not of vector bundles. Consequently, he works with Adams operations on G-theory, which rationally identify with Borel–Moore motivic *homology* rather than motivic cohomology: $(G_i(X) \otimes \mathbb{Q})_{(j)} \cong H^{\mathrm{BM,mot}}_{2j+i}(X; \mathbb{Q}(j))$. I am then alluding to a possible Poincaré duality statement in motivic cohomology for regular schemes of dimension d of finite type over $\mathbb{Z}$ which might be incorrect in general – but is correct for $X = \mathrm{Spec}(\mathcal{O}_F)$ for a number ring F, this is the case we then use:.

cosinus, and sinus functions). From this functional equation, and the fact that $\zeta_F(s)$ indeed has a simple pole at $s = 1$, one can show that Soules̉ conjecture is true for $\mathrm{Spec}(\mathcal{O}_F)$.

Lichtenbaum then conjectured the following about the special values of the ζ-function at non-positive integers, i.e. the coefficient of the leading term for a Taylor expansion around a non-positive integer. Concretely, this special value at $-n$ can be computed as

$$\mathrm{sv}(\zeta_F)(-n) = \lim_{s\to -n} (s+n)^{\mathrm{ord}_{s=-n}\zeta_F(s)} \cdot \zeta_F(s)$$

and Lichtenbaum conjectures:

2.4. **Conjecture** (Lichtenbaum) Let F is a number field with ring of integers $\mathcal{O}_F$ and $n \geq 0$. Then

$$\mathrm{sv}(\zeta_K)(-n) = \pm \frac{|K_{2n}(\mathcal{O}_F)|}{|K_{2n+1}(\mathcal{O}_F)_{\mathrm{tors}}|} \cdot R_{n+1}(F)$$

where $R_{n+1}(F)$ is a transcendental number called the *Borel regulator*.[10]

For $n = 0$, by $K_{2n}(\mathcal{O}_F)$ we really mean the reduced K_0-group, obtained by modding out the subgroup generated by $\mathcal{O}_F$ itself, which is isomorphic to the Class group $\mathrm{Cl}(\mathcal{O}_F)$ or equivalently the Picard group $\mathrm{Pic}(\mathcal{O}_F)$. Moreover, $K_1(\mathcal{O}_F)$ is isomorphic to $\mathcal{O}_F^\times$ which is a finitely generated group of rank $r_1 + r_2 - 1$ by Dirichlet's unit theorem and the torsion elements are precisely the roots of unity $\mu(F)$ of F and therefore a cyclic group. The case $n = 0$ is therefore closely related to the class number formula discussed in a number theory course, see e.g. [Neu92, Korollar 5.11]. In loc. cit., the class number formula however relates the special value of the ζ-function at the simple pole $s = 1$ with something like the right hand side in the above equation; Using the functional equation for the ζ-function, this determines the special value at $s = 0$, and in fact, in this formulation, the formula simplifies a bit (for instance powers of 2, π, and the discriminant of F appear on the right hand side of the class number formula precisely because of the contribution coming from the functional equation); In particular, Lichtenbaum's conjecture is known for $n = 0$.

One reason to expect relations between special vaues of ζ-functions and quotients of orders of K-groups to hold is that K-groups of number rings like $\mathcal{O}_F$ tend to be describable in terms of étale cohomology groups, and relations between special values of ζ_X and étale cohomology appear for instance in work of Wiles and Mazur-Wiles. As a consequence, Lichtenbaum's conjecture is also known if F is an abelian extension of $\mathbb{Q}$, and it is also known for totally real number fields.

Let us also talk about the Kummer–Vandiver conjecture.

2.5. **Conjecture** (Kummer, Vandiver) If p is a prime number, then p does not divide the class number of the maximal real subfield $\mathbb{Q}(\zeta_p)^+$ of $\mathbb{Q}(\zeta_p)$.

Let us mention that the class number of number field F is $|\mathrm{Pic}(\mathcal{O}_F)|$ and $\mathrm{Pic}(\mathcal{O}_F) \cong \widetilde{K}_0(\mathcal{O}_F)$. The class number h of $\mathbb{Q}(\zeta_p)$ is known to be the product h_1h_2 of the class number h_1 of $\mathbb{Q}(\zeta_p)^+$ and a second number h_2; this second number h_2 is quite well understood, can be computed in terms of Bernoulli numbers and is typically quite large. It really is the other factor h_1 in the class number of $\mathbb{Q}(\zeta_p)$ that is the mysterious one. Recall also that a prime is called *regular*

[10]The sign in this conjecture can be made fully explicit: It is $(-1)^{\frac{n+1}{2}r_1+r_2}$ if n is odd and $(-1)^{\frac{n}{2}r_1}$ if n is even.

if it does not divide the class number of $\mathbb{Q}(\zeta_p)$; the first irregular prime is 37. The above conjecture is therefore true for all regular primes (it is conjectured that $\sim 60\%$ of all primes are regular) and it has been verified for all primes $p < 2^{31}$.

2.6. **Theorem** (Kurihara [Kur92]) *The Kummer–Vandiver conjecture is equivalent to the statement that $K_{4n}(\mathbb{Z}) = 0$ for all $n > 0$.*

Moreover, we know the following things about $K(\mathbb{Z})$:

(1) $K_{2k+1}(\mathbb{Z})$ is known explicitly for all $k \geq 0$.
(2) The orders of $K_{4k+2}(\mathbb{Z})$ are known explicitly for all $k \geq 0$; they are conjectured to be cyclic; this is implied by the Kummer–Vandiver conjecture, but a priori a weaker assertion.
(3) $K_4(\mathbb{Z}) = 0$ (Rognes [Rog00]) and $K_8(\mathbb{Z}) = 0$ (Kupers [Kup17]); but as of now, we do not know $K_{12}(\mathbb{Z})$.

Finally, we mention Clausen's K-theoretic approach to Artin maps [Cla17]. To that end, in class field theory, there appears for a global field F with ring of adèles $\mathbb{A}_F$, the Artin map for F: It is a homomorphism

$$\mathbb{A}_F^\times / F^\times \to \mathrm{Gal}(F)^{\mathrm{ab}}$$

where $\mathrm{Gal}(F)$ denotes the absolute Galois group of F. Similarly, there is an Artin map for a local field F, taking the form

$$F^\times \to \mathrm{Gal}(F)^{\mathrm{ab}}$$

as well as an Artin map for a finite field F, taking the form

$$\mathbb{Z} \to \mathrm{Gal}(F)^{\mathrm{ab}}.$$

The final map seems simple to define: It merely sends $1 \in \mathbb{Z}$ to the Frobenius of the finite field F. However, these Artin maps obey a certain functoriality in F, which uniquely characterises them. The fact that such a compatible, functorial set of Artin maps exists in a non-trivial result. Clausen constructs these maps via the following K-theoretic construction. Associated to a field F, (in fact more generally) he defines a category $\mathrm{LC}_F = \mathrm{Fun}_{\mathbb{Z}}(\mathrm{Perf}(F), \mathrm{Perf}(\mathrm{LCA}))$ of "$\mathrm{Perf}(F)$-modules in the derived category of (second countable) locally compact abelian groups" – whatever that is, it is something of which one can take K-theory, and considers $K(\mathrm{LC}_F)$. He shows that there are maps from the sources of all the above Artin maps to $\pi_1(K(\mathrm{LC}_F))$. On the other hand, he constructs another invariant which he calls Selmer K-homology: $dK^{\mathrm{Sel}}(F)$, which is more complicated to define at this point, as it uses on the one hand more sophisticated homotopy theory (some height one chromatic Anderson duality) as well as another invariant, called topological cyclic homology which is closely related to algebraic K-theory. He proves that $\pi_1(dK^{\mathrm{Sel}}(F)) \cong \mathrm{Gal}(F)^{\mathrm{ab}}$. Moreover, he constructs a natural map

$$K(\mathrm{LC}_F) \to dK^{\mathrm{Sel}}(F)$$

and this map induces all the above Artin maps on π_1, with its functoriality, at once. The above, I think, serves as good motivation that one also wants to study the K-theory of suitable *categories*, not "only" that of rings or schemes.

2.4. **Algebraic and geometric topology.** Algebraic K-theory also appears prominently in algebraic and geometric topology. In first instance, the relevant rings to consider are given by $\mathbb{Z}\pi_1(X)$ for X a space. For instance, suppose X is a compact anima (historically, one would say a finitely dominated space: I.e. one that is a retract up to homotopy of a finite

CW complex). Associated to such a space is an element $o(X)$ in $K_0(\mathbb{Z}\pi_1(X))$ often called the K-theory Euler characteristic of X. Indeed, under the map $K_0(\mathbb{Z}\pi_1(X)) \to K_0(\mathbb{Z}) \cong \mathbb{Z}$, $o(X)$ is sent to $\chi(X)$, the homological Euler characteristic of X. Define $\widetilde{K}_0(\mathbb{Z}\pi_1(X)) = \operatorname{coker}[K_0(\mathbb{Z}) \to K_0(\mathbb{Z}\pi_1(X))]$ and denote by $\tilde{o}(X)$ the image of $o(X)$ under the canonical projection. If X is a finite anima (i.e. can be represented by a finite CW complex), then $\tilde{o}(X)$ vanishes, one therefore refers to $\tilde{o}(X)$ as the *finiteness obstruction* of X. Wall then proved the following result.

2.7. **Theorem** (Wall) *The anima X is finite (i.e. can be represented by a finite CW complex) if and only if $\tilde{o}(X) = 0 \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$. Moreover, for every finitely presented group π, every element $\tilde{o} \in \widetilde{K}_0(\mathbb{Z}\pi)$ appears as the finiteness obstruction of a finitely dominated space X.*

Wall came to this theorem from surgery theory: In practice he was often to show the existence of a finitely dominated space (i.e. a compact anima) and would like to have in fact constructed a closed manifold. But the anima of a closed manifold is always finite[11], not only compact, and so he naturally came to study what the difference between finite and compact anima are.

Moving more towards differential topology, consider a closed smooth manifold M.[12] Let W be an h-cobordism from M to M', that is, W is a cobordism with one boundary piece identified with M (the other end we simply call M'), such that both inclusions $M \to W$ and $M' \to W$ are homotopy equivalences. Associated to this, one can associate the Whitehead torsion $\tau(W,M) \in K_1(\mathbb{Z}\pi_1(M))/\langle \pm g \rangle) = Wh(\pi_1(M))$. The following is known as the s-cobordism theorem:

2.8. **Theorem** (Smale, Barden, Mazur, Stallings) *Let M be a closed manifold of dimension ≥ 5. Then the association $(W, M, M') \mapsto \tau(W, M)$ induces a bijection between isomorphism classes of h-cobordisms W over M and $Wh(\pi_1(M))$.*

Since the cylinder $M \times [0,1]$ is an h-cobordism with trivial Whitehead torsion, the s-cobordism theorem implies that an h-cobordism (W, M, M') with trivial Whitehead torsion $\tau(W, M)$ is in fact diffeomorphic to the cylinder, and in particular, there exists a diffeomorphism $M \cong M'$.

We note that for any group π, there is a comparison map, called the *assembly map*

$$B\pi \otimes K(\mathbb{Z}) \to K(\mathbb{Z}\pi)$$

and that the groups $\widetilde{K_0}(\mathbb{Z}\pi)$ and $Wh(\pi)$ identify with the cokernel of the map induced by the assembly map on π_0 and π_1.

Waldhausen has realised that one should consider the variant where $\mathbb{Z}$ is replaced by the sphere spectrum $\mathbb{S}$ and where one uses the group in anima ΩX rather than its π_0 (which is $\pi_1(X)$). Doing so, one still obtains two maps, the latter of which is the assembly map and the former of which is induced by the unit of the ring spectrum $K(\mathbb{S})$:

$$X \otimes \mathbb{S} \to X \otimes K(\mathbb{S}) \to K(\mathbb{S}[\Omega X])$$

The cofibre of the composite is called the smooth Whitehead spectrum $\mathrm{Wh}^{\mathrm{sm}}(X)$ of X, and the cofibre of the composite is called the topological Whitehead spectrum $\mathrm{Wh}^{\mathrm{top}}(X)$ of X. One can then show that π_0 and π_1 of the two versions of Whitehead spectra agree, and that

[11]In fact, a compact ANR is a finite anima by a result of West. Topological manifolds are ANRs, so that compact manifolds are always finite anima.

[12]In fact, all I am about to say holds for topological manifolds as well.

their common π_0 and π_1 are given by $\widetilde{K}_0(\mathbb{Z}\pi_1(X))$ and $Wh(\pi_1(X))$, respectively (perhaps we will learn some of the ingredients that go into these computations this term, but perhaps also not).

He then indicated a proof of what is now called the stable parametrized s-cobordism theorem, the details of which were published in joint work of Waldhausen with Jahren and Rognes. To state it, one has to accept that there can be built a space $\mathcal{H}(M)$ of h-cobordisms over M (whose points evidently are h-cobordisms over M) which comes with stabilisation maps $\mathcal{H}(M) \to \mathcal{H}(M \times [0,1]) \to ..$ whose colimit $\mathcal{H}^s(M)$ is the stable h-cobordism space. The stable parametrized s-cobordism theorem then states:

2.9. **Theorem** *For a compact smooth/topological manifold of dimension ≥ 5, there is a canonical equivalence $\mathcal{H}^s(M) \simeq \Omega \mathrm{Wh}^{\mathrm{sm/top}}(M)$.*

The consequence that their π_0 agree then recovers the s-cobordism theorem described above. Moreover, the space $\Omega\mathcal{H}(M)$ itself is described as the stable pseudoisotopy or concordance space $C^s(M)$, and hence contains very interesting information on M-parametrized families and hence about certain automorphism groups of M, as Igusa proved that the maps $C(M) \to C(M \times [0,1]) \to \cdots \to C^s(M)$ are at least (roughly) $\dim(M)/3$-connected. Here,

$$C(M) = \{f\colon M \times [0,1] \xrightarrow{\cong} M \times [0,1] \mid f|_{M\times\{0\}\cup\partial M\times[0,1]} = \mathrm{id}\}.$$

As a consequence there is a (roughly) $\dim(M)/3$-connected map

$$C(M) \to \Omega^2 \mathrm{Wh}(M).$$

When $M = D^d$ is a disk, source and target are only interesting in the smooth case, and one obtains

$$C(D^d) \to \Omega \mathrm{fib}(\mathbb{S} \to \mathbb{K}(\mathbb{S}))$$

Rationally, it turns out that $\mathbb{S} \to \mathbb{Z}$ is an equivalence as we will learn in the Topology IV course. K-theory behaves so well in this situation that this implies $K(\mathbb{S}) \to K(\mathbb{Z})$ is also a rational equivalence, and the latter is famously calculated by Borel. Finally, there is a fibre sequence

$$\mathrm{Diff}_\partial(D^{d+1}) \to C(D^d) \to \mathrm{Diff}_\partial(D^d)$$

and it is known by work of Randal-Williams and Berglund–Madsen, that $\mathrm{Diff}_\partial(D^{2d})$ is rationally (roughly) $2d$-connected. Therefore, the map

$$\mathrm{Diff}_\partial(D^{2d+1})_{\mathbb{Q}} \to C(D^{2d})_{\mathbb{Q}} \to \Omega^2 K(\mathbb{Z})_{\mathbb{Q}}$$

is (roughly) $2d/3$-connected. Nowadays, much more is known about the rational homotopy groups of $\mathrm{Diff}_\partial(D^d)$, mainly due to work of Krannich [Kra22], Krannich–Randal-Williams [KRW21] and Kupers–Randal-Williams [KRW25].

2.5. **K-theory of group rings.** As the finiteness obstruction and the Whitehead torsion are of great geometric relevance, it makes good sense to study the group in which they live in detail. As those are controlled by the assembly map, it therefore makes sense to study the assembly map

$$BG \otimes K(R) \to K(RG)$$

for a ring R and a group G. The following conjecture is due to Farrell and Jones:

2.10. **Conjecture** (Farrell–Jones, I) Let R be a regular Noetherian ring and G be a torsion free group. Then the assembly map

$$BG \otimes K(R) \to K(RG)$$

is an equivalence.

In fact, this is just the special case of a conjecture for all rings and all groups:

2.11. **Conjecture** Let R be a ring and G be a group. Then the assembly map

$$\operatorname*{colim}_{H \in \mathrm{Orb}_{\mathrm{VCyc}}(G)} K(RH) \to K(RG)$$

is an equivalence.

Here $\mathrm{Orb}_{\mathrm{VCyc}}(G)$ denotes the full subcategory of the category of G-sets on G-sets of the form G/H where $H \subseteq G$ is a virtually cyclic group, that is, one which contains a cyclic group of finite index. They come in two families, the ones which admit a surjection onto $\mathbb{Z}$ (with finite kernel), or the ones which admit a surjection onto $D_\infty = \mathbb{Z}/2 \star \mathbb{Z}/2$ (again with finite kernel). When G is torsion free and virtually cyclic, it must therefore be isomorphic to $\mathbb{Z}$. For a regular ring R, it is a consequence of the fundamental theorem from the very beginning of this introduction, that the first version of the Farrell–Jones conjecture holds for R. It can then be shown that for R regular and G torsion free, the more sophisticated conjecture is really equivalent to the easier one. There is no counterexample known to the sophisticated Farrell–Jones conjecture, and it is known for a large class of groups. In particular, typically, for torsion free groups, the finiteness obstruction and the Whitehead torsion vanish, simply because the groups in which they live are the trivial groups.

Similarly, When M is an aspherical manifold, it follows from the Farrell–Jones conjecture that $\mathrm{Wh}^{\mathrm{top}}(M)_{\mathbb{Q}} \simeq 0$, hence one concludes information about the rational homotopy groups of the (stable) concordance space of M.

There are further interesting consequences of the Farrell–Jones conjecture that are more about the representation theory of non-finite groups: First, the comparison map

$$\operatorname*{colim}_{H \in \mathrm{Orb}_{\mathrm{Fin}}(G)} K(RH) \to \operatorname*{colim}_{H \in \mathrm{Orb}_{\mathrm{VCyc}}(G)} K(RH)$$

reducing to the orbits with *finite* stabilizers on the source, induces an isomorphism on negative homotopy groups; essentially one has to show that the result holds for virtually cyclic groups G in which case the target becomes $K(RG)$. Using the classification of virtually cyclic groups, this then follows from known long exact sequences in the algebraic K-theory of group rings of amalgamated products and semidirect products over $\mathbb{Z}$.

2.12. **Conjecture** Let G be any group. Then $K_{-n}(\mathbb{Z}G) = 0$ for $n \geq 2$ and there is an isomorphism

$$\operatorname*{colim}_{H \in \mathrm{Orb}_{\mathrm{Fin}}(G)} K_{-1}(\mathbb{Z}H) \to K_{-1}(\mathbb{Z}G).$$

The vanishing result is rather famously known for finite groups and is also true for virtually cyclic groups. The result therefore follows from the Farrell–Jones conjecture and the above comparison isomorphism in negative degrees. It should be noted that $K_{-1}(\mathbb{Z}G)$ for finite G is also classically studied in representation theory.

When the orders of finite all finite subgroups of G are invertible in a regular ring R, as is the case for $R = \mathbb{Q}$ and any group G, then one obtains:

2.13. **Conjecture** Let G be any group. Then $K_{-n}(\mathbb{Q}G) = 0$ for $n \geq 1$ and there is an isomorphism

$$\operatorname*{colim}_{H \in \mathrm{Orb}_{\mathrm{Fin}}(G)} K_0(\mathbb{Q}H) \to K_0(\mathbb{Q}G).$$

This is a reminiscent of Artin induction for finite groups; It should be noted that $K_0(\mathbb{C}G) = R_{\mathbb{C}}(G)$ is the complex representation ring.

For finite groups G, a theorem of Swan asserts that $\widetilde{K}_0(\mathbb{Z}G)$ is itself finite. But by Artin–Wedderburn, $\mathbb{Q}G$ is a product of matrix algebras over division rings D; since $K_0(-)$ commutes with finite products, swallows matrix algebras, and $K_0(D) = \mathbb{Z}$ (every module over a division ring is free), we see that $K_0(\mathbb{Q}G)$ is torsion free. In particular, the map $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$ is trivial. It was an open question whether this remains true for general groups, but this turns out not be true, a counterexample was provided by Lehner.

Understanding, for finite groups G, the groups $K_0(\mathbb{Z}G)$ is very complicated and not too much beyond the finiteness result mentioned above is known in general. The situation becomes surprisingly different for $Wh(G)$ for finite groups G, where Oliver has a program of determining the groups algorithmically.

2.6. Some prominent open problems.

2.14. **Conjecture** (Bass finiteness) Let X be of finite type over $\mathbb{Z}$. Then $G_i(X)$ is finitely generated for all $i \geq 0$. In particular, if X is regular and of finite type over $\mathbb{Z}$, then $K_i(X)$ is finitely generated for all $i \geq 0$.

We say "in particular" since under the regularity assumption $K(X) \simeq G(X)$. It turns out, however, that both of the above formulations are in fact equivalent: the regular locus of X which is of finite type over $\mathbb{Z}$ is dense and open, so one may appeal to suitable localization sequences in G-theory and Noetherian induction. This conjecture is really one of the basic and widely open problems in algebraic K-theory.

2.15. **Conjecture** (Beilinson–Soulé vanishing) For k a field and $n < 0$, $l \geq 0$, one has $(K_{2l-n}(k) \otimes \mathbb{Q})_{(l)} = 0$.

As indicated above, the K-group appearing above is equivalently given by the motivic cohomology group $H^n_{\mathrm{mot}}(k; \mathbb{Q}(l))$; Beilinson–Soulé vanishing can therefore be thought of as either a statement about K-theory or motivic cohomology; the latter is easier to remember because it says that there is no negative cohomology, something that perhaps sounds plausible. It is true that if the Beilinson–Soulé vanishing conjecture as stated above holds for all fields, then the same vanishing also holds for all regular schemes X. Moreover, it is also equivalent to the same vanishing to hold for all regular schemes X of finite type over $\mathbb{Z}$, i.e. that $H^n_{\mathrm{mot}}(X; \mathbb{Q}(l)) = 0$ for $n < 0$ and $l \geq 0$.

Because of this result, it is known that the Bass conjecture implies the Beilinson–Soulé vanishing conjecture, but it does rely on heavily non-trivial input, even for schemes of finite type over $\mathbb{Z}[\frac{1}{2}]$, namely the (resolved) Milnor conjecture and Bloch–Kato conjectures comparing mod 2 or mod l Milnor K-theory with Galois cohomology; These conjectures have been resolved by work of Voevodsky and Rost. They imply that that the K-theory of a regular scheme of finite type over $\mathbb{Z}$ admits a filtration with associated graded controlled by motivic cohomology; and eventually that 2-local motivic cohomology is finitely generated if the Bass conjecture holds. The idea is to consider the long exact sequence

$$\cdots \to H^n_{\mathrm{mot}}(X; \mathbb{Z}_{(2)}(l)) \xrightarrow{\cdot 2} H^n_{\mathrm{mot}}(X; \mathbb{Z}_{(2)}(l)) \to H^n_{\mathrm{mot}}(X; \mathbb{Z}/2(l)) \to \dots$$

and then to use the Milnor conjecture to conlude that $H^n(X; \mathbb{Z}/2(l))$ vanishes in the range of interest (this seems to be the place to use that 2 is invertible on X). But then $H^n_{\mathrm{mot}}(X; \mathbb{Z}_{(2)}(l))$ is finitely generated over $\mathbb{Z}_{(2)}$ and at the same time 2 is invertible on it, so it must be trivial.

2.16. **Conjecture** (Gersten's injectivity conjecture) Let R be a discrete valuation ring[13] with fraction field F. Then the map $K_n(R) \to K_n(F)$ is injective for all $n \geq 0$.

It turns out that this conjecture is equivalent to the same conjecture for all regular local rings R which are smooth over a DVR. Indeed, in this case, there is a localization map $R \to D$ with D a DVR and such that the map $K_n(R) \to K_n(D)$ is injective for all $n \geq 0$. Gersten's conjecture is known "with finite coefficients" and in the "equal characteristic" case, for $n = 0, 1, 2$, and in case the residue field is an algebraic extension of $\mathbb{F}_p$; Gersten injectivity is also known for (possibly non-local) Dedekind domains with global fraction field, in particular, for $\mathbb{Z}$. It is open in mixed characteristic in general, however.

3. Low algebraic K-theory

In this section, we recall the definitions of $K_0(R)$, $K_1(R)$ and $K_2(R)$ and the basic results about those. As many of the participants have seen these results in a seminar last term, I summarize only the statements and essentially leave out all the proofs. Over time, I will add proofs just so that these notes are self-contained.

3.1. **Preliminaries from algebra.** Throughout, a ring R is unital and associative, but not necessarily commutative. R-modules refer to right R-modules unless stated otherwise; we write $\mathrm{RMod}(R)$ for the category of right R-modules.

3.1. **Definition** An R-module M is called

(1) *finitely generated* or *finite* for short, if there exists an exact sequence $R^n \to M \to 0$ for some $n \geq 0$,
(2) *finitely presented*, if there exists an exact sequence $R^m \to R^n \to M \to 0$ for some $n, m \geq 0$.
(3) *coherent*, if it and all its finitely generated submodules are finitely presented.
(4) *finite projective*, if there exists another module N and an isomorphism $N \oplus M \cong R^n$ for some $n \geq 0$.
(5) *stably finite free*, if there exists an isomorphism $M \oplus R^m \cong R^n$ for some $n, m \geq 0$.

3.2. **Remark** For Noetherian rings, finite and finitely presented modules agree. A ring is called (right) coherent, if it is coherent as a module over itself. In particular, Noetherian rings are coherent. For coherent rings, finitely presented and coherent modules agree.

The category of coherent R-modules is an abelian subcategory of $\mathrm{RMod}(R)$. In particular, for a coherent ring, $\mathrm{Mod}^{\mathrm{fp}}(R)$ is an abelian subcategory, and for Noetherian R, $\mathrm{Mod}^{\mathrm{fg}}(R)$ is an abelian subcategory.

[13] i.e. a local Dedekind domain.

In general, finite projective modules are finitely presented. Finitely presented modules are precisely the compact objects in $\mathrm{RMod}(R)$. P is projective if and only if $\mathrm{Hom}_R(P,-)$ is (right) exact. Projective modules are flat. Conversely, finitely presented and flat modules are projective.

3.3. **Example** Over a PID, finitely generated projective modules are free as follows from the classification of finitely generated modules over PIDs.[14] Let K be a field and G a finite group, assume that $|G| \in K^\times$. Let V be a finite dimensional G-representation over K, i.e. a KG-module, finite dimensional over K. Then V is a projective KG-module. Moreover, by Artin–Wedderburn, we have

$$KG = \prod_{i=1}^{d} M_{n_i}(D_i)$$

where D_i are division algebras over K.

3.4. **Example** Let X be a compact Hausdorff space and $E \to X$ a vector bundle. Then $\Gamma(X;E)$ is a projective $C(X)$-module. This follows essentially from the fact that every vector-bundle over compact X embeds into a trivial bundle, that short exact sequence of vector-bundles split and that $\Gamma(X; X \times \mathbb{C}^n) \cong C(X)^n$ is free. Same holds for real vector bundles with $C(X;\mathbb{C})$ replaced by $C(X;\mathbb{R})$.

3.5. **Lemma** *Let R be commutative and $a_1, \ldots, a_n$ elements spanning the unit ideal. An R-module is finitely presented or finite if and only if for all $i = 1, \ldots, n$, the module $M[\frac{1}{a_i}]$ is finitely presented or finite over $R[\frac{1}{a_i}]$.*

Equivalently, M is finitely presented or finite if for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there is an $s \in R \setminus \mathfrak{p}$ such that $M[\frac{1}{s}]$ is a finitely presented or finite $R[\frac{1}{s}]$-module.

3.6. **Definition** Let R be a ring. Its Jacobson radical $\mathrm{Jac}(R)$ is the intersection of all maximal (right) ideals of R.

3.7. **Remark** One can show that $\mathrm{Jac}(R)$ is in fact a 2-sided ideal (see Exercise 2 Sheet 0), and equal to the intersection of all maximal left ideals of R.

3.8. **Lemma** *Let R be a ring. Then $x \in \mathrm{Jac}(R)$ iff $1 - xy$ is right-invertible for all $y \in R$ iff $Mx = 0$ for all simple[15] R-modules.*

Proof. Exercise 1 Sheet 0. □

3.9. **Example** If R is local, $\mathrm{Jac}(R) = \mathfrak{m}$ is the unique maximal ideal. If $I \subseteq R$ is an ideal such that R is I-adically complete (e.g. if I is nilpotent), then $I \subseteq \mathrm{Jac}(R)$.

3.10. **Proposition** (Nakayama's Lemma) *Let R be a ring, $J \subseteq \mathrm{Jac}(R)$ a right ideal and M a finite R-module. If $MJ = M$, then $M = 0$.*

Proof. Assume $M \neq 0$. Apply Zorn's lemma to the poset of *proper* submodules $N \subseteq M$ (for a chain of proper submodules, the union is again proper, as if it were not, then a finite set of generators must already be contained in a finite stage) to obtain a maximal proper submodule

[14] In fact, projective modules are free.

[15] A module M is called simple if its only submodules are M and 0.

$M' \subseteq M$. Then M/M' is simple so that $M/M'x = 0$ for $x \in \mathrm{Jac}(R)$, and hence $Mx \subseteq M'$. Consequently, $M = MJ \subseteq M'$, a contradiction. □

3.11. **Corollary** *Let R be a ring $J \subseteq \mathrm{Jac}(R)$ a 2-sided ideal. Let P, Q be finite projective R-modules and $g\colon P \otimes_R R/J \to Q \otimes_R R/J$ and isomorphism. Then there exists an isomorphism $f\colon P \to Q$ lifting g.*

Proof. Consider the diagram

$$\begin{tikzcd} P \arrow[r, dashed, "f"] \arrow[d] & Q \arrow[d] \\ P \otimes_R R/J \arrow[r, "g"] & Q \otimes_R R/J \end{tikzcd}$$

Since P is projective, there exists a dashed morphism f making the diagram commute. Exercise 3 of Sheet 0 implies the claim. □

3.12. **Lemma** *Let R be a ring and $I \subseteq R$ a 2-sided ideal such that R is I-adically complete. Let Q be a finite projective R/I-module. Then there exists a finite projective R-module P and an isomorphism $P \otimes_R R/I \cong Q$.*

Proof. We can write Q as the image of an idempotent element in $M_n(R/I)$. Now $M_n(I) \subseteq M_n(R)$ is an ideal such that $M_n(R)$ is $M_n(I)$-adically complete. Appealing to Exercise 4 Sheet 0 then gives the lemma. □

3.13. **Definition** Let R be a ring. We write $\mathrm{Proj}(R)$ for the (essentially small symmetric monoidal, under direct sum) category of finite projective R-modules, and $\tau_{\leq 0}\mathrm{Proj}(R)$ for the abelian monoid consisting of its isomorphism classes.

3.14. **Corollary** *Let R be a ring, $I \subseteq R$ an ideal such that R is I-adically complete. Then the canonical map*

$$\tau_{\leq 0}\mathrm{Proj}(R) \to \tau_{\leq 0}\mathrm{Proj}(R/I)$$

is a bijection.

3.15. **Remark** It is necessary to pass to isomorphism classes for this statement to hold: the map $R^\times \to (R/I)^\times$ is an isomorphism only if $I = 0$: Indeed, its kernel contains $1 + I$.

However, we see that completeness is not a necessary condition for the above corollary to be true:

3.16. **Proposition** *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Then a finite projective R-module is free.*

Proof. Let P be finite projective over R. Then $P \otimes_R R/\mathfrak{m}$ is finite free, and lifting a basis we obtain an $n \geq 0$ and a map $R^n \to P$ which becomes invertible upon applying $- \otimes_R R/\mathfrak{m}$. Appealing to Corollary 3.11, we obtain the proposition. □

3.17. **Remark** Let R be a ring and $J \subseteq \mathrm{Jac}(R)$. Then, in general, it is not true that

$$\tau_{\leq 0}\mathrm{Proj}(R) \to \tau_{\leq 0}\mathrm{Proj}(R/J)$$

is a bijection (though it is always injective). Indeed, Consider the ideal $(6) \in \mathbb{Z}$. Its complement is multiplicatively closed, so we may localize at (6) to obtain a semi-local ring $\mathbb{Z}_{(6)}$

with two maximal ideals (2) and (3). In particular $\mathrm{Jac}(\mathbb{Z}_{(6)})$ is (6). Therefore, the quotient $\mathbb{Z}_{(6)}/\mathrm{Jac}(\mathbb{Z}_{(6)})$ is simply $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ by the Chinese remainder theorem. Hence $\mathbb{Z}/2\mathbb{Z}$ is a projective module over $\mathbb{Z}_{(6)}/\mathrm{Jac}(\mathbb{Z}_{(6)})$, but it cannot be lifted to a finite projective module over $\mathbb{Z}_{(6)}$ (this is a PID, so all its finite projective modules are finite free).

3.18. **Definition** Let R be a commutative ring and M an R-module. It is called locally free if for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists $s \in R \setminus \mathfrak{p}$ such that $M[\frac{1}{s}]$ is free over $R[\frac{1}{s}]$.

3.19. **Lemma** *Let R be commutative and P an R-module. The following are equivalent.*

(1) *P is finite projective,*
(2) *P is finitely presented and $P_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$, and*
(3) *P is finite and locally free,*

Proof. (1)⇒(2) follows from Proposition 3.16 and the observation that finite projectives are finitely presented. (2)⇒(3): Lifting a basis of $P_\mathfrak{p}$, we obtain an $n \geq 0$ and a map $f \colon R^n \to P$ which becomes an isomorphism after localizing at $\mathfrak{p}$. Let K be the kernel of f and C be the cokernel of f. Then $C_\mathfrak{p} = 0$ and C is finitely generated, hence there exists $s \in R \setminus \mathfrak{p}$ such that $C[\frac{1}{s}] = 0$. Similarly, $K[\frac{1}{s}]$ is finitely generated and vanishes after localization at $\mathfrak{p}$, so there exists another $t \in R \setminus \mathfrak{p}$ such that $K[\frac{1}{st}] = 0$. Hence $P[\frac{1}{st}]$ is finite free over $R[\frac{1}{st}]$. (3)⇒(1): By Lemma 3.5, P is finitely presented. Let $M \to N$ be a surjection. We want to show that $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ is also surjective. Since P is finitely presented, it is a compact R module. This implies that for every R-module V, the canonical map

$$\mathrm{Hom}_R(P, V)_\mathfrak{p} \to \mathrm{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, V_\mathfrak{p})$$

is an isomorphism. Indeed, the localization at $\mathfrak{p}$ can be written as a filtered colimit, so the compactness of P gives an isomorphism

$$\mathrm{Hom}_R(P, V)_\mathfrak{p} \to \mathrm{Hom}_R(P, V_\mathfrak{p})$$

so the claim follows by adjunction. Using this, the implication under investigation follows from the fact that $P_\mathfrak{p}$ is free. $\square$

3.20. **Remark** Let $(X, \mathcal{O}_X)$ be a scheme. A vector bundle on X is an $\mathcal{O}_X$-module $\mathcal{F}$ which is locally free in the sense that every point x has an open neighborhood U such that $\mathcal{F}_{|U} \cong \mathcal{O}_U^n$ for some n. In particular, $\mathcal{F}$ is quasi-coherent, and if $X = \mathrm{Spec}(R)$ is affine, vector bundles are locally free modules in the above sense, and hence correspond precisely to finite projective R-modules.

For non-affine schemes X and vector bundles $\mathcal{F}$, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ is typically not exact, i.e. vector bundles are not projectives in the category of quasi-coherent $\mathcal{O}_X$-modules: for$\mathcal{F} = \mathcal{O}_X$ itself, the Hom functor just becomes the global sections functor. But the higher cohomology of quasi-coherent sheaves need not vanish (look e.g. at projective space).

3.21. **Remark** Let R be commutative and P a finite projective R-module. Then one can form its rank function $\mathrm{Spec}(R) \to \mathbb{N}$, sending $\mathfrak{p}$ to $\dim_{R_\mathfrak{p}}(P_\mathfrak{p})$. This function is locally constant as a consequence of Lemma 3.19, and we say that P has constant rank if its rank function is constant. This is automatic of $\mathrm{Spec}(R)$ is connected.

3.22. **Definition** Let R be a commutative ring. A line bundle on R is a finite projective module of constant rank 1.

3.23. **Lemma** *Let R be a commutative ring. Then line bundles are precisely the $\otimes_R$-invertible R-modules.*

Proof. Let L be a projective module of constant rank 1. Let $L^\vee = \mathrm{Hom}_R(L, R)$ be its dual. We claim that the evaluation map $L \otimes_R L^\vee \to R$ is an isomorphism, showing that L is $\otimes_R$-invertible. Indeed, localizing at all primes $\mathfrak{p}$, we see that $L \otimes_R L^\vee \to R$ is surjective, so its kernel is again finitely generated and vanishes after localization at all $\mathfrak{p}$, and is hence itself trivial.

Conversely, suppose that P is an $\otimes_R$-invertible R-module, then its inverse is necessarily its dual $P^\vee$ and we find $\mathrm{Hom}_R(P, -) = P^\vee \otimes_R -$ so that P is projective and finitely presented. Moreover, for all $\mathfrak{p} \in \mathrm{Spec}(R)$, $P_\mathfrak{p}$ is again $\otimes_{R_\mathfrak{p}}$-invertible, and is hence one-dimensional. It follows that P has constant rank 1. $\square$

3.24. **Definition** Let R be a commutative ring. The set of iso classes of line bundles is denoted by $\mathrm{Pic}(R)$, called the Picard group of R. It is naturally an abelian group under $\otimes$.

3.2. $K_0(R)$**.** We come to algebraic K-theory:

3.25. **Definition** Let R be a ring. Then we denote by $K_0(R)$ the group completion $[\tau_{\leq 0}\mathrm{Proj}(R)]^{\mathrm{gp}}$ of the abelian monoid of isomorphism classes of finite projective R-modules.

3.26. **Remark** There are two standard models for M^{gp} when $(M, \oplus)$ is an abelian monoid: The brutal one, which is $\mathbb{Z}[M]/\langle \sum_i m_i = \oplus_i m_i \rangle$ and a less brutal one, which is given as follows: Consider $M \times M$ and define an equivalence relation by $(m, m') \sim (n, n')$ if there exists k such that $m + n' + k = m' + n + k$. Then the cosets form an abelian group, the inverse of $[m, m']$ being $[m', m]$, and the association $m \mapsto [m, 0]$ gives a monoid homomorphism $M \to \mathcal{M}^{\mathrm{gp}}$ which satisfies the universal property a group completion: Given a monoid homomorphism $f \colon M \to A$ where A is an abelian group, since the only way to extend f to a group homomorphism $\bar{f}$ is to set $\bar{f}(m, m') = f(m) - f(m')$, and this indeed is a group homomorphism.

The second perspective on the group completion implies that the functor $(-)^{\mathrm{gp}}$ commutes with arbitrary products of abelian monoids.

3.27. **Remark** An abelian monoid M is called cancellative, if $m + m' = m + m''$ implies that $m' = m''$. If M is cancellative, then the natural map $M \to M^{\mathrm{gp}}$ is injective (and vice versa). In the opposite direction, suppose M admits an absorbing element, i.e. an element $\infty \in M$ such that for all $m \in M$, we have $m + \infty = \infty$. Then $M^{\mathrm{gp}} = 0$.

3.28. **Lemma** *Let $R \to S$ be a ring homomorphism. It induces an additive functor* $\mathrm{Proj}(R) \to \mathrm{Proj}(S)$ *and hence a group homomorphism $K_0(R) \to K_0(S)$.*

3.29. **Remark** Let M be an (R, S)-bimodule. Then $- \otimes_R M \colon \mathrm{RMod}(R) \to \mathrm{RMod}(S)$ determines an additive functor. If the underlying S-module of M is projective, then we obtain an additive functor $- \otimes_R M \colon \mathrm{Proj}(R) \to \mathrm{Proj}(S)$ and therefore an induced map $- \otimes_R M \colon K_0(R) \to K_0(S)$.

Exercise. Show that $K_0(R \times S) \cong K_0(R) \times K_0(S)$. Hint: Show that $\tau_{\leq 0}\mathrm{Proj}(R \times S) \to \tau_{\leq 0}\mathrm{Proj}(R) \times \tau_{\leq 0}\mathrm{Proj}(S)$ is an isomorphism of abelian monoids and that in general the canonical map $(M_0 \times M_1)^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \times M_1^{\mathrm{gp}}$ is an isomorphism.

Exercise. Let K be a field and V a countably infinite dimensional K-vector space. Let $R = \mathrm{End}_K(V)$. Show that $K_0(R) = 0$.

Exercise. Let R be a ring. Show that $K_0(R) \cong K_0(M_n(R))$. Hint: Show that R^n is an $(R, M_n(R))$-bimodule which implements an equivalence of categories $\mathrm{Proj}(R) \simeq \mathrm{Proj}(M_n(R))$.

Exercise. Let R be a ring and consider the canonical ring homomorphism $R \to M_n(R)$. Compute the composite
$$K_0(R) \to K_0(M_n(R)) \cong K_0(R)$$
obtained the exercise above.

Exercise. Show that if $I \ni i \mapsto R_i$ is a filtered diagram of rings with $\mathrm{colim}_i R_i = R$, then
$$\operatorname*{colim}_i K_0(R_i) \to K_0(R)$$
is an isomorphism. Construct a ring R with $K_0(R) \cong \mathbb{Q}$. Can such a ring be commutative? Are there commutative rings with $K_0(R) = \mathbb{Z}/n$?

3.30. **Example** Let R be a ring such that finite projectives are free, with unique dimension. Then $K_0(R) = \mathbb{Z}$. Examples include division rings, PIDs, or local commutative rings, and by a theorem of Quillen and Suslin the rings $k[X_1, \ldots, X_n]$ for fields k.

3.31. **Example** Let G be a finite group, K a field, and assume $|G| \in K^\times$. Then $KG = \prod_{i=1}^d M_{n_i}(D_i)$ with D_i division K-algebras. Hence $K_0(KG) \cong \mathbb{Z}^d$. We note that KG is semisimple, i.e. Artinian with trivial Jacobson radical.

3.32. **Example** Let R be an commutative Artinian ring. Then it has finitely many prime ideals, each of which is a maximal ideal and one has $R \cong \prod_{\mathfrak{m} \in \mathrm{Spec}_{\max}(R)} R_\mathfrak{m}$. Since $R_\mathfrak{m}$ is local, we then find $K_0(R) \cong \prod_\mathfrak{m} K_0(R_\mathfrak{m}) \cong \mathbb{Z}^{|\mathrm{Spec}_{\max}(R)|}$.

3.33. **Lemma** *If R is commutative, then $\otimes_\mathbb{R}$ makes $\tau_{\leq 0}\mathrm{Proj}(R)$ into a commutative semi-ring. Hence $K_0(R)$ becomes a commutative ring. A map of commutative rings $R \to S$ induces a map of commutative rings $K_0(R) \to K_0(S)$.*

Proof. Then tensor product defines a functor
$$\mathrm{Proj}(R) \times \mathrm{Proj}(R) \to \mathrm{Proj}(R), \quad (P, Q) \mapsto P \otimes_R Q$$
which induces a map
$$\tau_{\leq 0}\mathrm{Proj}(R) \times \tau_{\leq 0}\mathrm{Proj}(R) \to \tau_{\leq 0}\mathrm{Proj}(R)$$
Using the above exercise, this induces a map
$$K_0(R) \times K_0(R) \to K_0(R)$$
and this map is readily checked to be the multiplication of a commutative ring structure on $K_0(R)$. That $K_0(R) \to K_0(S)$ is then a ring homomorphism follows by direct inspection. $\square$

3.34. **Remark** Suppose given semi-ring homomorphism $\tau_{\leq 0}\mathrm{Proj}(R) \to S$, i.e. a monoid homomorphism both for the addition and multiplication. Then the unique extension to a group homomorphism $K_0(R) \to S$ is in fact a ring homomorphism. This again follows readily from the definitions.

3.35. **Construction** Let R be a commutative ring. In Remark 3.21, we have discussed the rank function $\tau_{\leq 0}\mathrm{Proj}(R) \to C(\mathrm{Spec}(R), \mathbb{N})$ which is a monoid homomorphism. Consider then the diagram

$$\begin{tikzcd} \tau_{\leq 0}\mathrm{Proj}(R) \arrow[r, "\mathrm{rk}"] \arrow[d] & C(\mathrm{Spec}(R), \mathbb{N}) \arrow[d] \\ K_0(R) \arrow[r, dashed] & C(\mathrm{Spec}(R), \mathbb{Z}) \end{tikzcd}$$

in which the dashed arrow exists by the universal property of the group completion.

In fact, the right vertical map in the above diagram is always group completion, even more generally for any topological space X in place of $\mathrm{Spec}(R)$. Indeed, a continuous function $f\colon X \to \mathbb{Z}$ decomposes X into a disjoint union of clopen subsets $X_{<0} \amalg X_{=0} \amalg X_{>0}$, the subset where f takes negative, zero, or positive values. Consequently, f can be written as a difference of two continuous functions on X taking values in $\mathbb{N}$ – from here, the claim follows readily.

3.36. **Construction** Given a locally constant function $n\colon \mathrm{Spec}(R) \to \mathbb{N}$ we construct a finite projective R-module R^n with rank function n as follows. The image of $n\colon \mathrm{Spec}(R) \to \mathbb{N}$ consists of finitely many points in $\mathbb{N}$, since $\mathrm{Spec}(R)$ is quasi-compact and n is continuous. Therefore, the preimages of these points write $\mathrm{Spec}(R)$ as a disjoint union of open and closed subsets of $\mathrm{Spec}(R)$. Since any closed subset of $\mathrm{Spec}(R)$ is affine, we find that $R = R_1 \times \cdots \times R_k$ for some rings R_i (which need not necessarily have connected spectrum). Let $n_i = n_{|\mathrm{Spec}(R_i)}$ which is by construction constant. Then we let R^n be the product $R_1^{n_1} \times \cdots \times R_k^{n_k}$. We note that the association $n \mapsto R^n$ is a monoid homomorphism, essentially by construction.

In particular the composite

$$C(\mathrm{Spec}(R), \mathbb{N}) \xrightarrow{n \mapsto R^n} \tau_{\leq 0}\mathrm{Proj}(R) \xrightarrow{\mathrm{rk}} C(\mathrm{Spec}(R), \mathbb{N})$$

is the identity, and we may apply $(-)^{\mathrm{gp}}$, after which the composite still is the identity. Consequently, we find

3.37. **Proposition** *For a commutative ring R, we have $K_0(R) \cong C(\mathrm{Spec}(R), \mathbb{Z}) \oplus \overline{K}_0(R)$ where $\overline{K}_0(R) = \ker(\mathrm{rk}\colon K_0(R) \to C(\mathrm{Spec}(R), \mathbb{Z}))$ is an ideal.*

3.38. **Remark** Given a commutative Noetherian ring R, it can be written as a finite product $R_1 \times \cdots \times R_n$ such that all $\mathrm{Spec}(R_i)$ are (Noetherian and) connected. Therefore $K_0(R) \cong K_0(R_1) \oplus \cdots \oplus K_0(R_n)$. Since $C(\mathrm{Spec}(R_i), \mathbb{Z}) \cong \mathbb{Z}$, we then find $\overline{K}_0(R_i) \cong \widetilde{K}_0(R_i)$ and $\overline{K}_0(R) \cong \prod_{i=1}^n \widetilde{K}_0(R_i)$. A non-Noetherian ring need not be a finite product over connected rings, since its spectrum need not be Noetherian in the topological sense and hence the connected components need not be open in general. For instance $R = \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is such that $\mathrm{Spec}(R)$ is uncoutable and totally disconnected.

3.39. **Lemma** *A finite projective R module P is stably finite free if and only if $[P] = 0 \in \widetilde{K}_0(R)$.*

Proof. Recall that $[P] = 0 \in \widetilde{K}_0(R)$ is equivalent to the statement that there exists an n such that $[P] = \pm[R^n]$ in $K_0(R)$. If the sign is $+$, this in turn is equivalent to the existence of an m such that $P \oplus R^m \cong R^{n+m}$, so P is stably finite free. If the sign in $-$, it is equivalent to the existence of an m such that $P \oplus R^{n+m} \oplus R^m$, so again, P is stably finite free. $\square$

Exercise. Let TS^2 be the tangent bundle of S^2. Show that $\Gamma(TS^2; S^2)$ is a stably free $C(S^2;\mathbb{C})$-module, but it is not free.

One may still wonder when stably freeness implies actual freeness of a module. For instance, one can show that stably isomorphic line bundles are in fact isomorphic (we will perhaps do so later, using the determinant). There is the following cancellation theorem:

3.40. **Theorem** (Bass–Serre cancellation) *Let R be a Noetherian ring of Krull dimension d and P a finite projective R-module of rank $> d$.*

(Serre) *There is an isomorphism $P \cong P' \oplus R$ for some projective P'.*
(Bass) *If P is stably isomorphic to Q, the $P \cong Q$.*

Serre's part of this theorem is a reminiscent of the following topological fact: Let X be a d-dimensional CW complex and $E \to X$ a rank n $\mathbb{R}$-vector-bundle. If $n > \dim(X)$, then $E = E' \oplus \mathbb{R}$. Indeed, this follows simply from the fact that $\mathrm{BO}(n-1) \to \mathrm{BO}(n)$ has homotopy fibre S^{n-1}, so obstruction theory says that there is a lift of the classifying map $X \to \mathrm{BO}(n)$.

3.41. **Corollary** *Let R be Noetherian commutative of dimension 1 (e.g. a Dedekind domain). If $0 \neq P$ is a finite projective module, then $P \cong L \oplus R^{\mathrm{rk}(P)-1}$ where L is a line bundle.*

In fact, this line bundle can be made very explicit. To that end, we introduce the following constructions.

3.42. **Construction** Let R be a commutative ring and M an R-module. Then Σ_n acts on $M^{\otimes_R n}$. We form

(1) the nth symmetric power $\mathrm{Sym}^n_R(M) = (M^{\otimes_R n})_{\Sigma_n}$,
(2) the nth divided power $\Gamma^n_R(M) = (M^{\otimes_R} n)^{\Sigma_n}$,
(3) the nth exterior power $\Lambda^n_R(M) = (M^{\otimes_R} n)/\langle m_1 \otimes \cdots \otimes m_n \mid m_i = m_j \text{ for some } i \neq j\rangle$.

For P finite projective we have isomorphisms $\mathrm{Sym}^n_R(P)^\vee \cong \Gamma^n_R(P^\vee)$ and $\Lambda^n_R(P) \cong \Lambda^n_R(P^\vee)^\vee$, where $(-)^\vee = \mathrm{Hom}_R(-, R)$ denotes the linear dual.

3.43. **Remark** If F is free of rank d, then $\mathrm{Sym}^n_R(F)$ and $\Gamma^n_R(F)$ are free of rank $\binom{n+d-1}{n}$ and $\Lambda^n_R(F)$ is free of rank $\binom{d}{n}$. As a consequence, if P is finite projective of rank d, then $\mathrm{Sym}^n_R(P)$ and $\Gamma^n_R(P)$ are finite projective (the functors $\mathrm{Sym}^n_R(-)$, $\Gamma^n_R(-)$ and $\Lambda^n_R(-)$ preserve retracts) of rank $\binom{n+d-1}{n}$ and $\Lambda^n_R(P)$ is finite projective of rank $\binom{d}{n}$; indeed to compute the ranks, we may apply $-\otimes_\kappa$ where κ is some residue field of R.

Exercise. Set $\Lambda^*_n(M) = \bigoplus_n \Lambda^n_R(M)$ and note that this is a graded R-algebra. Show that $\Lambda^*_R(M \oplus N) \cong \Lambda^*_R(M) \otimes_R \Lambda^*(N)$.

3.44. **Definition** Let R be commutative and P finite projective of rank n. We write $\det(P)$ for $\Lambda^n_R(P)$.

3.45. **Lemma** *Let P and Q be finite projective modules of rank p and q over a commutative ring R. Then $\det(P)$ is a line bundle and $\det(P \oplus Q) \cong \det(P) \otimes_R \det(Q)$.*

Proof. By Remark 3.43, $\det(P)$ is a finite projective of rank 1 and hence a line bundle. Moreover, $P \oplus Q$ has rank $p+q$, so we get

$$\det(P \oplus Q) = \Lambda^{p+q}(P \oplus Q) = \bigoplus_{k+l=p+q} \Lambda^k_R(P) \otimes_R \Lambda^l_R(Q)$$

If $k > p$, then $\Lambda^k_R(P) = 0$, so the right most term above is non-zero precisely if $(k,l) = (p,q)$, giving the claim. □

3.46. **Remark** Consider a Noetherian commutative ring R of dimension 1 and P a finite projective with $P \cong L \oplus R^{\mathrm{rk}(P)-1}$ as in Corollary 3.41. Then we find

$$\det(P) \cong \det(L \oplus R^{\mathrm{rk}(P)-1}) \cong \det(L) \otimes_R \det(R^{\mathrm{rk}(P)-1}) \cong L$$

since $\det(L) \cong L$ and hence $\det(R^{\mathrm{rk}(P)-1}) \cong R$. In particular, the pair $(\det(P), \mathrm{rk}(P))$ determines the isomorphism class of P.

3.47. **Remark** Note that there are canonical symmetry isomorphisms $\alpha\colon P \oplus Q \cong Q \oplus P$ and $\beta\colon \det(P) \otimes_R \det(Q) \cong \det(Q) \otimes_R \det(Q)$ induced simply by "switching symbols". However, the diagram

$$\begin{tikzcd} \det(P \oplus Q) \arrow[r, "\det(\alpha)"] \arrow[d] & \det(Q \oplus P) \arrow[d] \\ \det(P) \otimes_R \det(Q) \arrow[r, "\beta"] & \det(Q) \otimes_R \det(P) \end{tikzcd}$$

in which the vertical maps are the canonical isomorphisms from Lemma 3.45, only commutes up to a sign $(-1)^{pq}$. In particular, $\det\colon (\mathrm{Proj}(R), \oplus) \to (\mathrm{Pic}(R), \otimes_R)$ is monoidal, but not *symmetric monoidal*. This can, however, be remedied as follows. Let $\mathrm{Pic}_{\mathbb{Z}}(R)$ denote the groupoid of $\mathbb{Z}$-graded $\otimes$-invertible modules over R whose objects are pairs (L, f) where L is a line bundle and $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function. The morphisms in $\mathrm{Pic}_{\mathbb{Z}}(R)$ from (L, f) to (L', f') are empty unless $f = f'$ in which case they are given by isomorphisms from L to L'. Then $\mathrm{Pic}_{\mathbb{Z}}(R)$ is canonically a symmetric monoidal groupoid, with tensor product

$$(L, f) \otimes (L', f') = (L \otimes L', f + f')$$

and symmetry isomorphism determined by $(l \otimes l') \mapsto (-1)^{f \cdot f'} l' \otimes l$. Then the association $\mathrm{Proj}(R) \to \mathrm{Pic}_{\mathbb{Z}}(R)$, sending P to $(\det(P), \mathrm{rk}(P))$ indeed becomes symmetric monoidal for the Koszul sign symmetry isomorphism on $\mathrm{Pic}_{\mathbb{Z}}(R)$.

By the universal property, we obtain a map $\det_{\mathbb{Z}}\colon K_0(R) \to \tau_{\leq 0}\mathrm{Pic}_{\mathbb{Z}}(R) \cong C(\mathrm{Spec}(R); \mathbb{Z}) \oplus \mathrm{Pic}(R)$.

3.48. **Corollary** *Let R be a commutative ring. Then the map*

$$\det_{\mathbb{Z}}\colon K_0(R) \to C(\mathrm{Spec}(R), \mathbb{Z}) \oplus \mathrm{Pic}(R)$$

is surjective. If R is Noetherian commutative of dimension 1*, it is an isomorphism.*

Proof. The map of sets $(n, L) \mapsto [R^n] + ([L] - [R])$ is a section of $\det_{\mathbb{Z}}$, showing surjectivity. When R is Noetherian commutative of dimension 1, Remark 3.46 implies that the map $\det_{\mathbb{Z}}\colon \tau_{\leq 0}\mathrm{Proj}(R) \to C(\mathrm{Spec}(R), \mathbb{N}) \oplus \mathrm{Pic}(R)$, sending P to $(\mathrm{rk}(P), \det(P))$ is an isomorphism of abelian monoids, and hence remains an isomorphism upon group completion. □

3.49. **Remark** Suppose R is a Dedekind domain (in particular $\mathrm{Spec}(R)$ is connected). One can show that every line bundle L over R is isomorphic to an invertible ideal. Hence $\mathrm{Pic}(R)$ is isomorphic to the ideal class group $\mathrm{Cl}(R)$. We therefore find $K_0(R) = \mathbb{Z} \oplus \mathrm{Cl}(R)$. For $\mathcal{O}_F$, the ring of integers in a number field F, $\mathrm{Cl}(\mathcal{O}_F)$ is known to be a finite group, as is shown in most number theory courses, see e.g. [Neu92, Theorem I.6.3] or [Sta21] for the general case where F is any global field.

3.3. $K_1(R)$**.** There are several ways of motivating the definition of K_1; we will now simply introduce it, study some basic properties about it and then motivate it in two ways in hindsight: Once via the excision exact sequence and Milnor's patching argument, and once via the group completion theorem. The latter is, in my mind, the key approach, but the former makes clear why historically it was clear what K_1 ought to be, before having known about the concept of group completing $\mathbb{E}_\infty$-monoids and how to compute their π_1.

3.50. **Definition** Let R be a ring. We define $K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}}$, where $\mathrm{GL}(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$ is the infinite general linear group of R and $(-)^{\mathrm{ab}}$ denotes the abelianization functor.

We note that, by the (1-dimensional) Hurewicz theorem, we also have $K_1(R) = H_1(\mathrm{BGL}(R); \mathbb{Z})$.

3.51. **Definition** We let $E(R) \subseteq \mathrm{GL}(R)$ be the subgroup of *elementary matrices*, which is generated by matrices $E_{i,j}(r)$ with 1's on the diagonal and precisely one (possibly) non-zero entry r at spot (i,j) for $i \neq j$.

Exercise. The elementary matrices $E_{i,j}(r)$ satisfy the following relations:
(1) $E_{i,j}(r)E_{i,j}(r') = E_{i,k}(r + r')$
(2) $[E_{i,j}(r), E_{j,k}(r')] = E_{i,k}(rr')$, if $i \neq k$ and
(3) $[E_{i,j}(r), E_{k,l}(r')] = 1$ if $i \neq l$ and $j \neq k$.

Exercise. Show that the center $C(E(R))$ of $E(R)$ is trivial. Hint: Show that if $A \in \mathrm{GL}_n(R)$ commutes with $E_n(R)$, then A must be a diagonal matrix (whose entries are in the center of R). Then deduce that no element of $E_{n-1}(R)$ is in the center of $E_n(R)$ and finally the result.

3.52. **Lemma** (Whitehead) *The commutator* $[\mathrm{GL}(R), \mathrm{GL}(R)]$ *is given by* $E(R)$ *and* $E(R)$ *is perfect.*

Proof. For the latter, it suffices to show that generators of $E(R)$ can be written as commutators of generators of $E(R)$. This follows form (2) in the above exercise. In fact, for every $n \geq 3$, this gives that $E_n(R)$ is perfect.[16] In particular we find $E(R) \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)]$. Conversely, for $A, B \in \mathrm{GL}_n(R)$, we have an equality in $\mathrm{GL}_{2n}(R)$

$$ABA^{-1}B^{-1} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \cdot \begin{pmatrix} (AB)^{-1} & 0 \\ 0 & AB \end{pmatrix}$$

In general we have for $C \in \mathrm{GL}_n(R)$ the equality in $\mathrm{GL}_{2n}(R)$:

$$\begin{aligned} &\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -C^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & C \\ -C^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} \end{aligned}$$

so it suffices to see that all the matrices appearing in the first line are elementary; see the exercises. □

3.53. **Corollary** *There is a canonical equivalence* $K_1(R) = \mathrm{GL}(R)/E(R)$.

[16] Exercise: Think about the group $E_2(R)$.

For a commutative ring, one may consider the determinant of invertible matrices as a function $\mathrm{GL}(R) \to R^\times$. Since $R^\times$ is commutative, we obtain a well-defined map $\det\colon K_1(R) \to R^\times$, which is split by the inclusion $R^\times \subseteq \mathrm{GL}(R) \to K_1(R)$.

3.54. **Definition** For a commutative ring R, we denote by $SK_1(R)$ the kernel of the determinant map $K_1(R) \to R^\times$.

Equivalently, we have $SK_1(R) = \mathrm{SL}(R)/E(R)$ since $\mathrm{SL}(R) = \ker(\det\colon \mathrm{GL}(R) \to R^\times)$.

3.55. **Example** For a euclidean domain such as a field and certain PIDs including $\mathbb{Z}$, $\mathbb{Z}[i]$ and $k[T]$ for fields k, one can show that $E(R) = \mathrm{SL}(R)$. Hence, the map $\det\colon K_1(R) \to R^\times$ is an isomorphism and $SK_1(R) = 0$. However, being euclidean is not a necessary condition for the vanishing of $SK_1(R)$ as we will see below.

3.56. **Lemma** *Let R be a semi-local*[17] *commutative ring. Then $E(R) = \mathrm{SL}(R)$, so that $SK_1(R) = 0$ and $K_1(R) = R^\times$.*

Proof. We always have $E(R) \subseteq \mathrm{SL}(R)$, so the other inclusion is to be shown. To that end, let $A \in \mathrm{SL}(R)$. Recall that $R/\mathrm{Jac}(R)$ is a (finite) product of fields, so $\mathrm{SL}(R/\mathrm{Jac}(R)) = E(R/\mathrm{Jac}(R))$ as follows from Example 3.55. Hence $\bar{A}$ is a product $\bar{E}$ of elementary matrices. One can then lift $\bar{E}$ to a product E of elementary matrices over R. Then $A \cdot E^{-1}$ lives over the identity of $R/\mathrm{Jac}(R)$, so its diagonal entries are in $1+\mathrm{Jac}(R)$ and the off-diagonal entries are in $\mathrm{Jac}(R)$. Using row and column operations, since $1 + \mathrm{Jac}(R) \in R^\times$, one can therefore transform $A \cdot E^{-1}$ into a diagonal matrix D. Since its determinant is 1, one can then finally show that D is elementary, using again that matrices of the form

$$\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}$$

are elementary. So it suffices to note that all matrices on the right hand side are elementary. It follows that A is elementary and hence the lemma. $\square$

Of course, the group of units in a commutative ring can be quite complicated (think of $\mathbb{C}^\times$). An instance where it is very well understood is for the ring of integers in a number field.

3.57. **Theorem** (Dirichlet's unit theorem) *Let F be a number field with ring of integers $\mathcal{O}_F$. Then $\mathcal{O}_F^\times \cong \mu(F) \oplus \mathbb{Z}^{r_1+r_2-1}$ where $\mu(F)$ denotes the (cyclic) group of roots of unity in F, r_1 denotes the number of real embeddings of F and r_2 the number of complex conjugate pairs of complex embeddings of F.*

In addition, regardless of whether $\mathcal{O}_F$ admits a Euclidean algorithm, there is the following theorem:

3.58. **Theorem** (Bass, Milnor, Serre) *Let F be a number field with ring of integers $\mathcal{O}_F$. Then $SK_1(\mathcal{O}_F) = 0$ so that $K_1(\mathcal{O}_F) \cong \mu(F) \oplus \mathbb{Z}^{r_1+r_2-1}$.*

3.4. $K_2(R)$**.** We now come to the original definition of $K_2(R)$ due to Milnor.

3.59. **Definition** Let R be a ring. We define its Steinberg group $\mathrm{St}(R)$ to be the group generated by symbols $e_{i,j}(r)$, for $r \in R$ and $i \neq j$ natural numbers, subject to the *Steinberg relations*

[17] That is, it has finitely many maximal ideals.

(1) $e_{i,j}(r)e_{i,j}(r') = e_{i,k}(r+r')$
(2) $[e_{i,j}(r), e_{j,k}(r')] = e_{i,k}(rr')$, if $i \neq k$ and
(3) $[e_{i,j}(r), e_{k,l}(r')] = 1$ if $i \neq l$ and $j \neq k$.

By one of the exercises, we obtain a canonical surjection $\mathrm{St}(R) \to E(R)$.

Exercise. Show that the map $R^{n-1} \to \mathrm{St}(R)$ given by

$$(r_1, \dots, r_{n-1}) \mapsto e_{1,n}(r_1)e_{2,n}(r_2)\cdots e_{n-1,n}(r_{n-1})$$

is an injective group homomorphism. Hint: Show that its composition with $\mathrm{St}(R) \to \mathrm{GL}(R)$ is injective.

3.60. **Definition** (Milnor) Let R be a ring. We define $K_2(R) = \ker(\mathrm{St}(R) \to E(R))$.

As written, it is not clear that $K_2(R)$ is an abelian group, but by construction, there is an extension of groups:

$$1 \to K_2(R) \to \mathrm{St}(R) \xrightarrow{p} E(R) \to 1$$

3.61. **Theorem** (Steinberg) *The above extension is central, in fact, $K_2(R) = C(\mathrm{St}(R))$.*

Proof. Since $\mathrm{St}(R) \to E(R)$ is surjective, $C(\mathrm{St}(R)) \to C(E(R))$ is well-defined. But $C(E(R)) = \{1\}$ so $C(\mathrm{St}(R)) \subseteq K_2(R)$. Conversely, if $x \in K_2(R)$, i.e. $p(x) = 1$, then we find for all $y \in \mathrm{St}(R)$ that $p([x,y]) = 1$. Now, write x as a product of d many generators $e_{a_i,b_i}(\gamma_i)$, and let n be larger than all numbers a_i and b_i appearing in this product. Then it follows from the defining relations (2) and (3) that for all $i < n$, the commutator $[e_{a_i,b_i}(\gamma_i), e_{i,n}(r)]$ lies in the subgroup generated by $e_{j,n}(s)$, where $j < n$ and $s \in R$. This implies that $[x, e_{i,n}(r)]$ lies in the same subgroup, and is also in the kernel of p. Indeed, recall that in general one has

$$[ab, c] = a[b,c]a^{-1}[a,c].$$

Therefore

$$[x, e_{i,n}(r)] = [e_{a,b}(\gamma)\cdot x', e_{i,n}(r)] = e_{a,b}[x', e_{i,n}(r)]e_{a,b}(-\gamma)[e_{a,b}(\gamma), e_{i,n}(r)]$$

and inductively, $[x', e_{i,n}(r)]$ is in the wanted subgroup, as is the right hand commutator term, and then also the conjugated left hand term. Finally, using the above exercise, one can show that p is injective on this subgroup, so we conclude $[x, e_{j,n}(r)] = 1$ for all $j < n$. Similarly one argues that $[x, e_{n,j}(r)] = 1$ and concludes that x commutes with all generators of $\mathrm{St}(R)$ so that x is in the center of $\mathrm{St}(R)$. As $x \in K_2(R)$ was arbitrary, we find $K_2(R) \subseteq C(\mathrm{St}(R))$ as needed. $\square$

3.62. **Proposition** *Let G be a perfect group. Then there exists an universal (more precisely initial) central extension*

$$1 \to C(\widehat{G})_{\mathrm{univ}} \to \widehat{G}_{\mathrm{univ}} \to G \to 1$$

that is, given any central extension $\widehat{G} \to G$, there is a unique map $\widehat{G}_{\mathrm{univ}} \to \widehat{G}$ commuting with the respective projections to G. We have $C(\widehat{G})_{\mathrm{univ}} = H_2(BG;\mathbb{Z})$.

Proof. A central extension of groups

$$1 \to A \to \widehat{G} \to G \to 1$$

is equivalently described by a map $f\colon BG \to K(A,2)$; see Remark 3.63 below. When G is perfect, we have $H_1(BG;\mathbb{Z}) = 0$, and $H^2(BG;A) \cong \mathrm{Hom}(H_2(BG;\mathbb{Z}), A)$ by the universal coefficient theorem. Therefore, we see that for $A = H_2(BG;\mathbb{Z})$, there is a unique lift of $\mathrm{id}_{H_2(BG;\mathbb{Z})}$ to an element $u \in H^2(BG; H_2(BG;\mathbb{Z}))$ which then by the above characterization of central extensions classifies the initial central extension $\widehat{G}_{\mathrm{univ}}$. □

3.63. **Remark** Given a cohomology class $x \in H^2(BG;A)$, represented by a map $f\colon BG \to K(A,2)$, denote by $F = \mathrm{fib}(f)$. Then $\pi_1(F)$ is the extension group corresponding to x and the sequence

$$0 \to A \to \pi_1(F) \to G \to 1$$

is the central extension associated to x.

Conversely, a central extension as above gives a fibre sequence

$$BA \to B\widehat{G} \to BG$$

and the map $B\widehat{G} \to BG$ is a simple map in the sense of [Lan24, Def. 5.13]. Then, just as in [Lan24, Cor. 5.14], this fibration deloops to a fibration

$$B\widehat{G} \to BG \xrightarrow{x} K(A,2).$$

3.64. **Remark** The above proof in fact works more generally for groups G such that G^{ab} is a free abelian group. Indeed, then the universal coefficient theorem implies that for all abelian groups A, we have $H^2(BG;A) \cong \mathrm{Hom}(H_2(BG;\mathbb{Z}), A)$ which is all that was used above.

3.65. **Remark** If $C(G) = \{1\}$, the inclusion $C(\widehat{G}_{\mathrm{univ}})/C(\widehat{G})_{\mathrm{univ}} \to C(G)$ shows that $H_2(BG;\mathbb{Z})$ is then the full center of $\widehat{G}_{\mathrm{univ}}$. In particular this applies to $G = E(R)$. It seems to be true that for all perfect groups G that $H_2(BG;\mathbb{Z})$ is all of the center of $\widehat{G}_{\mathrm{univ}}$. The above proof doesn't immediately reveal this, however (there are perfect groups with non-trivial center). In particular, I am not certain whether the same is true for groups with free abelianization.

3.66. **Proposition** *Let $\widehat{G} \to G$ be a central extension with G perfect. Then the following are equivalent:*

(1) *$\widehat{G} \to G$ is the initial central extension,*
(2) *$\widehat{G}$ is perfect, and every central extension over $\widehat{G}$ is trivial.*
(3) *$H_1(\widehat{G};\mathbb{Z}) = H_2(\widehat{G};\mathbb{Z}) = 0$.*

Proof. If $H_1(B\widehat{G};\mathbb{Z}) = 0$, we find $H^2(\widehat{G};A) = \mathrm{Hom}(H_2(B\widehat{G};\mathbb{Z}), A)$, so (2) and (3) are clearly equivalent. Now let $1 \to A \to \widehat{G} \to G \to 1$ be a central extension and consider the fibration sequence

$$B\widehat{G} \to BG \xrightarrow{e} K(A,2)$$

where e equivalently is described by a homomorphism $\varphi_e\colon H_2(BG) \to A$. We have argued earlier that the initial central extension is the one where φ_e is the identity of $H_2(BG)$. Since G is perfect and $H_3(K(A,2)) = 0$, we find from the Serre spectral sequence an exact sequence

$$A \otimes H_1(B\widehat{G}) \to H_2(B\widehat{G}) \to H_2(BG) \xrightarrow{\varphi_e} [H_2(K(A,2)) \cong A] \to H_1(B\widehat{G}) \to 0.$$

It follows that (3) is equivalent to φ_e being an isomorphism, which is as discussed earlier, equivalent to (1). □

3.67. **Theorem** (Kervaire, Steinberg) *Let R be a ring. Then*

$$1 \to K_2(R) \to \mathrm{St}(R) \to E(R) \to 1$$

is the initial central extension over $E(R)$.

Proof. It remains to prove $H_1(\mathrm{St}(R);\mathbb{Z}) = H_2(\mathrm{St}(R);\mathbb{Z}) = 0$. For the first, just as for $E(R)$, we note that the defining relation (2) implies that every generator of $\mathrm{St}(R)$ is a commutator, so $\mathrm{St}(R)$ is perfect. Then, it suffices to show that $H^2(\mathrm{St}(R);A) = 0$ or equivalently, that every central extension over $\mathrm{St}(R)$ splits. This is a purely algebraic, but arguably involved argument using the explicit presentation of $\mathrm{St}(R)$; we omit the argument here but see [Wei13, III Prop. 5.5.1]. □

3.68. **Corollary** *For every ring R, there is an isomorphism $K_2(R) \cong H_2(BE(R);\mathbb{Z})$.*

3.69. **Theorem** (Milnor) *We have $K_2(\mathbb{Z}) = \mathbb{Z}/2$.*

Let us describe the non-trivial element in $K_2(\mathbb{Z})$. To that end, consider the element $e_{1,2}(1)\cdot e_{2,1}(-1)\cdot e_{1,2}(1) \in \mathrm{St}(\mathbb{Z})$. Its image in $E(\mathbb{Z})$ is computed to be the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

From $A^4 = \mathrm{id}$, we deduce that $t = (e_{1,2}(1)\cdot e_{2,1}(-1)\cdot e_{1,2}(1))^4 \in K_2(\mathbb{Z})$. It then turns out that $t \neq 0$ and $2t = 0$, so that t generates a $\mathbb{Z}/2 \subseteq K_2(\mathbb{Z})$. This inclusion then turns out to in fact be an isomorphism.

3.70. **Remark** Some context to the above: Recall that there is a natural group homomorphism $\Sigma_\infty \to \mathrm{GL}(\mathbb{Z})$ sending a permutation to its permutation matrix. On commutators, this gives a map $A_\infty \to E(\mathbb{Z})$, this map in turn induces a map $H_2(BA_\infty;\mathbb{Z}) \to H_2(BE(\mathbb{Z});\mathbb{Z}) \cong K_2(\mathbb{Z})$. Conversely, the map $BE(\mathbb{Z}) \to \mathrm{BGL}(\mathbb{Z}) \to \mathrm{BO}$ factors through a map $BE(\mathbb{Z}) \to \mathrm{BSO}$, and hence in turn induces a further map $H_2(BE(\mathbb{Z});\mathbb{Z}) \to H_2(\mathrm{BSO};\mathbb{Z}) \cong \pi_2(\mathrm{BSO}) \cong \pi_1(\mathrm{SO}) \cong \mathbb{Z}/2\mathbb{Z}$. One can also show that $H_2(BA_\infty;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and that the composite

$$H_2(BA_\infty;\mathbb{Z}) \to H_2(BE(\mathbb{Z});\mathbb{Z}) \to H_2(\mathrm{BSO};\mathbb{Z}) \cong \mathbb{Z}/2$$

is an isomorphism.

3.71. **Theorem** (Matsumoto) *For a field k, there is a canonical map $k^\times \otimes_{\mathbb{Z}} k^\times/\langle a\otimes(1-a)\rangle \to K_2(k)$ and this map is an isomorphism.*

3.72. **Corollary** *Let F be a finite field. Then $K_2(F) = 0$.*

Proof. Recall that Matsumoto's theorem says that

$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times/\langle a \otimes 1 - a \mid a \in F^\times \setminus \{1\}\rangle.$$

Since the right hand side involves a tensor product over $\mathbb{Z}$, it is more convenient to write the group $F^\times$ additively. However, since $F^\times \subseteq F$ this is confusing, so we follow Milnor's suggestion to give a name to the isomorphism $F^\times \to K_1(F)$, say ℓ, and then write $K_2(F)$ as the quotient of $K_1(F)\otimes_{\mathbb{Z}} K_1(F)$ by the subgroup generated by $\ell(a)\otimes\ell(1-a)$ for $a \in F^\times\setminus\{1\}$. With this notation, we have $\ell(ab) = \ell(a) + \ell(b)$, as we write $K_1(F)$ additively. First, some general relations that follow in $K_2(F)$:

(1) $\ell(a) \otimes \ell(-a) = 0$,
(2) $\ell(a) \otimes \ell(a) = \ell(a) \otimes \ell(-1)$, and
(3) $\ell(a) \otimes \ell(b) = -\ell(b) \otimes \ell(a)$.

Indeed, to see (1), note the equality in $F^\times$ given by $-a = \frac{1-a}{1-a^{-1}}$. Hence we have

$$\ell(a)\otimes\ell(-a) = \ell(a)\otimes[\ell(1-a)-\ell(1-a^{-1})] = \ell(a)\otimes\ell(1-a)+\ell(a^{-1})\otimes\ell(1-a^{-1}) = 0+0 = 0$$

giving (1). Then, since $a = (-1)(-a)$, we get

$$\ell(a)\otimes\ell(a) = \ell(a)\otimes[\ell(-1)+\ell(-a)] = \ell(a)\otimes\ell(-1)+\ell(a)\otimes\ell(-a) = \ell(a)\otimes\ell(-1)$$

showing (2). Moreover, using (1) three times, we obtain (3):

$$0 = \ell(ab)\otimes\ell(-ab) = \ell(a)\otimes[\ell(-a)+\ell(b)]+\ell(b)\otimes[\ell(a)+\ell(-b)] = \ell(a)\otimes\ell(b)+\ell(b)\otimes\ell(a)$$

Now for F a finite field, we have $F^\times$ is a cyclic group, say of order $q-1$ (i.e. $F = \mathbb{F}_q$ is a finite field with $q = p^n$ elements). Pick a generator ξ. Then any two units in F are given by ξ^n and ξ^m for some n, m. Then

$$\ell(\xi^n)\otimes\ell(\xi^m) = nm\cdot[\ell(\xi)\otimes\ell(\xi)].$$

But using (3) above, we find that $\ell(\xi)\otimes\ell(\xi)$ is of order 2. Thus $K_2(F)$ is generated by an element of order 2 and is hence either cyclic of order two or trivial. In fact, we also find $\ell(1) = \ell(\xi^{q-1}) = (q-1)\ell(\xi)$, showing that

$$(q-1)\cdot\ell(\xi)\otimes\ell(\xi) = \ell(1)\otimes\ell(\xi)$$

which is the trivial element in $K_2(F)$ as we recall that $\ell(1) = 0 \in K_1(F)$. However, if q is even, then $(q-1)\equiv 1 \mod 2$, so the fact that $\ell(\xi)\otimes\ell(\xi)$ has order two implies that it vanishes. It remains to argue the case where q is odd, in which, possibly $\ell(\xi)\otimes\ell(\xi)$ is a non-trivial element of order 2 in $K_2(F)$.

Now to treat the case where q is odd, we first claim that there are elements $u, v \in nS = F^\times \setminus (F^\times)^2$ such that $1 = u+v$. Indeed, consider the set nS and the set $1-nS$. Both these sets have $(q-1)/2$ many elements and are subsets of $F\setminus\{0,1\}$ which has $q-2$ many elements. Hence the intersection is non-trivial, showing that there is a non-square u for which $1-u = v$ is also a non-square. It follows that $\ell(u)\otimes\ell(v) = \ell(u)\otimes\ell(1-u) = 0$. Now since $\xi \in F^\times$ is a generator, it is not a square. Moreover, the multiplication by squares acts transitively on the non-squares. This implies that there exists $a, b \in F^\times$ such that $a^2u = \xi = b^2v$. Furthermore, as we have already argued that $K_2(F)$ is 2-torsion, we know that for any unit a, we have $0 = 2\cdot\ell(a)\otimes\ell(v) = \ell(a^2)\otimes\ell(v)$, and similarly, $0 = \ell(u)\otimes\ell(b^2)$. Consequently, we obtain

$$0 = \ell(u)\otimes\ell(v) = \ell(a^2u)\otimes\ell(v) = \ell(a^2u)\otimes\ell(b^2v) = \ell(\xi)\otimes\ell(\xi)$$

showing that the generator of $K_2(F)$ is trivial, and hence finally that $K_2(F) = 0$ as wanted. □

3.73. **Remark** Given Milnor's result on $K_2(\mathbb{Z})$, we will later see that

$$K_2(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p \text{ prime}} \mathbb{F}_p^\times.$$

Similarly, we will see for F a finite field, $K_2(F(t))$ is a big sum over non-trivial finite abelian groups. In contrast, $K_2(\overline{\mathbb{Q}}) = 0$.

4. Definition $K(R)$ and some properties

We now move towards the K-theory space of a ring R.

4.1. **Some higher categorical background.** In the following we will use some higher algebra, that is, that is algebra in higher category theory rather than in ordinary category theory. For this, it is useful to take note of the following dictionary. We list a couple of references [Lur09, §1], [Gro20, Gep20, Lan21].

(1) The role of the category Set of sets in higher algebra is taken by the ∞-category An of anima.[18] Just as Set, this is a bicomplete (presentable) category[19], with cartesian closed symmetric monoidal structure. We may therefore consider the categories CMon(An) and CGrp(An) of commutative monoids and groups in An; these are the analogs of commutative monoids CMon and abelian groups Ab in sets as we are used to.

(2) We have argued that CGrp(An) is the analog of Ab. It turns out that CGrp(An) is a full subcategory of another category, the category Sp of *spectra* or *spectrum objects in* An. Concretely,
$$\mathrm{Sp} = \lim_n(\dots \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*)$$
so a spectrum may be thought of as a sequence of pointed spaces $\{X_n\}$ equipped with equivalences $\Omega X_n \simeq X_{n-1}$. In particular, X_0 is what is called an infinite loop space, because it is an n-fold loop space for every $n \geq 0$. For a spectrum X, one can define homotopy groups $\pi_k(X)$ for *all* $k \in \mathbb{Z}$. A spectrum is called *connective* if $\pi_k(X) = 0$ for $k < 0$. One can show that Sp is additive, in fact, it is *stable.* The tautological functor $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}_*$, sending $X = \{X_n\}_{n\geq 0}$ to X_0 therefore canonically lifts to a functor $\mathrm{Sp} \to \mathrm{CGrp(An)}$. The so-called *recognition principle* implies that the induced functor
$$\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp} \to \mathrm{CGrp(An)}$$
is an equivalence of categories (we will discuss some things about this recognition principle below). Therefore, we may think of both Sp and $\mathrm{Sp}_{\geq 0}$ as valid replacements for the ordinary category Ab of abelian groups and of the functor $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}_{(*)}$ as a forgetful functor.

(3) Apart from the analogy with ordinary category theory, spectra are relevant to algebraic topologists because the represent (co)homology theories, and in fact, every (co)homology theory is represented by a spectrum. (This is often referred to as Brown's representability theorem).

(4) In ordinary algebra, we care about (commutative) rings (amongst others). Categorically, these are formed by taking (commutative) monoids for a new symmetric monoidal structure on Ab, the tensor product, or more precisely, the tensor product $\otimes_{\mathbb{Z}}$ over $\mathbb{Z}$ for which $\mathbb{Z}$ is the unit. The left adjoint of the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$ is given by the free abelian group functor $\mathbb{Z}[-]$. This functor turns out to be symmetric monoidal for the cartesian product on Set and the tensor product on $\mathbb{Z}$; In particular, it sends (commutative) monoids and groups to (commutative) rings. In higher algebra, a similar result is true. There is a canonical symmetric monoidal structure

[18] Aka the ∞-category of spaces. It can be described as the ∞-category associated to the Quillen model structure on topological spaces, or the Kan model structure on simplicial sets, or simply as the (Dwyer-Kan) localization of the category of CW complexes at the homotopy equivalences.

[19] It is also the free cocompletion of a point, that is, for any cocomplete ∞-category $\mathcal{C}$, the evaluation at $*$-functor $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, \mathcal{C}) \to \mathcal{C}$ is an equivalence.

on Sp, denoted by $\otimes$ whose unit is called the sphere spectrum $\mathbb{S}$.[20] In analogy to the notation in ordinary algebra, we denote by $\mathbb{S}[-]\colon \mathrm{An}_{(*)} \to \mathrm{Sp}$ the left adjoint of the forgetful functor Ω^∞.[21] Objects of CAlg(Sp) then represent (coherently commutative) multiplicative cohomology theories, in particular ones that come with a graded commutative cup product, just like singular cohomology.

(5) The inclusion $\mathrm{Ab} \subseteq \mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ is canonically lax symmetric monoidal for $\otimes_\mathbb{Z}$ and $\otimes$; the inclusion $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ in fact is a symmetric monoidal subcategory. In particular, $\mathbb{S}$ is connective (this is a consequence of the computations $\pi_k(S^n) = 0$ for $k < n$). In particular, $\mathbb{Z}$ is a commutative ring spectrum, i.e. an object of $\mathrm{CAlg}(\mathrm{Sp}, \otimes)$, and therfore comes with a canonical map $\mathbb{S}$. We then think of the functor $\mathrm{Ab} \to \mathrm{Sp}$ as the restriction of scalars functor $\mathrm{Mod}(\mathbb{Z}) \to \mathrm{Mod}(\mathbb{S})$ associated to the map $\mathbb{S} \to \mathbb{Z}$. Similarly, any (commutative) ring R gives rise to a (connective) (commutative) ring spectrum again denoted by R. The cohomology theory represented by the spectrum R is singular cohomology $H^*(-;R)$ with coefficients in R.[22] For a (commutative) ring spectrum R, its graded homotopy groups $\pi_*(R)$ form a $\mathbb{Z}$-graded (commutative) ring, as the functor $\pi_*(-)\colon \mathrm{Sp} \to \mathrm{grAb}$ is naturally lax symmetric monoidal for the Koszul sign symmetric monoidal structure on grAb.

(6) The sphere spectrum $\mathbb{S}$ is *not* an Eilenberg-MacLane spectrum. Indeed $\mathrm{Ab} \subseteq \mathrm{Sp}$ is the full subcategory on spectra X having the property that $\pi_k(X) = 0$ for $k \neq 0$; the corresponding abelian group which fully characterises X is then of course $\pi_0(X)$. The fact that $\mathbb{S}$ is not Eilenberg-MacLane simply means that $\mathbb{S}$ has more non-trivial homotopy groups; these homotopy groups identify with the stable homotopy groups of spheres, and are highly non-trivial. More examples of commutative algebras in Sp which are not Eilenberg-MacLane spectra are given by real and complex topological K-theory spectra KO and KU, as well as many bordism spectra like MO, MSO,[23] MSpin, MU. There are also further examples which lie at the interface of homotopy theory and algebra, for instance the Lubin–Tate theories $E(k,\Gamma)$ where k is a perfect field of characteristic k and Γ is a fomal group (of height n) over k; These are commutative ring spectra whose homotopy ring is an even and 2-graded Laurent polynomial ring over the universal deformation ring of this formal group, which is given by $W(k)[\![u_1, \dots, u_{n-1}]\!]$. There are many more examples of commutative rings spectra; among them the algebraic K-theory spectra of commutative rings we shall define momentarily.

(7) Unlike in ordinary algebra, where monoid objects in monoids are commutative monoids (this is called the Eckmann-Hilton trick), in higher algebra, this is not the case. Rather, for each $n \geq 1 \cup \{\infty\}$, there is the notion of an $\mathbb{E}_n$-algebra in Sp, we write the category of such as $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$; then $\mathrm{Alg}(\mathrm{Sp}) = \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ and $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}) = \mathrm{CAlg}(\mathrm{Sp})$.

[20] There is a notion of pre-spectra, these are families of pointed spaces $\{X_n\}_{n\geq 0}$ equipped with maps $X_n \to \Omega X_{n+1}$ (which need not necessarily be equivalences). Then Sp forms a full subcategory of pre-spectra, and this inclusion admits a left adjoint, called *spectrification*. Examples of pre-spectra are the families $\{S^n \wedge X\}_{n\geq 0}$ with structure maps adjoint to the identity; these are called suspension spectra. For $X = S^0$, one obtains the sphere pre-spectrum, which spectrifies to the sphere spectrum $\mathbb{S}$.

[21] Classically, $\mathbb{S}[-]$ is denoted $\Sigma^\infty_{(+)}$.

[22] Classically, this spectrum is denoted by HR and is called the Eilenberg-MacLane spectrum on R.

[23] Though the underlying associative algebras of MO and $\mathrm{MSO}_{(2)}$ are what are called generalized Eilenberg–Mac Lane spectra, that is, $\mathbb{Z}$-algebra spectra.

An $\mathbb{E}_n$-algebra is informally given by n-many algebra structures which satisfy an appropriate compatibility condition. One has $\mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathrm{Sp}) = \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Sp}))$. Thus, unlike in the ordinary world, there is an infinite hierarchy between associative and commutative higher algebras.[24]

(8) Much of the usual algebra one discusses in a course on (commutative) algebra hold equally in the context of ring spectra. For instance, the notion of localising elements in a ring has an analog in ring spectra (there, "elements of R" refer to "elements of $\pi_*(R)$". For a set $S \subseteq \pi_*(R)$ of homogenous elements, a localization $R[S^{-1}]$ of R away from S is then the universal object under R in which the elements of S become invertible (as in the ordinary case, some care must be taken in the non-commutative situation). In particular, this means that the restriction map $\mathrm{Map}_{\mathrm{Alg}}(R[S^{-1}], T) \to \mathrm{Map}_{\mathrm{Alg}}(R, T)$ is the inclusion of those path components corresponding to ring maps $R \to T$ which send all elements of S to invertible elements of $\pi_*(T)$.

Having now surveyed some basic principles in higher category, let us also add some more details. First, we shall make use of the following description of ∞-categories.

4.1. **Definition** Given an ∞-category $\mathcal{C}$, let us consider the simplicial anima $\mathrm{N}(\mathcal{C})$ given by

$$[n] \mapsto \mathrm{Map}_{\mathrm{Cat}_\infty}([n], \mathcal{C}).$$

$\mathrm{N}(\mathcal{C})$ is called the Rezk nerve of $\mathcal{C}$, the formation of Rezk nerves is the right adjoint of the unique colimit preserving functor asscat: $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Cat}_\infty$ whose restriction along the Yoneda embedding gives the tautological inclusion $\Delta \subseteq \mathrm{Cat}_\infty$. This left adjoint is called the *associated category* functor.

To describe properties of the Rezk nerve functor it will be worthwhile to discuss the following properties of simplicial animae.

4.2. **Definition** Let X be a simplicial anima, i.e. an object of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) = \mathrm{sAn}$. Then X is called *Segal* if for all $[n] \in \Delta$, the maps $\rho_i\colon [1] \to [n]$ given for $1 \leq i \leq n$ by $0 \mapsto i-1$ and $1 \mapsto i$ induce equivalences

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

For a Segal anima X and $x, x' \in X_0$ we define $\mathrm{Map}_X(x, x')$ as the fibre of $X_1 \to X_0 \times X_0$ over (x, x') and obtain composition maps

$$\mathrm{Map}_X(x', x'') \times \mathrm{Map}_X(x', x) \subseteq X_1 \times_{X_0} X_1 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1$$

observe: This canonically factors through the inclusion $\mathrm{Map}_X(x'', x) \subseteq X_1$. The map $s_0\colon X_0 \to X_1$ provides for each $x \in X_0$ the identity morphism $\mathrm{id}_x \in \mathrm{Map}_X(x, x)$; check: this indeed serves as identities for the above defined composition law. For a Segal anima, we therefore think of X_n as collections of n composable morphisms in X_1. We write $\mathrm{SAn} \subseteq \mathrm{sAn}$ for the full subcategory of Segal animae.

Given a Segal anima, one defines a new simplicial anima $X^\times$ as follows: First, define $X_1^\times$ to be the collection of components in X_1 consisting of morphisms which have a left and right inverse (with respect to the just defined composition law), we call the morphisms in $X_1^\times$

[24]This might not be unfamiliar: Indeed, there is the notion of a monoidal category and the notion of a symmetric monoidal category, and these two notions have something lying strictly between them: The notion of a braided monoidal category. Indeed, the latter is simply an $\mathbb{E}_2$-algebra in the (2,1)-category Cat of ordinary categories. The fact that this is a (2,1)-category rather than an $(\infty,1)$-category is the reason that $\mathbb{E}_3$-algebras in Cat are already $\mathbb{E}_\infty$, i.e. symmetric monoidal categories.

isomorphisms. Note that $X_0 \to X_1$ factors through $X_1^\times$. For all $n \geq 1$, let $X_n^\times$ denote the collection of components of X_n such that for all $f\colon [1] \to [n]$, the image under $f^*\colon X_n \to X_1$ lands in $X_1^\times$; that is $X_n^\times$ consists of the collection of n composable isomorphisms in X_1.

A Segal anima X is called complete if the unique degeneracy $X_0 \to X_1$ induces an equivalence $X_0 \to X_1^\times$.

Exercise. The following condition on a Segal anima are equivalent.

(1) X is complete,
(2) $X^\times$ is a constant simplicial anima,
(3) the diagram

$$\begin{tikzcd} X_0 \arrow[r, "s"] \arrow[d, "\Delta"] & X_3 \arrow[d, "f\times g"] \\ X_0 \times X_0 \arrow[r, "s\times s"] & X_1 \times X_1 \end{tikzcd}$$

is a pullback, where f and g are induced by the two maps $[1] \to [3]$ in Δ given by $0 \mapsto 0$, $1 \mapsto 2$ and $0 \mapsto 1$, $1 \mapsto 3$, respectively.
(4) The map $X_0 \to \mathrm{Map}_{\mathrm{sAn}}(J, X)$ induced by $J \to *$ is an equivalence. Here, J denotes the nerve of the contractible groupoid with two elements.

4.3. **Theorem** *The Rezk nerve functor* $\mathrm{N}\colon \mathrm{Cat}_\infty \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ *is fully faithful with essential image consisting of the complete Segal anima.*[25]

We will not prove this theorem here. But, as a consequence, we therefore find an equivalence $\mathrm{N}\colon \mathrm{Cat}_\infty \simeq \mathrm{cSAn}$. Under this equivalence, we denote the left adjoint of $\mathrm{cSAn} \subseteq \mathrm{sAn}$ by comp, the *completion* functor. The inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ has left adjoint $|-|$ given by inverting all morphisms in an ∞-category. Under the equivalence $\mathrm{Cat}_\infty \simeq \mathrm{cSAn}$, the inclusion $\mathrm{An} \to \mathrm{Cat}_\infty$ becomes the inclusion via constant diagrams. It then follows that the left adjoint $|-|$ can equivalently be thought of as taking the Rezk nerve, viewing it as a simplicial object and then taking the colimit over this simplicial object (such colimits are called geometric realizations and hence written $|-|$).

Some more information is useful (we will not prove this here, though):

4.4. **Proposition** *For a Segal anima X, we have:*

(1) $\iota(\mathrm{asscat}(X)) = |X^\times|$, *where* $|-|$ *denotes the colimit over the underlying simplicial anima. In particular,* $\pi_0(X_0) \to \pi_0(\iota(\mathrm{asscat}(X)))$ *is surjective.*
(2) *For* $x, x' \in X_0$, *the map* $\mathrm{Map}_X(x, x') \to \mathrm{Map}_{\mathrm{asscat}(X)}(x, x')$ *is an equivalence.*

To move on, we also give definitions of (cartesian) monoids and commutative monoids in an ∞-category with finite products.

4.5. **Definition** (Associative monoids) Let $\mathcal{C}$ be an ∞-category with finite products. We define the category $\mathrm{Mon}(\mathcal{C})$ of monoids in $\mathcal{C}$ as $\mathrm{Fun}^{\mathrm{red,Seg}}(\Delta^{\mathrm{op}}, \mathcal{C})$. Here, the superscripts refer to *reduced* functors M, that is, those where $M_0 \simeq *$, and *Segal* as in our above definition of Segal animae, then means that $M_n \to M_1^{\times n}$ is an equivalence.

[25]This is a higher categorical version of the possibly more familiar fact that the ordinary nerve functor from categories to simplicial sets is fully faithful with explicit image.

4.6. **Remark** Being a Segal object, we may interpret M_1 as morphisms which we can compose. Hence, in this situation, composition defines a multiplication on M_1 with identity given by the image of the degeneracy $* \to M_1$. Exercise: For $\mathcal{C} = \mathrm{Set}$, the above really is an equivalent way of defining an ordinary monoid.

Exercise. Let $M \in \mathrm{Mon}(\mathrm{An})$ be a monoid. When is M complete in the sense of our earlier definitions?

4.7. **Definition** (Commutative monoids) Let $\mathcal{C}$ be an ∞-category with finite products. We define the category $\mathrm{CMon}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$ as $\mathrm{Fun}^{\mathrm{red,Seg}}(\mathrm{Fin}_p, \mathcal{C})$. Here, Fin_p denotes the category of finite sets with partially defined maps.[26] Reduced and Segal simply means that $A(\emptyset) = *$ and that for every finite set I, the collection of partially defined maps $I \supseteq \{i\} \to \{i\}$ induce equivalences $A(I) \to \prod_{i\in I} A(i)$.

4.8. **Lemma** *There is a canonical functor* $\mathrm{Cut}\colon \Delta^{\mathrm{op}} \to \mathrm{Fin}_p$ *sending* $[n]$ *to the set of Dedekind cuts, that is, the subset of* $\mathrm{Hom}([n],[1])$ *where the preimage of* 0 and 1 *are non-emtpy. A functor* $A\colon \mathrm{Fin}_p \to \mathcal{C}$ *is a commutative monoid, i.e. is reduced and Segal, if and only if its restriction* $\Delta^{\mathrm{op}} \to \mathrm{Fin}_p \to \mathcal{C}$ *is a monoid, i.e. reduced and Segal.*

Proof. $\mathrm{Hom}_\Delta(-,[1])$ is a functor $\Delta^{\mathrm{op}} \to \mathrm{Fin}$; For $\alpha\colon [m] \to [n]$, note that the induced map $\mathrm{Hom}([n],[1]) \to \mathrm{Hom}([m],[1])$ as the preimage of a non-empty subset of $[n]$ under α need not be non-empty. But one can consider the subset of $\mathrm{Cut}([n])$ where this is the case, and the taking the preimage under α defines a partially defined map $\mathrm{Cut}([n]) \to \mathrm{Cut}([m])$ as needed. Now, there is a canonical bijection $\mathrm{Cut}([n]) = \langle n\rangle = \{1,\dots,n\}$, where $j \in \langle n\rangle$ corresponds to the decomposition $[n] = \{0,\dots,n\} = \{0,\dots,j-1\} \cup \{j,\dots,n\}$. Let us then consider the Segal maps $\rho_i\colon [n] \to [1]$ in Δ. Under the Dedekind cuts functor, this gives rise to a partially defined map $\langle n\rangle \to \langle 1\rangle = \{*\}$; to see on which subset of $\langle n\rangle$ it is defined, we recall that it is defined on the subset of cuts of $[n]$ where the preimage under ρ_i is still a Dedekind cut, concretely, this means that $i-1$ and i have to be separated in the given cut of $[n]$. There is a unique such cut, and under the bijection $\mathrm{Cut}([n]) \cong \langle n\rangle$ this is the element $i \in \langle n\rangle$. It follows that the Segal maps in Δ are sent to the Segal maps in Fin_p, showing the final claim. $\square$

The functor Cut therefore induces a functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$ which we refer to as the forgetful functor (it forgets the commutative structure and remembers only the underlying associative structure of the multiplication).

4.9. **Remark** For any finite set I, we may consider the map $I \supseteq I \to *$ in Fin_p (it is in fact fully defined, not only partially so). For a commutative monoid $A \in \mathrm{CMon}(\mathcal{C})$, we obtain an induced map

$$\prod_I A(*) \xleftarrow{\cong} A(I) \to A(*)$$

where the left hand equivalence comes from the Segal condition. The map $A(I) \to A(*)$ then identifies with the multiplication map of the underlying associative monoid. In addition, we

[26] That is a morphisms from $I \to J$ consists of a pair $I_0 \subseteq I$ and a map $I_0 \to J$. An equivalent category is the category Fin_* of finite pointed sets, by sending a set I to the pointed set I_+ and a partially defined map to the same map which on the complement of the subseteq of definition takes everything to the basepoint. The inverse functor is given by taking a pointed set to the complement of the basepoint.

see that the map $I \to *$ is invariant under the permutation action of Σ_I on I. This expresses the fact that the multiplication map of a commutative monoid is in fact commutative.

4.10. **Remark** The cartesian product of finite sets and partially defined maps defines a symmetric monoidal structure on Fin_p. Under the equivalence to Fin_*, this is the smash product of pointed sets, im particular, this is not the cartesian monoidal structure. In particular, there are associated functors $\mathrm{Fin}_p \times \mathrm{Fin}_p \to \mathrm{Fin}_p$ and $\Delta^0 \to \mathrm{Fin}_p$ such that both composites $\Delta^0 \times \mathrm{Fin}_p \to \mathrm{Fin}_p$ and $\mathrm{Fin}_p \times \Delta^0 \to \mathrm{Fin}_p$ are canonically equivalent to the identity. One checks that the functor $\mathrm{Fin}_p \times \mathrm{Fin}_p \to \mathrm{Fin}_p$ sends products of Segal maps to Segal maps and deduces from this that one obtains a diagram

$$\mathrm{CMon}(\mathcal{C}) \longrightarrow \mathrm{CMon}(\mathrm{CMon}(\mathcal{C})) \rightrightarrows \mathrm{CMon}(\mathcal{C})$$

such that both composites are canonically identified with the identity functor.[27] Here, the two right maps are the two forgetful maps. It follows that the composite

$$\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathrm{CMon}(\mathcal{C})) \to \mathrm{Mon}(\mathrm{CMon}(\mathcal{C})) \to \mathrm{Mon}(\mathcal{C})$$

is the canonical forgetful functor: Indeed, unravelling the definitions, this functor is induced by the restriction along

$$\Delta^{\mathrm{op}} \times \Delta^0 \to \Delta^{\mathrm{op}} \times \mathrm{Fin}_p \to \mathrm{Fin}_p \times \mathrm{Fin}_p \to \mathrm{Fin}_p$$

which is just the canonical map $\Delta^{\mathrm{op}} \to \mathrm{Fin}_p$. In particular, the simplicial diagram in $\mathcal{C}$ given by the underlying associative monoid of a commutative monoid M in $\mathcal{C}$ is in fact a simplicial diagram in commutative monoids in $\mathcal{C}$. This will become relevant later, when comparing two a priori different definitions of K-theory anima.

4.11. **Remark** Denoting by $\mathrm{Cat} \subseteq \mathrm{Cat}_\infty$ and $\mathrm{Gpd} \subseteq \mathrm{An}$ the full subcategories on ordinary categories and ordinary groupoids, one finds that Mon(Cat) and Mon(Gpd) are the categories of monoidal categories and monoidal groupoids, respectively, and similarly that CMon(Cat) and CMon(Gpd) are the categories of symmetric monoidal categories and symmetric monoidal groupoids, respectively. Moreover, there are fully faithful inclusions $\mathrm{Mon}(\mathrm{Gpd}) \subseteq \mathrm{Mon}(\mathrm{An})$ and $\mathrm{CMon}(\mathrm{Gpd}) \subseteq \mathrm{CMon}(\mathrm{An})$.

A categorial construction we have used in the definition of $K_0(R)$ is the group completion, i.e. the fact that the inclusion $\mathrm{Ab} \subseteq \mathrm{CMon}$ has a left adjoint. The higher categorical version of this is also true, e.g. by means of the adjoint functor theorem. We will give some perspectives on it a bit later.

4.12. **Lemma** *The inclusion* $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ *(and* $\mathrm{Grp}(\mathrm{An}) \subseteq \mathrm{Mon}(\mathrm{An})$*) admits a left adjoint which we again denote by* $(-)^{\mathrm{gp}}$.

Now recall from Remark 4.11 that $(\iota\mathrm{Proj}(R), \oplus)$ is naturally a symmetric monoidal groupoid, and hence a commutative monoid in An.

4.13. **Definition** For a ring R, we denote by $K(R) \in \mathrm{CGrp}(\mathrm{An})$ the group completion $\iota\mathrm{Proj}(R)^{\mathrm{gp}}$ of the commutative monoid $\iota\mathrm{Proj}(R)$ in An and for $n \geq 0$, define $\pi_n K(R)$ as the K-groups of R.

[27] And in fact it turns out that all three functors in this diagram are equivalences.

4.14. **Example** There are many more examples of group completion that are interesting to study:

(1) Let $\mathbb{K}$ be $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider the topological category $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{K}}$. It is symmetric monoidal under direct sum. Its groupoid core is then equivalent to the space $\coprod_n \mathrm{BGL}_n(\mathbb{K})$ but $\mathrm{GL}_n(\mathbb{K})$ carries the topology induced from the euclidean norm. Concretely, these are equivalent to $\coprod_n \mathrm{BO}(n)$, $\coprod_n \mathrm{BU}(n)$ and $\coprod_n \mathrm{BSp}(n)$. We denote the group completions by ko, ku, and ksp, respectively.
(2) Consider the topological groupoid Euc whose objects are the euclidean spaces $\mathbb{R}^d$ and whose mapping spaces are given by the mapping spaces of homeomorphisms (in the compact open topology). Again, this is a symmetric monoidal topological groupoid under direct sum. Its underlying anima is equivalent to $\coprod_n \mathrm{BTop}(n)$ and we denote by ktop its group completion.
(3) Consider the symmetric monoidal ∞-groupoid Sph given by the full subgroupoid of $\mathrm{An}^{\simeq}$ on objects of the form S^{d-1}. This is symmetric monoidal under the join. Its underlying anima is given by $\coprod_n \mathrm{BG}(n)$. We denote by $\mathrm{Pic}(\mathbb{S})$ its group completion.
(4) the forgetful functors give maps

$$\mathrm{Vect}_{\mathbb{H}} \to \mathrm{Vect}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{R}} \to \mathrm{Euc} \to \mathrm{Sph}$$

which are symmetric monoida. In particular, they group complete to maps in CGrp(An):

$$\mathrm{ksp} \to \mathrm{ku} \to \mathrm{ko} \to \mathrm{ktop} \to \mathrm{Pic}(\mathbb{S}).$$

which are often referred to as J-homomorphisms.

4.15. **Remark** Recall that for R commutative, we have indicated a symmetric monoidal functor $\det\colon \mathrm{Proj}(R) \to \mathrm{Pic}_{\mathbb{Z}}(R)$ and that the latter is a symmetric monoidal groupoid whose associated commutative monoid in spaces is in fact a commutative group (such things are also called symmetric monoidal Picard groupoids). It follows that there exists a unique extension of $\det_{\mathbb{Z}}$ to a map of commutative groups in anima $\det_{\mathbb{Z}}\colon K(R) \to \mathrm{Pic}_{\mathbb{Z}}(R)$. By construction, we have $\pi_0(\mathrm{Pic}_{\mathbb{Z}}(R)) = C(\mathrm{Spec}(R), \mathbb{Z}) \oplus \mathrm{Pic}(R)$ and $\pi_1(\mathrm{Pic}_{\mathbb{Z}}(R)) = R^\times$. The map $\det_{\mathbb{Z}}$ hence induces maps

$$\pi_0 K(R) \to C(\mathrm{Spec}(R), \mathbb{Z}) \oplus \mathrm{Pic}(R), \quad \text{and } \pi_1 K(R) \to R^\times$$

which of course bears resemblence with what we have discussed earlier.

4.16. **Remark** Here are some remarks on formulas for the group completion which will serve useful. There is a functor $\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \to (\mathrm{Cat}_\infty)_*$, informally given by sending M to the ∞-category $\mathrm{B}M$ with a single object and M as endomorphisms (we give a more formal argument below). This functor has a right adjoint given by taking a pointed category $(\mathcal{C}, x)$ to the monoid $\mathrm{End}_{\mathcal{C}}(x)$. The unit of the adjunction is then the identity so B is fully faithful. By construction, this functor participates in a commutative diagram

$$\begin{tikzcd} \mathrm{Grp}(\mathrm{An}) \arrow[r] \arrow[d, hook] & \mathrm{An}_* \arrow[d, hook] \\ \mathrm{Mon}(\mathrm{An}) \arrow[r] & (\mathrm{Cat}_\infty)_* \end{tikzcd}$$

where the vertical maps are full inclusions (recall that $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ is the full subcategory on ∞-groupoids). It follows that $\mathrm{B}\colon \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*$ is fully faithful and has right adjoint given by $(X,x) \mapsto \Omega_x X$. For connected anima X, the canonical map $\mathrm{B}\Omega X \simeq X$ is an equivalence, as it is fully faithful by construction and essentially surjective by the connectivity assumption on X. Hence, B and Ω implement inverse equivalences between $\mathrm{Grp}(\mathrm{An})$ and $\mathrm{An}_*^{\geq 1}$. This is often referred to as the recognition principle for loop spaces. We obtain the following computation

$$\begin{aligned}\mathrm{Map}_{\mathrm{Grp}(\mathrm{An})}(\Omega|\mathrm{B}M|, G) &= \mathrm{Map}_{\mathrm{An}_*}(|\mathrm{B}M|, \mathrm{B}G) \\ &= \mathrm{Map}_{(\mathrm{Cat}_\infty)_*}(\mathrm{B}M, \mathrm{B}G) \\ &= \mathrm{Map}_{\mathrm{Mon}(\mathrm{An})}(M, G)\end{aligned}$$

showing that $\Omega|\mathrm{B}(-)|$ is the left adjoint to the inclusion $\mathrm{Grp}(\mathrm{An}) \subseteq \mathrm{Mon}(\mathrm{An})$ where $|-|\colon \mathrm{Cat}_\infty \to \mathrm{An}$ is the left adjoint of the inclusion.

4.17. **Remark** Both functors in the adjunction

$$\mathrm{Mon}(\mathrm{An}) \underset{i}{\overset{\Omega|\mathrm{B}(-)|}{\rightleftarrows}} \mathrm{Grp}(\mathrm{An})$$

preserves finite products and hence induces an adjunction

$$\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \underset{i}{\overset{\Omega|\mathrm{B}(-)|}{\rightleftarrows}} \mathrm{CMon}(\mathrm{Grp}(\mathrm{An}))$$

compatible with the above adjunction upon forgetting commutative monoid structures. It turns out that the forgetful map $\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \to \mathrm{CMon}(\mathrm{An})$ is an equivalence, and that, likewise, the forgetful map $\mathrm{CMon}(\mathrm{Grp}(\mathrm{An})) \to \mathrm{CMon}(\mathrm{An})$ factors through an equivalence $\mathrm{CMon}(\mathrm{Grp}(\mathrm{An})) \to \mathrm{CGrp}(\mathrm{An})$. Under these equivalences, the right adjoint functor $\mathrm{CMon}(\mathrm{Grp}(\mathrm{An})) \subseteq \mathrm{CMon}(\mathrm{Mon}(\mathrm{An}))$ identifies with the inclusion $\mathrm{CGrp}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{An})$. In particular, one obtains a commutative diagram

$$\begin{tikzcd} \mathrm{CMon}(\mathrm{An}) \arrow[r, "(-)^{\mathrm{gp}}"] \arrow[d] & \mathrm{CGrp}(\mathrm{An}) \arrow[d] \\ \mathrm{Mon}(\mathrm{An}) \arrow[r, "(-)^{\mathrm{gp}}"] & \mathrm{Grp}(\mathrm{An}) \end{tikzcd}$$

in which both functors $(-)^{\mathrm{gp}}$ can be identified with $M \mapsto \Omega|\mathrm{B}M|$.

4.18. **Remark** Let us explain the following description of $|\mathrm{B}M|$ more closely related to the definition. Namely, the anima $|\mathrm{B}M|$ can be computed as the geometric realization $|\mathrm{Bar}(M)|$ where $\mathrm{Bar}(M)$ denotes the reduced Segal anima defining the monoid M. Often, we think of a monoid M as an anima with the structure of multiplication maps encoded in the simplicial object $\mathrm{Bar}(M)$, but of course, $M = \mathrm{Bar}(M)$ are in our definitions the same objects – nevertheless, we find it useful to have a different name for the simplicial object as to not overload notation all the time. Then it turns out that $\mathrm{asscat}(\mathrm{Bar}(M)) = \mathrm{B}M$ (this is either a definition, or if one takes for granted that $\mathrm{B}(-)$ as a left adjoint exists as claimed, there is a canonical functor from right to left which is essentially surjective and fully faithful as a consequence of Proposition 4.4, so that it is an equivalence.) See Remark 4.19 for details. As discussed earlier, we then find

$$|\mathrm{Bar}(M)| = |\mathrm{asscat}(\mathrm{Bar}(M))| = |\mathrm{B}M|.$$

4.19. **Remark** In this remark, we explain in more detail the fact that the functor $\mathrm{Mon}(\mathrm{An}) \to (\mathrm{Cat}_\infty)_{*/}$, $M \mapsto \mathrm{asscat}(\mathrm{Bar}(M))$ admits a right adjoint, taking $(\mathcal{C}, x)$ to $\mathrm{End}_{\mathcal{C}}(x)$. Via Theorem 4.3, the functor is equivalently given by $\mathrm{Mon}(\mathrm{An}) \to \mathrm{cSAn}_{*/}$, $M \mapsto \mathrm{comp}(\bar{(}M))$. Since the inclusion $\mathrm{cSAn}_{*/} \subseteq \mathrm{SAn}_{*/}$ is a right adjoint (with completion as left adjoint), the desired statement is implied by the statement that the functor $\mathrm{Mon}(\mathrm{An}) \to \mathrm{SAn}_{*/}$, $M \mapsto \bar{(}M)$ admits a right adjoint. We describe the adjoint as follows: Let $(X, x) \in \mathrm{SAn}_{*/}$ and consider $X_0 \colon \Delta^{\mathrm{op}}_{\leq 0} \to \mathrm{An}$ as a functor. We may right Kan extend it as follows:

$$\begin{tikzcd} \Delta^{\mathrm{op}}_{\leq 0} \arrow[r, "X_0"] \arrow[d] & \mathrm{An} \\ \Delta^{\mathrm{op}} \arrow[ur, "RX_0"'] & \end{tikzcd}$$

and the pointwise limit formula for Kan extensions gives $(RX_0)_n = X_0^{\times n+1}$. The identity of X_0 then adjoins to a canonical map $X \to RX_0$ and the basepoint x of X can be viewed as a map $* \to X \to RX_0$. We define the putative right adjoint $R(X, x)$ by the pullback

$$\begin{tikzcd} R(X,x) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & RX_0 \end{tikzcd}$$

whose left vertical map is equipped with a splitting, making $R(X, x)$ at the very least a pointed simplicial anima. We need to check that $R(X, x)$ is in fact reduced and Segal, i.e. a monoid in anima. To that end, one checks that the three terms defining the pullback are all Segal anima, from which it follows that so is $R(X, x)$. Since the right vertical map is an equivalence on 0-simplices, so is the left vertical map, showing that $R(X, x)$ is reduced. To see that the association $(X, x) \mapsto R(X, x)$ assembles into a right adjoint of the inclusion, it then suffices to argue that for every $M \in \mathrm{Mon}(\mathrm{An})$, the map induced by the top horizontal map $R(X, x) \to X$

$$\mathrm{Map}_{\mathrm{sAn}_{*/}}(M, R(X, x)) \to \mathrm{Map}_{\mathrm{sAn}_{*/}}(M, X)$$

is an equivalence. Since the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{sAn}_{*/}}(M, R(X,x)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{sAn}_{*/}}(M, X) \arrow[d] \\ \mathrm{Map}_{\mathrm{sAn}_{*/}}(M, *) \arrow[r] & \mathrm{Map}_{\mathrm{sAn}_{*/}}(M, RX_0) \end{tikzcd}$$

is a pullback, it suffice to argue that $\mathrm{Map}_{\mathrm{sAn}_{*/}}(M, RX_0)$ is contractible. Now since RX_0 is right Kan extended, this mapping anima is equivalent to $\mathrm{Map}_{\mathrm{An}_{*/}}(*, X_0)$ which is indeed contractible. It then remains to compute $R(X, x)_1$, which is the pullback

$$\begin{tikzcd} R(X,x)_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ * \arrow[r] & X_0 \times X_0 \end{tikzcd}$$

in which the right vertical map is given by (d_1, d_0). Consequently, we find $R(X,x) = \mathrm{Map}_X(x,x)$ as claimed.

4.20. **Remark** When M is commutative (as will be the case in all our situations), the Bar construction simplifies: Indeed, for commutative M, the Bar construction $\mathrm{Bar}(M)$ canonically upgrades to a reduced Segal object in CMon(An) as noted in Remark 4.10. As such, it is the left Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq 1}$ (and also of its restriction to $\Delta^{\mathrm{op}}_{\leq 1,\mathrm{inj}}$), which is the diagram

$$M \rightleftarrows\!\!\!\!\rightarrow *$$

(or the same diagram with section removed), as one can check by a direct inspection of the slice categories $(\Delta^{\mathrm{op}}_{\leq 1})_{/[n]}$ involved in the pointwise formula for the left Kan extension that we work out on the exercise sheet.

Here is an independent way of proving that in this situation, the Bar construction implements the group completion.

On general grounds, one can then consider the décalage $\mathrm{dec}(\mathrm{Bar}(M))$ obtained by precomposing the simplicial object with the (opposite of the) functor $\Delta \to \Delta$, $[n] \mapsto [1+n]$. The maps $d_0\colon [n] \to [1+n]$ induce a natural map $\mathrm{dec}(X) \to X$ for any simplicial object, and for a Segal object X, there is then a pullback diagram

$$\begin{tikzcd}
\mathrm{const}(X_1) \arrow[r] \arrow[d] & \mathrm{dec}(X) \arrow[d] \\
\mathrm{const}(X_0) \arrow[r] & X
\end{tikzcd}$$

showing that the fibre of $\mathrm{dec}(X) \to X$ is given by $\mathrm{const}(\mathrm{fib}(X_1 \to X_0))$. Moreover, it is a general fact that $|\mathrm{dec}(X)| \simeq X_0$ (see Exercises). Hence in our case we obtain fibre sequence of simplicial objects in CMon(An)

$$\mathrm{const}(M) \to \mathrm{dec}(\mathrm{Bar}(M)) \to \mathrm{Bar}(M)$$

which gives, upon realisation, a commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & |\mathrm{Bar}(M)|
\end{tikzcd}$$

This diagram turns out to be cocartesian in CMon(An):[28] use that Bar is Kan extended from $\Delta^{\mathrm{op}}_{\leq 1,\mathrm{inj}}$ to see that its colimit is the pushout defining the internal suspension functor of CMon(An). The above diagram also gives a canonical map $M \to \Omega|\mathrm{Bar}(M)|$, which is the unit of the adjunction.[29] It is a consequence of Rezk's equifibrancy lemma, that if $M \in \mathrm{CGrp}(\mathrm{An})$, then the geometric realization preserves the above fibre sequence (see Exercise Sheets). From this, it is a formal consequence that $M \mapsto \Omega|\mathrm{Bar}(M)|$ is a group completion, i.e. a left adjoint to the inclusion $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ (Exercise Sheet).

[28] Note that this really only makes sense for commutative monoids: For associative monoids, the square is only one of anima, and is *not* cocartesian in anima.

[29] This map exists, however, for a general associative monoid.

We now set out to prove that the just defined K-groups coincide with the ones we have already defined. The first case is easy and requires almost nothing:

4.21. **Corollary** *For every ring R, we have $K_0(R) \cong \pi_0 K(R)$.*

Proof. First note that for $X \in \mathrm{An}$ there is an equivalence $\pi_0(X) \simeq \tau_{\leq 0}(X)$ where we view $\mathrm{Set} \subseteq \mathrm{An}$ as the full subcategory on the 0-truncated objects. We now show more generally that the square

$$\begin{array}{ccc} \mathrm{CMon}(\mathrm{An}) & \xrightarrow{(-)^{\mathrm{gp}}} & \mathrm{CGrp}(\mathrm{An}) \\ {\scriptstyle \tau_{\leq 0}}\downarrow & & \downarrow{\scriptstyle \tau_{\leq 0}} \\ \mathrm{CMon}(\mathrm{Set}) & \xrightarrow{(-)^{\mathrm{gp}}} & \mathrm{CGrp}(\mathrm{Set}) \end{array}$$

commutes and then simply apply this to $\iota\mathrm{Proj}(R)$. Then the composite over the lower left corner is $K_0(R)$ while the composite over the top right corner is $\pi_0 K(R)$. To see that this diagram commutes, we note that $\tau_{\leq 0}$ is the left adjoint of the fully faithful inclusion $\mathrm{Set} \to \mathrm{An}$. Hence it is equivalent to show that the induced diagram of right adjoints commutes:

$$\begin{array}{ccc} \mathrm{CMon}(\mathrm{An}) & \longleftarrow & \mathrm{CGrp}(\mathrm{An}) \\ \uparrow & & \uparrow \\ \mathrm{CMon}(\mathrm{Set}) & \longleftarrow & \mathrm{CGrp}(\mathrm{Set}) \end{array}$$

This is obvious as all functors are just the inclusions. $\square$

To show that also the next two K-groups agree with our earlier ad hoc definitions, we will need some more heavy machine.

4.2. **The group completion theorem.**

The aim of this section is to prove the group completion theorem and to use it to identify more K-groups. To state it, let us first recall some notation. For a commutative ring spectrum $A \in \mathrm{CAlg}(\mathrm{Sp})$ and $M \in \mathrm{CMon}(\mathrm{An})$, we denote by $E[M] = E \otimes \mathbb{S}[M]$ the monoid algebra over E. Note that there is a ring homormophism $\pi_0(M) \to E_0(M) = \pi_0(M \otimes E)$, this is just the map induced by the ring map $\mathbb{S}[M] \to E[M]$ on π_0.

Next, let $M \in \mathrm{CMon}(\mathrm{An})$ and for ease of notation assume that there exists an element m such that inverting m in $\pi_0(M)$ results in a group (which is then necessarily given by $\pi_0(M)^{\mathrm{gp}}$). In the case $\mathrm{Proj}(R)$, such an element is given by the π_0-class of R. Denote by $M_\infty = \mathrm{colim}_{\cdot m} M$. To see what kind of object M_∞ is, let us think of the category of left M-modules in anima, $\mathrm{Mod}_M(\mathrm{An})$, and let us read our definition of M_∞ literally: Since M is commutative, it is in particular an M-bimodule, and $\cdot m$ refers to the right multiplication map by m which is canonically left M-linear. Since colimits in $\mathrm{Mod}_M(\mathrm{An})$ are compatible with the forgetful functor $\mathrm{Mod}_M(\mathrm{An}) \to \mathrm{An}$, we find that M_∞ canonically refines to an object of $\mathrm{Mod}_M(\mathrm{An})$. We also note that M is also a commutative monoid in $\mathrm{CMon}(\mathrm{An})$ by Remark 4.10. Then $\mathrm{Mod}_M(\mathrm{CMon}(\mathrm{An})) \simeq \mathrm{CMon}(\mathrm{An})_{M/}$ as one is used to from ordinary algebra where $\mathrm{CAlg}_E = \mathrm{CAlg}_{E/}$. In particular, the map $M \to M^{\mathrm{gp}}$ of commutative monoids makes M^{gp} into an M-module as well. We then obtain the following diagram of M-modules in anima:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_\infty \arrow[d] \\ M^{\mathrm{gp}} \arrow[r, "\simeq"] & (M^{\mathrm{gp}})_\infty \end{tikzcd}$$

in which the lower horizontal map is an equivalence, as the image of m under $M \to M^{\mathrm{gp}}$ becomes invertible since M^{gp} is a group. We therefore obtain a factorization of the group completion map $M \to M^{\mathrm{gp}}$ (which is a map of commutative monoids) as a composite

$$M \to M_\infty \to M^{\mathrm{gp}}$$

where now both maps are (a priori) only maps of M-modules in anima.

4.22. **Theorem** (The group completion theorem I) *Let $M \in \mathrm{CMon}(\mathrm{An})$ and let $E \in \mathrm{CAlg}(\mathrm{Sp})$.*

(1) *Then the map $M \to M^{\mathrm{gp}}$ induces an equivalence*

$$E[M][\pi_0(M)^{-1}] \to E[M^{\mathrm{gp}}]$$

of commutative E-algebras.

(2) *The map $M_\infty \to M^{\mathrm{gp}}$ induces an isomorphism on E-homology.*

4.23. **Remark** Since we assume that everything is commutative, we obtain

$$\pi_*(E[M][\pi_0(M)^{-1}]) = E_*(M)[\pi_0(M)^{-1}]$$

where the second localization is in the purely algebraic sense. The same statements as in Theorem 4.22 and also the above isomorphism of homotopy groups, hold in fact more generally for (not necessarily commutative) monoids in An (and not necessarily commutative ring spectra E) but such that $\pi_0(M) \subseteq \pi_*(E[M])$ satisfies the left (or right) Ore condition.

Moreover, it is a formal consequence that under the assumptions of Theorem 4.22, (1) and (2) hold true upon replacing E with an arbitrary E-module spectrum A. Indeed, applying the functor $- \otimes_E A$ to the map appearing in (1), we obtain the map

$$A[M][\pi_0(M)^{-1}] \to A[M^{\mathrm{gp}}]$$

which is then again an equivalence. Similarly, we have a commutative diagram

$$\begin{tikzcd} E[M_\infty] \otimes_E A \arrow[r] \arrow[d] & E[M^{\mathrm{gp}}] \otimes_E A \arrow[d] \\ A[M_\infty] \arrow[r] & A[M^{\mathrm{gp}}] \end{tikzcd}$$

whose vertical maps are isomorphisms.

Proof of Theorem 4.22. The map appearing in (1) is obtained from the map $\mathbb{S}[M][\pi_0(M)^{-1}] \to \mathbb{S}[M^{\mathrm{gp}}]$ upon tensoring with E, since the functor $E \otimes -\colon \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{CAlg}_E(\mathrm{Sp})$ is a left adjoint. Now we simply compare universal properties: Consider a test commutative ring spectrum S, then by definition of localizations and adjunction, we find a fully faithful inclusion

$$\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[M][\pi_0(M)^{-1}], S) \subseteq \mathrm{Map}_{\mathrm{CMon(An)}}(M, \Omega^\infty_\times(S))$$

where $\Omega^\infty_\times(S)$ denotes the anima $\Omega^\infty(S)$ equipped with the commutative monoid structure coming from the multiplication on S[30] and it is given by the collection of components on

[30]Formally, the fact that $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ is symmetric monoidal gives the left adjoint $\mathbb{S}[-]\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CAlg}(\mathrm{Sp})$ whose right adjoint we denote by $\Omega^\infty_\times$.

all those morphisms $M \to \Omega^\infty_\times(S)$ which send every element of M to an invertible element of $\Omega^\infty_\times(S)$, or equivalently to $\mathrm{gl}_1(S)$, since $\mathrm{gl}_1(S)$ is obtained from $\Omega^\infty_\times(S)$ by applying the right adjoint to the inclusion $\mathrm{CGrp(An)} \subseteq \mathrm{CMon(An)}$ (which simply takes all connected components whose π_0 class is invertible). We arrive at

$$\mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[M][\pi_0(M)^{-1}], S) \simeq \mathrm{Map}_{\mathrm{CMon(An)}}(M, \mathrm{gl}_1(S)) \simeq \mathrm{Map}_{\mathrm{CMon(An)}}(M^{\mathrm{gp}}, \mathrm{gl}_1(S)).$$

Running the same argument backwards then similarly gives

$$\mathrm{Map}_{\mathrm{CMon(An)}}(M^{\mathrm{gp}}, \mathrm{gl}_1(S)) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[M^{\mathrm{gp}}], S)$$

as needed.

For (2), we now prove that the map $E[M] \to E[M_\infty]$ exhibits the target as the localization of the source at $\pi_0(M)$, so that in the commutative triangle

$$\begin{tikzcd} E[M] \arrow[r] \arrow[dr] & E[M_\infty] \arrow[d] \\ & E[M^{\mathrm{gp}}] \end{tikzcd}$$

both the horizontal and the diagonal map are localizations at $\pi_0(M)$, and hence the vertical map is an equivalence. Now, by our standing assumption, it suffices to see that this map exhibits the target as $E[M][\frac{1}{m}]$. But for this, we have to believe that the (correct version of the) usual proof for existence of localisations of commutative rings holds in the context of commutative ring spectra. Indeed, one way to proceed is to define the notion of m-local modules, and to show that the inclusion admits an adjoint which can be implemented by taking the filtered colimit over m. $\square$

As a nice application of the group completion theorem we include the following:

4.24. **Corollary** *Let M be a commutative monoid in sets, viewed as a commutative monoid in anima. Then M^{gp} is again discrete (and coincides with the usual group completion of M).*

Proof. By the above, we have $H_*(M^{\mathrm{gp}};\mathbb{Z}) \cong H_*(M;\mathbb{Z})[\pi_0(M)^{-1}]$. But $H_*(M;\mathbb{Z}) \cong H_0(M;\mathbb{Z}) = \mathbb{Z}[M]$, so the localisation at the elements of M is still just an ordinary ring, so we find that $H_n(M^{\mathrm{gp}};\mathbb{Z}) = 0$ for $n > 0$. From this we conclude that all components of M^{gp} have the same homology as a point. Since these components are simple, it follows that they are in fact contractible, so M^{gp} is again discrete. $\square$

4.25. **Example** To see concrete examples where the map $M_\infty \to M^{\mathrm{gp}}$ fails to be an equivalence, consider the symmetric monoidal groupoid $\iota\mathrm{Fin}$ of finite sets with bijections. Then we find $\iota\mathrm{Fin} = \coprod_n \mathrm{B}\Sigma_n$. We find that M_∞ is then given by $\mathbb{Z} \times \mathrm{B}\Sigma_\infty$, and the generator $1 \in B\Sigma_1$ induces the map $\Sigma_\infty \to \Sigma_\infty$ which sends a permutation ρ to $(\mathrm{id} \amalg \rho)$ using $1 + \infty = \infty$. This map is not surjective, and hence the map $\cdot 1\colon \{0\} \times \mathrm{B}\Sigma_\infty \to \{1\} \times \mathrm{B}\Sigma_\infty$ is not an equivalence.

4.26. **Example** Consider $\mathrm{Proj}(R)$ and the element R in it. Then $\mathrm{Proj}(R)_\infty$ has many components (in fact, $K_0(R)$ many) and the component of $P \in \mathrm{Proj}(R)$ is given by $\mathrm{colim}_n \mathrm{BAut}_R(P \oplus R^n)$. In particular, the component of 0 itself is given by $\mathrm{BGL}(R) = \mathrm{colim}\, \mathrm{BGL}_n(R)$. It follows that the map $\mathrm{BGL}(R) \to \Omega^\infty_0 K(R)$, where the latter denotes the component of 0 in the space $K(R)$, induces an isomorphism on every homology theory; in particular on integral ordinary homology.

4.27. **Corollary** *For a ring R, we have $K_1(R) \cong \pi_1 K(R)$.*

Proof. As discussed in Example 4.26, $\mathrm{Proj}(R)_\infty \to \mathrm{Proj}(R)^{\mathrm{gp}} = K(R)$ induces a bijection on path components, and for the component of 0, we obtain the map $\mathrm{BGL}(R) \to \Omega^\infty_0 K(R)$. It fits into a commutative diagram

$$\begin{tikzcd} \pi_1(\mathrm{BGL}(R)) \arrow[r] \arrow[d] & \pi_1(K(R)) \arrow[d, "\cong"] \\ H_1(\mathrm{BGL}(R);\mathbb{Z}) \arrow[r, "\cong"] & H_1(\Omega^\infty_0 K(R);\mathbb{Z}) \end{tikzcd}$$

where the lower horizontal map is an isomorphism again by Example 4.26. The right vertical map is also an isomorphism by the Hurewicz theorem (recall that $\pi_1 K(R)$ is abelian since $K(R)$ is a group), and the lower left corner canonically identifies by another application of the Hurewicz theorem with $\mathrm{GL}(R)^{\mathrm{ab}} = K_1(R)$. □

4.28. **Example** Let R be a ring. As discussed in Example 4.26, $\mathrm{Proj}(R)_\infty \to \mathrm{Proj}(R)^{\mathrm{gp}} = K(R)$ induces a bijection on path components, and for the component of 0, we obtain the map $\mathrm{BGL}(R) \to \Omega^\infty_0 K(R)$ which on π_1 induces the canonical map $\mathrm{GL}(R) \to \mathrm{GL}(R)^{\mathrm{ab}}$. This map is an isomorphism if and only if $E(R) = \{1\}$ which in turn is the case if and only if R is the zero ring. This shows that for $M = \mathrm{Proj}(R)$, the map $M_\infty \to M^{\mathrm{gp}}$ is (essentially) never an equivalence.

Having seen these examples, one might wonder whether M_∞ is ever equivalent to M^{gp}. This can be characterized precisely, as we show below. To formulate the result, recall first that M_∞ is an M-module. Let us say that an M-module M' is *$\pi_0(M)$-local* if for all $m \in \pi_0(M)$, the induced map $\cdot m\colon M' \to M'$ is an equivalence. Moreover, recall that part of the definition of a commutative monoid gives a Σ_n-equivariant multiplication map $M^{\times n} \to M$ of anima, or equivalently, a map $M^{\times n}_{h\Sigma_n} \to M$. Since for $m \in M$ the map $* \xrightarrow{(m,\dots,m)} M^{\times n}$ is also Σ_n-equivariant, one obtains a map $\mathrm{B}\Sigma_n = *_{h\Sigma_n} \to M^{\times n}_{h\Sigma_n} \to M$. On π_1, we this induces a map $\Sigma_n = \pi_1(\mathrm{B}\Sigma_n) \to \pi_1(M, m^n)$. With these preliminaries out of the way, we have the following second part of the group completion theorem:

4.29. **Theorem** (Group completion theorem II) *Then the following statements are equivalent:*

1. *M_∞ is $\pi_0(M)$-local,*
2. *$M \to M_\infty$ is the initial map to a $\pi_0(M)$-local space,*
3. *the map $M_\infty \to M^{\mathrm{gp}}$ is an equivalence,*
4. *for all $m \in M$, $\pi_1(M_\infty, m)$ is abelian,*
5. *for all $m \in M$, $\pi_1(M_\infty, m)$ is hypoabelian, that is, every perfect subgroup is trivial*[31]
6. *for all $m \in M$, the map $C_3 \subseteq \Sigma_3 \to \pi_1(M, m^3) \to \pi_1(M_\infty, m^3)$ is trivial,*
7. *for all $m \in M$ there is an $n \geq 2$ such that the map $C_n \to \Sigma_n \to \pi_1(M_\infty, m^n)$ is trivial.*

Proof sketch. (1)⇒(2): We need to show that for a map $M \to Y$ with Y $\pi_0(M)$-local, there exists a unique extension to M_∞. I.e. we need to show that the map

$$\mathrm{Map}_M(M_\infty, Y) \to \mathrm{Map}_M(M, Y)$$

is an equivalence. But its source is equivalent to $\mathrm{Map}_M(M, \lim_{\cdot m} Y) \simeq \mathrm{Map}_M(M, Y)$ since Y is $\pi_0(M)$-local and hence all the transition maps in the diagram computing $\lim_{\cdot m} Y$ are

[31] Equivalently the maximal perfect subgroup is trivial.

equivalences. (2)⇒(3): It is a general fact that a module localization is also the localization in CMon, i.e. that the initial map to a $\pi_0(M)$-local space is a map of commutative monoids (with target a group) and initial among such.[32] (3)⇒(4)⇒(5) are clear. (4)⇒(6): Follows since $C_3 \to \Sigma_3 \to \Sigma_3^{\mathrm{ab}} = \mathbb{Z}/2$ is trivial. (5)⇒(7): Note that $A_n \subseteq \Sigma_n$ is simple for $n \geq 5$, and hence in particular perfect. It follows that the hypoabelianization of Σ_n is $\mathbb{Z}/2$, at least when $n \geq 5$. Again, $C_n \to \Sigma_n \to \mathbb{Z}/2$ is trivial for odd n, giving the result. (7)⇒(1): we need to show that $\cdot m\colon M_\infty \to M_\infty$ is an equivalence. For this, we need to recall how to define this induced map. In fact, it will be more insightful to recall how to define the map $\cdot x\colon M_\infty \to M_\infty$ for general $x \in M$. It is induced by the following map of diagrams:

$$\begin{tikzcd} M \arrow[r, "\cdot m"] \arrow[d, "\cdot x"] & M \arrow[r, "\cdot m"] \arrow[d, "\cdot x"] & M \arrow[r] \arrow[d, "\cdot x"] & \dots \\ M \arrow[r, "\cdot m"] & M \arrow[r, "\cdot m"] & M \arrow[r] & \dots \end{tikzcd}$$

and the 2-cells witnessing commutativity come from the commutativity of the multiplication on M, that is from the general homotopy witnessing $xm \simeq mx$. Now, for $x = m$, these this is in general not the trivial homotopy witnessing the tautological equality $x^2 \simeq x^2$. Rather, The trivial homotopy, for $x = m$, would be the map inducing the identity upon passing to horizontal colimits. Now, glueing together $(n-1)$-many of the adjacent squares gives the 2-cell witnessing $m^n = m^n$ via the cycle permutation $C_n \subseteq \Sigma_n \to \pi_1(M, m^n)$. If this map becomes trivial upon mapping to M_∞, one can then deduce that the map on horizontal colimits is an equivalence as needed. □

4.30. **Example** To see that the equivalent conditions of Theorem 4.29 do happen in practice, consider rather than $\mathrm{Proj}(R)$ the commutative monoids in anima discussed in Example 4.14: There, we discussed in particular the monoids M being $\mathrm{Vect}_{\mathbb{K}}$, Euc, and Sph. In these cases, M_∞ is (as an anima) given as follows:

(1) $(\mathrm{Vect}_{\mathbb{R}})_\infty = \mathbb{Z} \times \mathrm{BO}$,
(2) $(\mathrm{Vect}_{\mathbb{C}})_\infty = \mathbb{Z} \times \mathrm{BU}$,
(3) $(\mathrm{Vect}_{\mathbb{H}})_\infty = \mathbb{Z} \times \mathrm{BSp}$,
(4) $\mathrm{Euc}_\infty = \mathbb{Z} \times \mathrm{BTop}$, and
(5) $\mathrm{Sph}_\infty = \mathbb{Z} \times \mathrm{BG}$.

Now, $\pi_1(\mathrm{BO}) = \pi_1(\mathrm{BTop}) = \pi_1(\mathrm{BG}) = \mathbb{Z}/2$ and $\pi_1(\mathrm{BU}) = \pi_1(\mathrm{BSp}) = 0$ (Exercise). In particular, all these groups are abelian. From Theorem 4.29 (4)⇒(3), we deduce the following equivalences of anima:

(1) $\mathrm{ko} \simeq \mathbb{Z} \times \mathrm{BO}$,
(2) $\mathrm{ku} \simeq \mathbb{Z} \times \mathrm{BU}$,
(3) $\mathrm{ksp} \simeq \mathbb{Z} \times \mathrm{BSp}$,
(4) $\mathrm{ktop} \simeq \mathbb{Z} \times \mathrm{BTop}$, and
(5) $\mathrm{Pic}(\mathbb{S}) \simeq \mathbb{Z} \times \mathrm{BG}$.

[32]In the category of M-modules, one can perform the operation $X \mapsto X_\infty$. Under assumption (2), it follows that X_∞ is $\pi_0(M)$-local for all X and that the map $X \to X_\infty$ is the initial map to a $\pi_0(M)$-local M-module. This then shows that operation $X \mapsto X_\infty$ refines to a symmetric monoidal localisation from M-modules to $\pi_0(M)$-modules. In particular, the image of M is itself a commutative monoid and one checks that it is then given by M^{gp} by comparing universal properties.

To move on, we next aim to describe the map $M_\infty \to M^{\mathrm{gp}}$ in terms of Quillen's plus construction. We recall this construction first.

4.31. **Theorem** (Quillen) *For every anima X, there is a map $t\colon X \to X^+$ satisfying the following properties:*

(1) *X^+ is hypoabelian,*
(2) *the map t induces an isomorphism on homology.*

Proof. We will define X^+ for each component of X individually, so may assume X is connected. Let P be the maximal perfect subgroup of $\pi_1(X)$. We construct X^+ in two steps. First, define X' via the pushout

$$\begin{tikzcd} \coprod_{s\in P} S^1 \arrow[r, "\amalg s"] \arrow[d] & X \arrow[d, "i"] \\ \coprod_{s\in P} * \arrow[r] & X' \end{tikzcd}$$

Then, by Seifert–van Kampen, we find $\pi_1(X') \cong \pi_1(X)/P$. We show that $\pi_1(X)/P$ is hypoabelian. Indeed, subgroups H of $\pi_1(X)/P$ correspond bijectively to a subgroups $P \subseteq \bar{H} \subseteq \pi_1(X)$ by taking preimages along the projection $\pi_1(X) \to \pi_1(X)/P$. In particular, we have a short exact sequence

$$1 \to P \to \bar{H} \to H \to 1$$

and consequently (since $P^{\mathrm{ab}} = 1$) an isomorphism $\bar{H}^{\mathrm{ab}} \to H^{\mathrm{ab}}$ so that H is perfect if and only if $\bar{H}$ is perfect which is the case if and only if $P = \bar{H}$ by the maximality of P.

Now, the map $i\colon X \to X'$ is not acyclic, but has only relative homology in degree 2. Indeed, the long exact sequence of the defining pushout for X' reads as follows.

$$H_2(X) \to H_2(X') \to H_2(\bigvee_P S^2) \to H_1(X) \to H_1(X')$$

but we have just argued that the map is an isomorphism, as it is isomorphic to the map $\pi_1(X)^{\mathrm{ab}} \to [\pi_1(X)/P]^{\mathrm{ab}}$. We can therefore lift the basis elements of the free abelian group in the middle of this sequence to elements of $H_2(X')$. In fact, we can lift them to elements of $\pi_2(X')$ as we show now.

To that end, let $Y' \to X'$ be the universal cover of X', i.e. the fibre of the canonical map $X' \to \tau_{\leq 1}X'$ and define $Y \to X$ via the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "i"] & X' \end{tikzcd}$$

It follows that $\pi_1(Y) \to \pi_1(X)$ identifies with the inclusion $P \subseteq \pi_1(X)$. Moreover, we may consider the defining pushout square for X' and pull it back along the map $Y' \to X'$. Since the pullback functor $\mathrm{An}_{/X'} \to \mathrm{An}_{Y'}$ identifies with the functor $\mathrm{Fun}(X', \mathrm{An}) \to \mathrm{Fun}(Y', \mathrm{An})$, it preserves colimits. We therefore find a pushout

$$\begin{tikzcd} \coprod_{s\in P} S^1 \times_{X'} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ \coprod_{s\in P} * \times_{X'} Y' \arrow[r] & Y' \end{tikzcd}$$

and we have pullback diagrams

$$\begin{tikzcd} S^1 \times_{X'} Y' \arrow[r] \arrow[d] & * \times_{X'} Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ S^1 \arrow[r] & * \arrow[r] & X' \end{tikzcd}$$

so from $* \times_{X'} Y' = \pi_1(X') = \pi_1(X)/P$, we find an equivalence $S^1 \times_{X'} Y' \simeq S^1 \times \pi_1(X)/P$. In total, we therefore have a pushout

$$\begin{tikzcd} \coprod_{P\times\pi_1(X)/P} S^1 \arrow[r] \arrow[d] & Y \arrow[d] \\ \coprod_{P\times\pi_1(X)/P} * \arrow[r] & Y' \end{tikzcd}$$

and consequently, we obtain the following (diagram of) exact sequence

$$\begin{tikzcd} H_2(Y) \arrow[r] \arrow[d] & H_2(Y') \arrow[r, two heads] \arrow[d] & H_2(\bigvee_{P\times\pi_1(X)/P} S^2) \arrow[r] \arrow[d, two heads] & H_1(Y) = 0 \arrow[r] \arrow[d] & 0 \\ H_2(X) \arrow[r] & H_2(X') \arrow[r] & H_2(\bigvee_P S^2) \arrow[r] & H_1(X) \arrow[r, "\cong"] & H_1(Y) \end{tikzcd}$$

since $H_1(Y) = \pi_1(Y)^{\mathrm{ab}} = P^{\mathrm{ab}}$ and P is perfect. Here, the middle vertical map is the map induced by the projection $P \times \pi_1(X)/P \to P$ and is therefore the projection onto a direct summand. Consider then the commutative diagram

$$\begin{tikzcd} \pi_2(Y') \arrow[r, "\cong"] \arrow[d, "\cong"] & H_2(Y') \arrow[d] \\ \pi_2(X') \arrow[r] & H_2(X') \end{tikzcd}$$

in which the left vertical map is an isomorphism since $Y' \to X'$ is a covering map and the top horizontal map is an isomorphism by the Hurewicz theorem, as we recall that Y' is simply-connected. We deduce that the composite

$$\pi_2(X') \to H_2(X') \to H_2(\bigvee_P S^2)$$

is surjective, so that for each element $p \in P$, we can choose a (pointed) map $\alpha_p \colon S^2 \to X'$ whose composite with the map $X' \to \bigvee_P S^2$ is the inclusion of the wedge summand indexed by $p \in P$. Then we define X^+ as the pushout

$$\begin{tikzcd} \bigvee_P S^2 \arrow[r, "\vee_p \alpha_p"] \arrow[d] & X' \arrow[d] \\ * \arrow[r] & X^+ \end{tikzcd}$$

Using Seifert–van Kampen again, we see that the map $\pi_1(X') \to \pi_1(X^+)$ is an isomorphism, so that X^+ is indeed hypoabelian. Consider then the diagram consisting of pushout squares

$$\begin{tikzcd}
 & \bigvee_{p\in P} S^2 \arrow[r] \arrow[d, "\vee_p \alpha_p"'] & * \arrow[d] \\
X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X^+ \arrow[d] \\
* \arrow[r] & \bigvee_P S^2 \arrow[r] & X^+/X
\end{tikzcd}$$

By construction, the middle vertical composite is the identity, so we deduce that $X^+/X \simeq *$, showing that $X \to X^+$ indeed induces an isomorphism on homology. □

In fact, the map $t\colon X \to X^+$ is *acyclic*, as we explain next. First, we have the following lemma.

4.32. **Lemma** *Let $f\colon X \to Y$ be a map of anima. Then the following conditions are equivalent.*

1. *For every local coefficient system $\mathcal{L}$ on Y, the map $f_*\colon H_*(X; f^*\mathcal{L}) \to H_*(Y;\mathcal{L})$ is an isomorphism,*
2. *the map $X \times_Y \widetilde{Y} \to \widetilde{Y}$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$; here $\widetilde{Y} \to Y$ denotes the universal cover, and*
3. *for every point $y \in Y$, the map $\mathrm{fib}_y(f) \to *$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$.*

A map satisfying any of the above conditions is called acyclic.

Proof. Exercise 1 Sheet 5. □

4.33. **Remark** In this remark, we explain that the map $t\colon X \to X^+$ is in fact acyclic by showing that the map $Y = X \times_{X^+} \widetilde{X^+} \to \widetilde{X^+}$ induces an isomorphism on homology. To see this, consider the following pullback diagrams

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & \widetilde{X^+} \arrow[d] \\
X \arrow[r] & X' \arrow[r] & X^+
\end{tikzcd}$$

and note that $Y' \to X'$ is a universal cover since $X' \to X^+$ is a π_1-isomorphism. Now, as in the proof above, one the shows that there is a commutative diagram of pushout squares

$$\begin{tikzcd}
 & \coprod_{\pi_1(X)/P} \bigvee_P S^2 \arrow[r] \arrow[d] & \coprod_{\pi_1(X)/P} * \arrow[d] \\
 & \bigvee_{P\times\pi_1(X)/P} S^2 \arrow[r] \arrow[d] & * \arrow[d] \\
Y \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & \widetilde{X^+} \arrow[d] \\
* \arrow[r] & \bigvee_{P\times\pi_1(X)/P} S^2 \arrow[r] & \widetilde{X^+}/Y
\end{tikzcd}$$

and that the middle the middle vertical map $\bigvee_{P\times\pi_1(X)/P} S^2 \to \bigvee_{P\times\pi_1(X)/P} S^2$ is an equivalence, showing that $\widetilde{X^+}/Y$ is contractible, and therefore that $Y \to \widetilde{X^+}$ induces an isomorphism on homology.

4.34. **Corollary** *The map t exhibits $X \mapsto X^+$ as left adjoint to the inclusion* $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ *of hypoabelian anima into all anima.*

Proof. For a hypoabelian space Y we need to show that the induced map

$$\mathrm{Map}(X^+, Y) \to \mathrm{Map}(X, Y)$$

is an equivalence. Since $X \to X^+$ induces a bijection on π_0, we may assume that X, X^+ and Y connected. We induct over the Postnikov tower of Y. The start case is $Y = BG$ with G hypoabelian. In this case, the map under consideration is given by $\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X)/P, G)_{hG} \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)_{hG}$ where G acts via conjugation on the set of group homomorphisms, thought of as an anima. Indeed, this follows from the fibre sequence

$$\mathrm{Map}_*(T, BG) \to \mathrm{Map}(T, BG) \to BG$$

which exists for every anima T and the fact that for T connected, we have $\mathrm{Map}_*(T, BG) \simeq \mathrm{Map}_*(B\pi_1(T), BG)$ which is discrete and its π_0 is canonically bijective to $\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(T), G)$, see [Lan24, Lemma 4.35]. See also [Lan24, Remark 4.36] for a more direct argument. Now, before taking homotopy orbits of the conjugation action on group homomorphisms, the restriction map $\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X^+), G) \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)$ is injective with image those group homomorphisms $f\colon \pi_1(X) \to G$ that are trivial upon restriction to P. But $f(P) \subseteq G$ is a subgroup which is a quotient of the perfect group P, and hence itself perfect. Since G is hypoabelian, $f(P) = \{1\}$, so that the map $\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X^+), G) \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)$ is a bijection. Consequently, upon applying homotopy orbits, the map remains an equivalence and the induction start is shown. In the inductive step, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(X^+, K(\pi_n(Y), n)) \arrow[r] \arrow[d] & \mathrm{Map}(X^+, \tau_{\leq n} Y) \arrow[r] \arrow[d] & \mathrm{Map}(X^+, \tau_{\leq n-1} Y) \arrow[d] \\
\mathrm{Map}(X, K(\pi_n(Y), n)) \arrow[r] & \mathrm{Map}(X, \tau_{\leq n} Y) \arrow[r] & \mathrm{Map}(X, \tau_{\leq n-1} Y)
\end{tikzcd}$$

in which the right vertical map is an equivalence by induction and the left vertical map is an equivalence, since its induced map on π_k is given by $H^{n-k}(X^+;\pi_n(Y)) \to H^{n-k}(X;\pi_n(Y))$ which is an isomorphism because the plus construction induces an isomorphism on homology. Finally, we use that in the commutative diagram

$$\begin{tikzcd} \mathrm{Map}(X^+, Y) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Map}(X, Y) \arrow[d, "\simeq"] \\ \lim_n \mathrm{Map}(X^+, \tau_{\leq n} Y) \arrow[r, "\simeq"] & \lim_n \mathrm{Map}(X, \tau_{\leq n} Y) \end{tikzcd}$$

the vertical maps are equivalences as the functor $\mathrm{Map}(X, -)$ commutes with limits, and the lower horizontal map is an equivalence as it is an inverse limit of equivalences as we have just shown. $\square$

Exercise. Without using the construction, show that if the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ admits a left adjoint L, then the unit map $X \to LX$ induces an isomorphism on homology. Hint: Show that a map which induces an isomorphism on cohomology also induces an isomorphism on homology.

Exercise. Show that the canonical map $(X \times Y)^+ \to X^+ \times Y^+$ is an equivalence for all anima X and Y.

As a consequence, if $M \in \mathrm{CMon}(\mathrm{An})$, we find that the map $M \to M^+$ is one of commutative monoids, in particular, M^+ is canonically an M-module. Moreover, for X an M-module, we find that X^+ is canonically an M^+-module.

4.35. **Theorem** (Group completion theorem III) *Let $M \in \mathrm{CMon}(\mathrm{An})$. Then there are canonical equivalences $(M^+)_\infty \simeq (M_\infty)^+ \simeq M^{\mathrm{gp}}$ and therefore $(M^{\mathrm{gp}})_0 = [(M_\infty)_0]^+$.*

Proof. Let us consider the following square

$$\begin{tikzcd} (M_\infty)^+ \arrow[r] \arrow[d] & M^{\mathrm{gp}} \arrow[d] \\ (M^+)_\infty \arrow[r, "\simeq"] & (M^+)^{\mathrm{gp}} \end{tikzcd}$$

in which the top horizontal map is induced by $M_\infty \to M^{\mathrm{gp}}$ using that M^{gp} is in particular hypoabelian. The lower horizontal map is an equivalence due to Theorem 4.29: Indeed, since the map $C_n \to \pi_1((M^+)_\infty, m^n)$ factors through $\pi_1(M^+, m^n)$ which is hypoabelian, the claim follows from the observation that $\Sigma_n \to C_2$ is a hypoabelianization for $n \geq 5$ and has C_n in its kernel when n is odd. In particular, $(M^+)_\infty$ is hypoabelian. The left vertical map is then the colimit-interchange, i.e. induced by the map $M_\infty \to (M^+)_\infty$ using that $(-)^+$ is left adjoint to the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$; this is an equivalence because $(-)^+$ is a left adjoint and hence commutes with the colimit describing $(-)_\infty$. It then remains to prove that the map $M^{\mathrm{gp}} \to (M^+)^{\mathrm{gp}}$ is also an equivalence. The map is one of commutative groups and is hence simple. Consequently, it suffices to show that the map induces an isomorphism on homology. But by Theorem 4.22, on homology this map induces the map

$$H_*(M)[\pi_0(M)^{-1}] \to H_*(M^+)[\pi_0(M)^{-1}]$$

which is an isomorphism since the map $H_*(M) \to H_*(M^+)$ is an isomorphism of commutative rings. $\square$

4.36. **Example** Coming back to Example 4.36, it turns out that $\mathbb{Z} \times \mathrm{B}\Sigma_\infty^+ \simeq (\mathrm{Fin}^{\simeq})^{\mathrm{gp}} = \mathbb{S}$ is the sphere spectrum; this is called the Barratt–Priddy–Quillen theorem. Indeed, consider the free commutative monoid functor $\mathrm{An} \to \mathrm{CMon}(\mathrm{An})$, i.e. the left adjoint of the inclusion functor. It is a general fact that this admits a formula as follows: $X \mapsto \coprod_{n\geq 0}(X_n^{\times})_{h\Sigma_n}$. For $X = *$, we obtain $\coprod_{n\geq 0} B\Sigma_n \simeq \mathrm{Fin}^{\simeq}$. Hence, we find that $\iota\mathrm{Fin}^{\mathrm{gp}}$ is the free commutative group on $*$. Using that $\mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$, the free commutative group functor identifies with the functor Ω^∞, so we deduce that $\iota\mathrm{Fin}^{\mathrm{gp}} = \mathbb{S}[*] = \mathbb{S}$.

4.37. **Example** As anima, we have $K(R) = K_0(R) \times K(R)_0$ and $K(R)_0 \simeq \mathrm{BGL}(R)^+$. We warn that this is really only an equivalence of anima, and is not compatible with the group structure in general. Here, we regard $\mathrm{BGL}(R)^+ \simeq K(R)_0$ as a group. Indeed, such product decompositions are not true on the level of spectra. For instance, we claim that $\mathbb{S} \to K(\mathbb{Z})$ induces an equivalence $\tau_{\leq 1}\mathbb{S} \to \tau_{\leq 1}K(\mathbb{Z})$. Indeed, we have just argued in Example 4.36 that $\mathbb{S} = \mathbb{Z} \times B\Sigma_\infty^+$ and $K(\mathbb{Z}) = \mathbb{Z} \times \mathrm{BGL}(\mathbb{Z})^+$, so it suffices to show that $\Sigma_\infty \to \mathrm{GL}(\mathbb{Z})$ induces an isomorphism on abelianizations. But as spectrum one can show that

$$\tau_{\leq 1}\mathbb{S} = \mathrm{fib}(\mathbb{Z} \xrightarrow{\mathrm{Sq}^2 \circ \mathrm{red}_2} \Sigma^2\mathbb{Z}/2)$$

which is not equivalent to $\mathbb{Z} \oplus \Sigma\mathbb{Z}/2$ (this is equivalent to saying that $\mathrm{Sq}^2\mathrm{red}_2$ is not null-homotopic). Therefore, there is also no equivalence of spectra $K(\mathbb{Z}) \simeq K_0(\mathbb{Z}) \times \mathrm{BGL}(\mathbb{Z})^+$.

4.38. **Lemma** *Let $f\colon X \to Y$ be acyclic. Then for all $x \in X$, the map $\pi_1(X) \to \pi_1(Y)$ is surjective with perfect kernel. Moreover, if f induces an isomorphism on π_1 of all components, then f is an equivalence.*

Proof. Exercise 4 Sheet 5. □

4.39. **Lemma** *Let $F \to E \to B$ be a fibre sequence with B hypoabelian. Then $F^+ \to E^+ \to B$ is again a fibre sequence.*

Finally, let us also identify $\pi_2 K(R)$. To that end we have:

4.40. **Proposition** *Let G be a discrete group with maximal perfect subgroup P. Let $f\colon BG \to (BG)^+$ be the plus construction of its classifying space and let $F = \mathrm{fib}(f)$. Then there is an induced short exact sequence*

$$1 \to \pi_2((BG)^+) \to \pi_1(F) \to P \to 1$$

and this is the universal central extension of P.

Proof. The long exact sequence in homotopy groups gives an exact sequence

$$\pi_2(BG) \to \pi_2((BG)^+) \to \pi_1(F) \to G \to G/P \to 1$$

since $BG \to (BG)^+$ induces $G \to G/P$ on π_1, the kernel of which is P. Moreover, $\pi_2(BG) = 0$ so we have the claimed short exact sequence; here we use that we know that the image of the boundary map $\pi_2((BG)^+) \to \pi_1(F)$ is central, which is in fact true for any fibre sequence. By Proposition 3.66 it now remains to show $H_1(\pi_1(F)) = H_2(\pi_1(F)) = 0$. By Remark 4.33, the plus construction is acyclic, so we deduce $H_*(F) = H_*(*)$. Now in general, $H_1(X) \to H_1(\pi_1(X))$ is an isomorphism, and $H_2(X) \to H_2(\pi_1(X))$ is surjective, since $B\pi_1(X)$ is obtained from X by attaching cells of dimension ≥ 3. Hence we deduce $H_1(\pi_1(F)) = H_2(\pi_1(F)) = 0$ as needed. □

Applying this to $\mathrm{GL}(R)$, we obtain:

4.41. **Corollary** *For a ring R, there is a canonical equivalence $K_2(R) \cong \pi_2(K(R))$.*

Proof. Indeed, we have just argued that $\pi_2 K(R)$ is the kernel group of the universal central extension of $E(R)$ as this is the maximal perfect subgroup of $\mathrm{GL}(R)$. Earlier, we have argued that $K_2(R)$ is also the kernel of the universal central extension of $E(R)$, so we obtain the desired result. □

4.42. **Remark** Recall that $K_1(R) \cong H_1(\mathrm{GL}(R))$ and $K_2(R) \cong H_2(E(R))$. One may wonder whether $K_3(R) = \pi_3 K(R)$ also has such a homological description. The answer is yes: Gersten observed that one has $K_3(R) \cong H_3(\mathrm{St}(R))$. To see this, consider the fibre sequence

$$BE(R) \to \mathrm{BGL}(R) \to BK_1(R).$$

It is a general fact that applying the plus construction to a fibre sequence whose base anima is hypoabelian yields another fibre sequence, see the exercises. In particular, it follows that

$$BE(R)^+ \to \mathrm{BGL}(R)^+ \to BK_1(R)$$

is again a fibre sequence, so $BE(R)^+$ is 1-connected. Similarly, there is a fibre sequence

$$B\mathrm{St}(R)^+ \to BE(R)^+ \to K(K_2(R), 2)$$

so $B\mathrm{St}(R)^+$ is 2-connected. It follows that $H_3(B\mathrm{St}(R)) \cong H_3(B\mathrm{St}(R)^+) \cong \pi_3(B\mathrm{St}(R)^+) \cong \pi_3(BE(R)^+) \cong \pi_3(BGL(R)^+) \cong \pi_3 K(R)$.

4.43. **Remark** A slightly different way to obtain the comparison $K_2(R) \cong \pi_2 K(R)$ (but still using the isomorphism $K_2(R) = H_2(E(R))$) goes as follows. Use that the 2-truncation of $K(R)$ is, as space, a product, i.e. $\tau_{[1,2]}\Omega^\infty K(R) \simeq K(K_2(R), 2) \times K(K_1(R), 1)$.[33] It follows that

$$H_2(\Omega^\infty_0 K(R)) \cong H_2(K_2(R)) \oplus H_2(K_1(R))$$

and hence $K_2(R) \cong \ker[H_2(\Omega^\infty_0 K(R)) \to H_2(K_1(R))]$. Then one can deduce from the fibre sequence

$$BE(R) \to \mathrm{BGL}(R) \to BK_1(R)$$

and the Serre spectral sequence that $\pi_2 K(R) \cong H_2(E(R))$.

4.3. **Milnor patching.** In this section, we want to further motivate the definition of K_1 by establishing the excision exact sequence of Bass, Milnor, Murthy. The argument is based on Milnor's patching construction. The basic setup uses the following definition. We will consider a commutative square of rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ A' & \xrightarrow{f'} & B' \end{array}$$

4.44. **Definition** A square of rings as above is called a *Milnor square* if it is a pullback and the map $p\colon B \to B'$ is surjective.

[33]This is a result about arbitrary 2-fold loop sapces, but it is not quite obvious, see [Arl90].

4.45. **Remark** Assume $I \subseteq A$ is an ideal of A and that $f\colon I \to f(I)$ is an isomorphism onto an ideal $f(I) \subseteq B$. Then setting $A' = A/I$ and $B' = B/f(I)$ is a Milnor square. Conversely, if a square is a Milnor square, then $A \to A'$ is also surjective, and for $I = \ker(p'\colon A \to A')$, the map $f\colon I \to f(I)$ is an isomorphism (since the diagram is a pullback), and $f(I) = \ker(p\colon B \to B')$.

Moreover, for a Milnor square, the underlying diagram of abelian groups is also a pushout, i.e. the sequence

$$0 \to A \xrightarrow{(p',f)} A' \oplus B \xrightarrow{f'-p} B' \to 0$$

is exact.

4.46. **Example** (The coordinate axes) Let k be a commutative ring. Then the square

$$\begin{tikzcd} k[X,Y]/(XY) \arrow[r] \arrow[d] & k[Y] \arrow[d] \\ k[X] \arrow[r] & k \end{tikzcd}$$

is a Milnor square.

4.47. **Example** (The nodal curve) Let k be a commutative ring with $2 \in k^\times$. Then the square

$$\begin{tikzcd} k[X,Y]/(Y^2 - X^3 - X^2) \arrow[r, "f"] \arrow[d, "p"] & k[T] \arrow[d, "(\mathrm{ev}_1, \mathrm{ev}_{-1})"] \\ k \arrow[r, "\Delta"] & k \times k \end{tikzcd}$$

where $p(X) = p(Y) = 0$ and $f(X) = T^2 - 1$ and $f(Y) = T(T^2 - 1)$, is a Milnor square.

4.48. **Example** (The cuspidal curve) Let k be a commutative ring. Then the square

$$\begin{tikzcd} k[X,Y]/(Y^2 - X^3) \arrow[r, "f"] \arrow[d, "q"] & k[T] \arrow[d, "\mathrm{pr}"] \\ k \arrow[r] & k[T]/(T^2) \end{tikzcd}$$

where $q(X) = q(Y) = 0$ and $f(X) = T^2$ and $f(Y) = T^3$, is a Milnor square.

4.49. **Example** (Rim's square) Let p be a prime. Then

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta_p] \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

is a Milnor square.

4.50. **Example** (Group rings) Let G be a finite group of order N. Let $\mathfrak{M}$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$. Then $I = N \cdot \mathfrak{M}$ turns out to be a common ideal in $\mathbb{Z}[G]$

and $\mathfrak{M}$, and the square

$$\begin{tikzcd} \mathbb{Z}[G] \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ \mathbb{Z}[G]/I \arrow[r] & \mathfrak{M}/I \end{tikzcd}$$

is a Milnor square.

Now suppose given a Milnor square as above. Then we may apply $\mathrm{Proj}(-)$ to it to obtain a commutative diagram of (symmetric monoidal, under direct sum) categories

$$\begin{tikzcd} \mathrm{Proj}(A) \arrow[r] \arrow[d] & \mathrm{Proj}(B) \arrow[d] \\ \mathrm{Proj}(A') \arrow[r] & \mathrm{Proj}(B') \end{tikzcd}$$

Consequently, we obtain a functor $\theta\colon \mathrm{Proj}(A) \to \mathrm{Proj}(A') \times_{\mathrm{Proj}(B')} \mathrm{Proj}(B)$. An object in the target of this functor consist of a triple (P, Q, α) where $P \in \mathrm{Proj}(A')$, $Q \in \mathrm{Proj}(B)$ and $\alpha\colon P \otimes_{A'} B' \xrightarrow{\cong} Q \otimes_B B'$ is an isomorphism of B'-modules. Morphisms between such objects are given as the pullback

$$\begin{tikzcd} \mathrm{Hom}_\times((P,Q,\alpha),(P',Q',\alpha')) \arrow[r] \arrow[d] & \mathrm{Hom}_B(Q,Q') \arrow[d] \\ \mathrm{Hom}_{A'}(P,P') \arrow[r] & \mathrm{Hom}_{B'}(P \otimes_{A'} B', Q' \otimes_B B') \end{tikzcd}$$

in which the right vertical map sends $f\colon Q \to Q'$ to the composite

$$P \otimes_A B' \xrightarrow{\alpha} Q \otimes_B B' \xrightarrow{f \otimes_B B'} Q' \otimes_B B'$$

and the lower horizontal map sends $g\colon P \to P'$ to the composite

$$P \otimes_{A'} B' \xrightarrow{g \otimes_{A'} B'} P' \otimes_{A'} B' \xrightarrow{\alpha'} Q' \otimes_{A'} B'.$$

Given this, we now have the following theorem due to Milnor.

4.51. **Theorem** (Milnor patching) *The just described functor*

$$\theta\colon \mathrm{Proj}(A) \to \mathrm{Proj}(A') \times_{\mathrm{Proj}(B')} \mathrm{Proj}(B)$$

is an equivalence of categories.

Proof. To prove fully faithfulness, we need to show that for $P, P' \in \mathrm{Proj}(A)$, the square

$$\begin{tikzcd} \mathrm{Hom}_A(P,P') \arrow[r] \arrow[d] & \mathrm{Hom}_B(P \otimes_A B, P' \otimes_A B) \arrow[d] \\ \mathrm{Hom}_{A'}(P \otimes_A A', P' \otimes_A A') \arrow[r] & \mathrm{Hom}_{B'}(P \otimes_A B', P \otimes_A B') \end{tikzcd}$$

is a pullback diagram. But using the extensions-and-restrictions of scalars adjunctions, this square is isomorphic to the square

$$\begin{tikzcd} \mathrm{Hom}_A(P,P') \arrow[r] \arrow[d] & \mathrm{Hom}_A(P,P'\otimes_A B) \arrow[d] \\ \mathrm{Hom}_A(P,P'\otimes_A A') \arrow[r] & \mathrm{Hom}_A(P,P'\otimes_A B') \end{tikzcd}$$

which in turn is obtained by applying the (exact) functor $\mathrm{Hom}_A(P,-)$ to the square obtained from the original Milnor square upon applying $P'\otimes_A -$. Since P' is projective, the tensored square is again cartesian (because it can equivalently be described as applying the left exact functor $\mathrm{Hom}_A(DP',-)$) and again, it remains cartesian upon further applying $\mathrm{Hom}_A(P,-)$.

The meat in Milnor's theorem is therefore to prove that θ is essentially surjective. To that end, pick an element in its target (P,Q,α) consisting of a finite projective A'-module P, a finite projective B-module Q and an isomorphism of B'-modules $\alpha\colon P\otimes_{A'} B'\to Q\otimes_B B'$. Consider then the pullback diagram

$$\begin{tikzcd} M(P,Q,\alpha) \arrow[r] \arrow[d] & Q \arrow[d, "i"] \\ P \arrow[r, "f"] & Q\otimes_B B' \end{tikzcd}$$

where f is the composite $P\to P\otimes_{A'} B'\xrightarrow{\alpha} Q\otimes_B B'$. Then i and f are tautologically A-linear, so $M(P,Q,\alpha)$ is canonically and A-module and the square is one of A-modules. On the exercise sheet we will prove:

(1) $M(P,Q,\alpha)$ is finite projective over A,
(2) the canonical maps $M(P,Q,\alpha)\otimes_A A'\to P$ and $M(P,Q,\alpha)\otimes_A B\to Q$ are isomorphisms, and
(3) the resulting isomorphism
$$P\otimes_{A'} B'\cong M(P,Q,\alpha)\otimes_A B'\cong Q\otimes_B B'$$
is precisely the isomorphism α.

These then give the essential surjectivity of θ. □

4.52. **Remark** One of the ingredients of the steps in the proof above is the following result which we will also make explicit use of: Assume that S is in the image of an invertible matrix $T\in M_{n,m}(B)$. Then $M(A'^n,B^m,\alpha)\cong A^n$ is finite free.

4.53. **Corollary** *For a Milnor square, the associated sequence*

$$K_0(A)\xrightarrow{(f,p')} K_0(A')\oplus K_0(B)\xrightarrow{f'-p} K_0(B')$$

is exact.

Proof. That the composite is trivial is immediate. Let $(x,y)\in K_0(A')\oplus K_0(B)$ be such that $f'_*(x)=p_*(y)$. Write $x=[P]-[A'^n]$ and $y=[Q]-[B^m]$. Replacing then x by $x+[A'^{n+m}]$ and y by $y+[B^{n+m}]$ (and noting that $[A^{n+m}]$ lifts $[A'^{n+m}]$ and $[B^{n+m}]$, we see that we may assume that $x=[P]$ and $y=[Q]$. In that case, we obtain that $P\otimes_{A'} B'$ and $Q\otimes_B B'$ represent the same element in $K_0(B')$, so they become isomorphic after adding B'^k for sufficiently large k, via an isomorphism α. Therefore, $(P \oplus A'^k, Q \oplus B^k, \alpha)$ is an object of $\mathrm{Proj}(A) \otimes_{\mathrm{Proj}(B')} \mathrm{Proj}(B)$. By Milnor's theorem, there is then a finite projective A-module M such that

$$f_*[M] = [P] + [A'^k] \quad \text{and} \quad p'_*[M] = [Q] + [B^k].$$

We find that $[M] - [A^k] \in K_0(A)$ is a preimage of $([P], [Q])$, giving the desired exactness. □

4.54. **Remark** In general, the map $K_0(A') \oplus K_0(B) \to K_0(B')$ need not be surjective. Indeed, in the case of the nodal curve over a field of characteristic $\neq 2$, we have the maps

$$K_0(k) \oplus K_0(k[T]) \xrightarrow{\Delta_* - (\mathrm{ev}_1, \mathrm{ev}_{-1})_*} K_0(k \times k)$$

Now recall that $k[T]$ is a PID, so $K_0(k) \to K_0(k[T])$ is an isomorphism. We then see that the map in question sends a pair (x, y) to the pair $(x - y, x - y)$, and is consequently not surjective. This phenomenon leads to the introduction of negative K-groups, and once they are in place, the above sequence extends to the right indefinitely. We will come to this later.

4.55. **Remark** In general, the map $K_0(A) \to K_0(A') \oplus K_0(B)$ need not be injective. Indeed, let us analyze its kernel concretely. So assume $[P] - [A^n] \in K_0(A)$ maps to zero. This simply means that $P \otimes_A A'$ and $P \otimes_A B$ are stably isomorphic to A'^n and B^n, respectively. Therefore, for suitable $k \geq 0$, we have $(P \oplus A^k) \otimes_A A' \cong A'^{n+k}$ and $(P \oplus A^k) \otimes_A B \cong B^{n+k}$. Hence, by Milnor's patching Theorem 4.51 we obtain

$$P \oplus A^k \cong M(A'^{n+k}, B^{n+k}, \alpha)$$

for a suitable isomorphism $\alpha \colon B'^{n+k} \to B'^{n+k}$.

4.56. **Lemma** *For a Milnor square, there is a canonical map*

$$\partial \colon K_1(B') \to K_0(A)$$

induced by sending an invertible matrix $S \in \mathrm{GL}_n(B')$ to $[M(A'^n, B^n, S)] - [A^n]$. This map surjects onto the kernel of $K_0(A) \to K_0(A') \oplus K_0(B)$.

Proof. First we note that

$$\partial[S] = [M(A'^n, B^n, S)] - [A^n] = [M(A'^{n+1}, B^{n+1}, S \oplus 1] - [A^{n+1}] = \partial[S \oplus \mathrm{id}]$$

so the described map gives a well-defined map $\mathrm{GL}(B') \to K_0(A)$. We claim that this map is a monoid homomorphism, this will also be part of the exercise sheet. Hence $\partial \colon \mathrm{GL}(B') \to K_0(A)$ as described above is a group homomorphism and hence induces the map $\partial \colon K_1(B') \to K_0(A)$ by the universal property of abelianizations. The fact that the image of this map is the kernel of $K_0(A) \to K_0(A') \oplus K_0(B)$ is the argument of Remark 4.55. □

4.57. **Theorem** (Bass, Milnor, Murthy) *For a Milnor square, the associated sequence*

$$K_1(A) \xrightarrow{(p', f)} K_1(A') \oplus K_1(B) \xrightarrow{f' - p} K_1(B') \xrightarrow{\partial} K_0(A) \xrightarrow{(p', f)} K_0(A') \oplus K_0(B) \xrightarrow{f' - p} K_0(B')$$

is exact.

Proof. First, we show that the composite $K_1(A') \oplus K_1(B) \to K_1(B') \to K_0(A)$ is trivial. To do so, it suffices to show that $K_1(A') \to K_1(B') \to K_0(A)$ and $K_1(B) \to K_1(B') \to K_0(A)$ are trivial. To see this, let $S \in \mathrm{GL}_n(A')$ represent an element of $K_1(A')$. Then by definition, the image of the composite is given by $[M(A'^n, B^n, S)] - [A^n]$ which vanishes by Remark 4.52; note that here, we also denote by S the image under the map $\mathrm{GL}(A') \to \mathrm{GL}(B')$. The same argument applies to the second composite under investigation. Now let $S \in \mathrm{GL}_n(B')$

represent an element of $\ker(\partial)$. This means that for $k \geq 0$ large enough $M(A'^k, B^k, S)$ – where we abusively denote $S \oplus \mathrm{id}$ again by S – is isormorphic to A^k, via an isomorphism ϕ. Then the maps

$$A'^k \xrightarrow{\phi \otimes_A A'} M(A'^k, B^k, S \oplus \mathrm{id}) \otimes_A A' \cong A'^k$$

and

$$B^k \xrightarrow{\phi \otimes_A B} M(A'^k, B^k, S \oplus \mathrm{id}) \otimes_A B' \cong B^k$$

are again isomorphisms $\phi_{A'}$ and ϕ_B, which fit into a commutative diagram

$$\begin{array}{ccc} A^k & \xrightarrow{(p',f)} & A'^k \oplus B^k \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle (\phi_{A'}, \phi_B)} \\ M(A'^k, B^k, S) & \longrightarrow & A'^k \oplus B^k \end{array}$$

where the horizontal maps are the canonical inclusions part of the Milnor square and defining pullback square for $M(A'^k, B^k, S)$. The claim is that this implies that $S = [\phi_B \otimes_B B'] \cdot [\phi_{A'} \otimes_{A'} B']^{-1}$, showing that S is indeed in the image of the map $K_1(A') \oplus K_1(B) \to K_1(B')$. Finally, for exactness of the sequence at $K_1(A') \oplus K_1(B')$, it is clear that the relevant composite is trivial. So consider a pair of matrices $(S, T) \in \mathrm{GL}(A') \times \mathrm{GL}(B)$ such that $f'(S)p(T)^{-1} = \mathrm{id}_{B'}$ or in other words, where $f'(S) = p(T)$. Then this equation holds for all entries of the matrix, so that the exact sequence associated to the Milnor square

$$A \xrightarrow{(p',f)} A' \oplus B \xrightarrow{f'-p} B'$$

shows that there is a matrix $U \in M_n(A)$ with $p'(U) = S$ and $f(U) = T$. The same argument applies to the respective inverses of S and T, showing that U is in fact invertible. □

4.58. **Remark** Consider a map between Milnor squares, that is, a commutative cube where to parallel faces are Milnor squares:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \quad \Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C' & \longrightarrow & D' \end{array}$$

Then the diagram of Bass–Milnor–Murthy sequences

$$\begin{array}{ccccccccccc} K_1(A) & \longrightarrow & K_1(A') \oplus K_1(B) & \longrightarrow & K_1(B') & \xrightarrow{\partial} & K_0(A) & \longrightarrow & K_0(A') \oplus K_0(B) & \longrightarrow & K_0(B') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(C) & \longrightarrow & K_1(C') \oplus K_1(D) & \longrightarrow & K_1(D') & \xrightarrow{\partial} & K_0(C) & \longrightarrow & K_0(C') \oplus K_0(D) & \longrightarrow & K_0(D') \end{array}$$

commutes. This is a direct consequence of the functoriality of $K_1(-)$ and $K_0(-)$ except for the diagram involving the boundary maps ∂. Denote by φ the map $A \to C$ and by $\psi \colon B' \to D'$ the maps part of the map of Milnor squares. To see that the remaining diagram commutes, consider $S \in \mathrm{GL}_n(B')$. Then we claim that there is an isomorphism

$$M(A'^n, B^n, S) \otimes_A C \cong M(C'^n, D^n, \psi(S))$$

from which the desired commutativity follows readily. To see this isomorphism, recall that $M(A'^n, B^n, S)$ is a finite projective A-module. It follows that the square

$$\begin{array}{ccc} M(A'^n, B^n, S) \otimes_A C & \longrightarrow & M(A'^n, B^n, S) \otimes_A D \\ \downarrow & & \downarrow \\ M(A'^n, B^n, S) \otimes_A C' & \longrightarrow & M(A'^n, B^n, S) \otimes_A D' \end{array}$$

is again a pullback square. Moreover, we have

$$M(A'^n, B^n, S) \otimes_A D \cong M(A'^n, B^n, S) \otimes_A B \otimes_B D \cong B^n \otimes_B D \cong D^n$$

and similarly

$$M(A'^n, B^n, S) \otimes_A C \cong C^n \quad \text{and} \quad M(A'^n, B^n, S) \otimes_A D' \cong D'^n$$

as follows from point (2) in the list appearing in the proof of Milnor's patching Theorem 4.51. Consequently, there is a pullback square

$$\begin{array}{ccc} M(A'^n, B^n, S) \otimes_A C & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ C'^n & \longrightarrow & D'^n \end{array}$$

and it follows from point (3) in the same list that the vertical map $D^n \to D'^n$ is the canonical one, while the map $C'^n \to D'^n$ in the above diagram is the composite $C'^n \to D'^n \xrightarrow{\psi(S)} D'^n$. Consequently, we find

$$M(A'^n, B^n, S) \otimes_A C \cong M(C'^n, D^n, \psi(S))$$

as desired.

4.59. **Remark** We will later define negative K-groups $K_n(R)$ for $n < 0$ and any ring R. The Bass–Milnor–Murthy excision sequence then continues indefinitely to the right with these negative K-groups; compare with Remark 4.54. Indeed, we recall that $K_{-n-1}(R)$ is a direct summand of $K_{-n}(R[t^{\pm 1}])$ and given by the cokernel of the map

$$K_{-n}(R[t]) \oplus K_{-n}(R[t^{-1}]) \to K_{-n}(R[t^{\pm 1}]).$$

Then we observe that tensoring a Milnor square (over $\mathbb{Z}$) with $\mathbb{Z}[t^{\pm 1}]$, $\mathbb{Z}[t]$, or $\mathbb{Z}[t^{-1}]$ yields another Milnor square. Consequently, we obtain an exact sequence

$$K_1(A[t^{\pm 1}]) \to K_1(A'[t^{\pm 1}]) \oplus K_1(B[t^{\pm 1}]) \to K_1(B'[t^{\pm 1}]) \to K_0(A[t^{\pm 1}]) \to \dots$$

and similarly for polynomial rings in place of Laurent polynomials; the BMM sequence for the polynomial rings maps to the above BMM sequence for the Laurent polynomial rings by Remark 4.58, so passing to the cokernels, using that they form direct summands, we obtain an exact sequence

$$K_0(A) \to K_0(A') \oplus K_0(B) \to K_0(B') \to K_{-1}(A) \to K_{-1}(A') \oplus K_{-1}(B) \to K_{-1}(B')$$

This procedure may be continued inductively, so that the Bass–Milnor–Murthy sequence associated to a Milnor square continues indefinitely to the right.

4.60. **Example** We compute K_0 of the coordinate axes, the nodal curve, and the cusp over a field k (of characteristic different from 2 is case of the nodal curve) using the BMM sequence. The outcome is

(1) $K_0(k[X,Y]/(XY)) = \mathbb{Z}$,
(2) $K_0(k[X,Y]/(Y^2 - X^3 - X^2)) = k^\times \oplus \mathbb{Z}$,
(3) $K_0(k[X,Y]/(Y^2 - X^3)) = k \oplus \mathbb{Z}$.

Taking for granted that $K_{-n}(k) = K_{-n}(k[T]) = K_{-n}(k[T]/T^2) = 0$ for all $n > 0$ (we will perhaps prove this later), we also find

(1) $K_{-1}(k[X,Y]/(XY)) = 0$,
(2) $K_{-1}(k[X,Y]/(Y^2 - X^3 - X^2)) = \mathbb{Z}$,
(3) $K_{-1}(k[X,Y]/(Y^2 - X^3)) = 0$.

All even lower K-groups vanish as follows immediately from the above fact. This is no accident: Recall that Weibel's conjecture (which is now a theorem) says that there are no non-trivial negative K-groups beyond $-\dim(X)$ in case X is Noetherian of finite Krull dimension. In the case of regular Noetherian schemes, we might also show that all negative K-groups vanish.

4.61. **Example** Let us analyze the BMM sequence for the Rim square. It reads as follows:

$$K_1(\mathbb{Z}) \oplus K_1(\mathbb{Z}[\zeta_p]) \to K_1(\mathbb{F}_p) \to K_0(\mathbb{Z}[C_p]) \to K_0(\mathbb{Z}) \oplus K_0(\mathbb{Z}[\zeta_p]) \to K_0(\mathbb{F}_p) \to \dots$$

Since $\mathbb{Z}[\zeta_p]^\times \to \mathbb{F}_p^\times$ is surjective and $K_0(\mathbb{Z}) \to K_0(\mathbb{F}_p)$ is an isomorphism, the map $K_0(\mathbb{Z}[C_p]) \to K_0(\mathbb{Z}[\zeta_p]) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathbb{Z}[\zeta_p])$ is an isomorphism and $K_{-n}(\mathbb{Z}[C_p]) = 0$ for all $n > 0$ since $\mathbb{Z}, \mathbb{F}_p$ and $\mathbb{Z}[\zeta_p]$ are all regular Noetherian and hence have trivial negative K-groups.

We end this part on lower K-groups with a theorem of Swan's, saying that the above excision sequence cannot be extended to the left in general.

4.62. **Theorem** (Swan) *There is no functor $F\colon \mathrm{Rings} \to \mathrm{Ab}$ such that for any Milnor square, there is an associated exact sequence*

$$F(A) \to F(A') \oplus F(B) \to F(B') \xrightarrow{\partial} K_1(A) \to K_1(A') \oplus K_1(B) \to \dots$$

Proof. Fix a commutative ring $k \neq 0$. Consider then the rings

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in k \right\} \quad \text{and} \quad S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in k \right\}$$

and note that R is commutative. There are then ring maps $R \to k$ and $S \to k \times k$ given by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto a \quad \text{and} \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c).$$

Both these maps admit sections, by setting the off-diagonal entry zero. There is then a Milnor square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ k \arrow[r] & k \times k \end{tikzcd}$$

If there exists a functor F with the prescribed property, the split surjectivity of $S \to k \times k$ shows that $F(S) \to F(k \times k)$ is also surjective, resulting in the injectivity of the map $K_1(R) \to$

$K_1(S) \oplus K_1(k)$. This, however, is not the case. To see this, pick $0 \neq x \in k$ and consider the element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in $R^\times \setminus \{1\}$. Since R is commutative, this element is non-trivial in $K_1(R)$. Its image in $K_1(k)$ is represented by $1 \in k$ and hence trivial. Now assume that k is such that there exists $x \in k \setminus \{0\}$ such that $1 - x \in k^\times$ is invertible; e.g. let k be a field with at least 3 elements and pick $x \in k \setminus \{0, 1\}$. Then we have that

$$\begin{pmatrix} 1-x & -1 \\ 0 & 1 \end{pmatrix}$$

is invertible with inverse given by

$$\begin{pmatrix} (1-x)^{-1} & (1-x)^{-1} \\ 0 & 1 \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-x & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1-x)^{-1} & (1-x)^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

showing that the right hand element is a commutator of invertible elements in S, and is hence in particular trivial in $K_1(S)$. $\square$

4.63. **Remark** In fact, for any commutative ring k with $x \in k \setminus \{0\}$, the invertible element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in S represents the trivial element in $K_1(S)$. You can think about this as an exercise, or wait a bit. I'll add the argument here soon.

The fact that the BMM-sequence therefore cannot be extended to the left has been a hindering fact for further computations in higher K-groups. In joint work with Tamme [LT19], this problem has been rectified to some extend. We might come to this theorem and some of its applications later.

5. K-theory for ∞-categories

It is a crucial fact about K-theory that it is really defined for reasonably general categories; for instance at this point it is not clear how to define $K(X)$ for a scheme X: Group completing vector bundles over X, for instance, already has the wrong π_0, since in $K_0(X)$, we want to convert short exact sequences of vector bundles into sums – and this is in general not true when X is not affine. In principle, we could try to formally extend K-theory from affine schemes to all schemes, say via right Kan extension along $\mathrm{Aff}^{\mathrm{op}} \hookrightarrow \mathrm{Sch}^{\mathrm{op}}$, or we could consider K-theory as a presheaf on Aff and sheafifiy with respect to the Zariski topology; the result then formally extends to a Zariski sheaf on schemes; both these definitions in fact are the "correct" ones, but this is largely due to the fact that $K \colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ *is* already a Zariski sheaf. However, to prove this, it is very convenient to have a definition of K-theory for suitable categories. With the group completion definition we gave, it is not so clear that K-theory is a sheaf (at least, I do not know of a way of proving this directly from, say, the group completion theorem).

5.1. **K-theory of exact ∞-categories.** There are different categorical setups one can choose for the domain of a K-theory functor on categories. I've chosen to use that of exact ∞-categories for this course. These contain exact 1-categories in Quillen's sense as well as stable ∞-categories as examples; this is the reason for my choice: I like to think in stable ∞-categories, but Quillen's original arguments are in terms of exact 1-categories. Most of the following material on exact ∞-categories is taken from [SW25, Sau23]. I thank Christoph Winges for answering several questions of mine.

5.1. **Definition** An ∞-category $\mathcal{C}$ is called

(1) *pointed* if it admits a zero object 0; that is, an initial object which is also terminal.
(2) *semi-additive* if it is pointed, admits finite coproducts which are also finite products. We write $\oplus$ for the resulting binary (co)product.
(3) *additive* if it is semi-additive and every object $x \in \mathcal{C}$ is group-like, that is, the map $x \oplus x \to x \oplus x$ given by the fold map in the first product factor of the target and the projection in the second product factor of the target, is an equivalence.
(4) *stable* if it is additive, admits finite colimits, and a square in $\mathcal{C}$ is a pushout if and only if it is a pullback.[34]

We say that a functor between pointed categories is *reduced* if it preserves the zero object, between semi-additive categories is *additive* if it preserves finite (co)products, and between stable categories is *exact* if it is additive and preserves pushout squares.[35] We denote by Cat_∞^*, Cat_∞^Π, $\mathrm{Cat}_\infty^{\mathrm{st}}$ the ∞-categories of pointed, additive, and stable ∞-categories (with reduced, additive, and exact functors, respectively).

5.2. **Lemma** *The ∞-categories Cat_∞^Π and $\mathrm{Cat}_\infty^{\mathrm{st}}$ are semi-additive.*

Proof. Since (co)limits in product categories are formed factor-wise, it is readily checked that the product of two additive or stable categories is again additive or stable. This shows that Cat_∞^Π and $\mathrm{Cat}_\infty^{\mathrm{st}}$ have finite products. The empty product is given by the terminal category $*$; this is stable (exercise) and since exact and additive functors are in particular reduced, we find that $*$ is also initial in Cat_∞^Π and $\mathrm{Cat}_\infty^{\mathrm{st}}$; we then write 0 for this category. Now we have to show that if $\mathcal{C}, \mathcal{C}'$ are additive or stable, then the functors $\mathcal{C} \times 0 \to \mathcal{C} \times \mathcal{C}' \leftarrow 0 \times \mathcal{C}'$ make $\mathcal{C} \times \mathcal{C}'$ into a coproduct of $\mathcal{C}$ and $\mathcal{C}'$. For this, one observes the equivalence

$$(x, y) = (x, 0) \oplus (0, y) \in \mathcal{C} \times \mathcal{C}'$$

so that given additive or exact functors $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{C}' \to \mathcal{D}$, the only compatible additive extension to $\mathcal{C} \times \mathcal{C}'$ is given by $(x, y) \mapsto F(x) \oplus F'(y)$. Exercise: Check that this functor is in fact additive or exact. $\square$

5.3. **Definition** An *exact* ∞-category is an additive ∞-category $\mathcal{E}$ together with subcategories $\mathrm{in}\mathcal{E}$ and $\mathrm{pr}\mathcal{E}$ of *inclusions* and *projections* satisfying the following conditions.

(1) For all $x \in \mathcal{E}$, the map $0 \to x$ is an inclusion and the map $x \to 0$ is a projection.
(2) the class of inclusions is stable under pushouts (along arbitrary maps)
(3) the class of projections is stable under pullbacks (along arbitrary maps)

[34]It follows from this definition that a stable ∞-category also has finite limits and that the suspension and loop functors are inverse equivalences.

[35]Equivalently, an exact functor between stable ∞-categories preserves finite colimits. Moreover, a functor between stable ∞-categories is exact if and only if it preserves finite limits.

(4) For a commutative square

$$\begin{tikzcd} x \arrow[r, "i"] \arrow[d, "p"'] & y \arrow[d, "q"] \\ x' \arrow[r, "j"'] & y' \end{tikzcd}$$

the following are equivalent:
(a) i is an inclusion, p is a projection, and the square is a pushout,
(b) j is an inclusion, q is a projection, and the square is a pullback,

Any such square is called *exact*. Exact squares with $x' = 0$ are called exact sequences.

For $\mathcal{E}, \mathcal{E}'$ exact ∞-categories, we let $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{E}')$ be the full subcategory on functors which preserves exact squares. An exact subcategory of an exact ∞-category $\mathcal{E}$ is an additive subcategory $\mathcal{A} \subseteq \mathcal{E}$ such that for any exact sequence

$$X \hookrightarrow Y \twoheadrightarrow Z$$

with $X, Z \in \mathcal{A}$, it follows that $Y \in \mathcal{A}$. With the restricted inclusions and projections, $\mathcal{A} \subseteq \mathcal{E}$ then becomes an exact functor. We denote by $\mathrm{Cat}_\infty^{\mathrm{ex}}$ the ∞-category of exact ∞-categories.

5.4. **Remark** Let $\mathcal{E}$ be an exact ∞-category. Then $\mathcal{E}^{\mathrm{op}}$ is canonically exact with $\mathrm{in}\mathcal{E}^{\mathrm{op}} = \mathrm{pr}\mathcal{E}$ and $\mathrm{pr}\mathcal{E}^{\mathrm{op}} = \mathrm{in}\mathcal{E}$. Similarly, the opposite of an exact functor is again exact.

5.5. **Remark** A Waldhausen ∞-category is a pointed ∞-category $\mathcal{W}$ equipped with a subcategory $\mathrm{in}\mathcal{W}$ of inclusions satisfying the following axioms:
(1) for all $x \in \mathcal{W}$, the map $0 \to x$ is an inclusion,
(2) the class of inclusions is closed under pushouts.

A functor between Waldhausen ∞-categories is called exact if it is reduced and preserves inclusions and pushouts along inclusions. In particular, an exact ∞-category $(\mathcal{E}, \mathrm{in}\mathcal{E}, \mathrm{pr}\mathcal{E})$ gives rise to two Waldhausen categories: $(\mathcal{E}, \mathrm{in}\mathcal{E})$ and $(\mathcal{E}^{\mathrm{op}}, \mathrm{pr}(\mathcal{E})^{\mathrm{op}} = \mathrm{in}(\mathcal{E}^{\mathrm{op}}))$ and it follows from Lemma 5.12 that a functor between exact ∞-categories is exact if and only if the same functor viewed as functors between the above *two* induced Waldhausen categories is exact.

5.6. **Remark** The ∞-category $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is also semi-additive: For this, note that if $\mathcal{E}$ and $\mathcal{E}'$ are exact ∞-categories, $\mathcal{E} \times \mathcal{E}'$ carries an induced exact structure given by $\mathrm{in}(\mathcal{E} \times \mathcal{E}') = \mathrm{in}\mathcal{E} \times \mathrm{in}\mathcal{E}'$ and $\mathrm{pr}(\mathcal{E} \times \mathcal{E}') = \mathrm{pr}\mathcal{E} \times \mathrm{pr}\mathcal{E}'$. With this definition, it is immediate to see that $\mathcal{E} \times \mathcal{E}'$ is a product in $\mathrm{Cat}_\infty^{\mathrm{ex}}$. Therefore, we need to show again that it is also a coproduct. As in the case of stable ∞-categories, this follows from the fact that the sum of two exact squares is exact (exercise).

5.7. **Example** A stable ∞-category $\mathcal{C}$ can be viewed as an exact ∞-category by letting $\mathrm{in}\mathcal{C} = \mathrm{pr}\mathcal{C} = \mathcal{C}$. This yields a fully faithful inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}$: Indeed, a functor between stable categories is exact if and only if it is so when viewed as a functor between exact ∞-categories.

5.8. **Example** There is a fully faithful inclusion $\mathrm{Cat}_\infty^{\Pi} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}$ whose image consists of exact ∞-categories with *minimal exact structure*, i.e. where inclusions are maps of the form $A \to A \oplus B$ and where the projections are maps of the form $A \oplus B \to B$. In fact, this functor is left adjoint to the forgetful functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\Pi}$, which forgets the exact structure of $\mathcal{E}$ (in other words, this forgetful functor is a Bousfield colocalization). Indeed, an exact functor out of a minimal exact ∞-category is really just an additive functor (Exercise. Hint: Use Lemma 5.12).

5.9. **Example** An abelian category $\mathcal{A}$ is canonically an exact category where inclusions are the monomorphisms and projections are the epimorphisms (see Exercise Sheet 8). Exact sequences in an abelian category are then the same datum as exact sequences in its associated exact category.

5.10. **Remark** Finally, an exact (and additive) ∞-category can well be an ordinary category. In fact, ordinary exact categories in the sense of Quillen are examples of exact ∞-categories (Exercise). However, a stable ∞-category is an ordinary category if and only if it is the zero category 0, so stable categories are really a phenomenon of higher category theory having no direct classical counterpart.[36]

5.11. **Lemma** *Let $\mathcal{E}$ be an exact ∞-category and consider a commutative square in $\mathcal{E}$*

$$\begin{tikzcd} x \arrow[r, hook, "i"] \arrow[d, "f"] & y \arrow[d, "g"] \\ z \arrow[r, hook, "j"] & w \end{tikzcd}$$

in which i and j are inclusions. Then the square is a pushout if and only if $y/x \to w/z$ is an equivalence. Here $y/x = \operatorname{cofib}(i)$ and $w/z = \operatorname{cofib}(j)$ are the cofibres of i and j, respectively.

Proof. If the square is a pushout, then the map on horizontal cofibres is indeed an equivalence. So let us prove the converse. First, we note that the square under investigation is a pullback: Indeed, it participates in a commutative diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] \arrow[d] & w \arrow[d] \\ 0 \arrow[r] & w/z \end{tikzcd}$$

whose outer and lower squares are exact by the assumption that $y/x \to w/z$ is an equivalence. Hence, they are pullbacks, so pasting pullbacks, we find that the square in question is a pullback. Denote by $p = z \amalg_x y$ the pushout of the above square. We obtain an induced map $p \to w$ whose composite with $z \to p$ is j. Moreover, $z \to p$ is again an inclusion, since inclusions are closed under pushouts. Hence, if we show the conclusion of the lemma in the case where $x \to z$ is the identity of x, then the general case follows. In this case, we note that

[36] Its closest such counterpart is that of a triangulated category, but this is of different nature: Being stable is a property, whereas a triangulated category is an additive category equipped with extra structure satisfying various axioms.

the square participates in the following big commutative diagram

$$\begin{tikzcd}
x \arrow[r, "{(\mathrm{id},-\mathrm{id})}"] \arrow[d, equal] & x \oplus x \arrow[r, "\mathrm{id}\oplus i"] \arrow[d, "\mathrm{sh}"] & x \oplus y \arrow[d, "\mathrm{sh}'"] \\
x \arrow[r, "i_1"] \arrow[d] & x \oplus x \arrow[r, "\mathrm{id}\oplus i"] \arrow[d, "\mathrm{pr}_2"] & x \oplus y \arrow[d, "\mathrm{pr}_2"] \\
0 \arrow[r] \arrow[d] & x \arrow[r, hook, "i"] \arrow[d, equal] & y \arrow[d, "g"] \\
0 \arrow[r] & x \arrow[r, hook, "j"] & w
\end{tikzcd}$$

where sh is given by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and sh' is given by the matrix $\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$; note that both are equivalences since $\mathcal{E}$ is in particular additive. All squares in this diagram except for the bottom right one are exact squares and in particular pullbacks and pushouts and we have already observed that the bottom right square is a pullback.

The right most vertical composite is also the left vertical composite in the diagram

$$\begin{tikzcd}
x \oplus y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\oplus g"] & y \arrow[d, "g"] \\
x \oplus w \arrow[r, "\mathrm{pr}_2"] \arrow[d, "j+\mathrm{id}"] & w \arrow[d] \\
w \arrow[r] & w/x
\end{tikzcd}$$

whose top square is both a pullback and a pushout (it is a sum of two such squares). Exercise: In an additive ∞-category, a square is a pullback if and only if the induced map on (vertical or horizontal) fibres is an equivalence. It follows that the bottom square is a pullback as well, as the induced map on horizontal fibres is the identity of x. Moreover, the right vertical composite is equivalent to $y \to y/x \to w/x$ whose latter map is an equivalence by assumption and whose former map is a projection. We deduce that in the big upper diagram, the right most vertical composite is a projection. It follows that the big combined square is an exact square, and hence also a pushout. It follows that also the square under investigation is a pushout as claimed. □

5.12. **Lemma** *An exact functor between exact ∞-categories preserves inclusions, projections, pushouts along inclusions, pullbacks along projections, and is additive.*

Proof. Let $F\colon \mathcal{E} \to \mathcal{E}'$ be an exact functor and consider an inclusion $x \hookrightarrow y$ in $\mathcal{E}$. Then the associated squares

$$\begin{tikzcd}
x \arrow[r, hook] \arrow[d, two heads] & y \arrow[d] \\
0 \arrow[r] & y/x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Fx \arrow[r] \arrow[d] & Fy \arrow[d] \\
0 \arrow[r] & F(y/x)
\end{tikzcd}$$

are exact. Hence by axiom (4), we find that in the right square $Fx \to Fy$ is an inclusion, $Fy \to F(y/x) \simeq Fy/Fx$ is a projection, and the square is also a pullback square. It follows that F

preserves inclusions, the same argument applied to $F^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{E}'^{\mathrm{op}}$, compare Remark 5.4, shows that F also preserves projections.

Now consider in addition an arbitrary morphism $x \to z$ and the diagrams

$$\begin{tikzcd} x \arrow[r, hook] \arrow[d] & y \arrow[d] \\ z \arrow[r, hook] & w \end{tikzcd} \qquad \begin{tikzcd} Fx \arrow[r, hook] \arrow[d] & Fy \arrow[d] \\ Fz \arrow[r, hook] & Fw \end{tikzcd}$$

in which the left square is a pushout. Since F is exact, it preserves cofibres, so the right square above is a pushout as a consequence of Lemma 5.11. The same argument applied to F^{op} shows that F preserves pullbacks along projections. Finally, to see that F is additive, we note that for $x, y \in \mathcal{E}$, we have a pushout diagram

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d] & y \arrow[d] \\ x \arrow[r, hook] & x \oplus y \end{tikzcd}$$

which is preserved by F as we have just shown. Since F is reduced, we deduce that $F(x \oplus y) \simeq Fx \oplus Fy$ as needed. $\square$

Let us now mention some concrete examples of exact categories.

5.13. **Example**
(1) Let $\mathcal{A}$ be an abelian category. Then it admits an exact structure with inclusions the monomorphisms and projections the epimorphisms. Exact sequences are then simply exact sequences in the usual sense. Hence, any exact subcategory of an abelian category is canonically an exact category.
(2) In particular, for a scheme X, the category $\mathrm{QCoh}(X)$ is an exact category. Concretely, the projections are the maps which are epimorphisms on stalks, and the inclusions are kernels of projections.
(3) Let X be a scheme. The category $\mathrm{Vect}(X)$ of vector bundles on X is an exact category; projections and inclusions are the same as in $\mathrm{QCoh}(X)$, i.e. $\mathrm{Vect}(X) \subseteq \mathrm{QCoh}(X)$ is an exact subcategory.
(4) Let X be a scheme with $\mathcal{O}_X$ coherent or R a coherent ring. The category $\mathrm{Coh}(X)$ or $\mathrm{Coh}(R)$ of coherent sheaves on X or coherent R-modules is an exact subcategory of $\mathrm{QCoh}(X)$, and we have $\mathrm{Vect}(X) \subseteq \mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X)$.

5.14. **Example** Examples of stable ∞-categories are:

(1) the category Sp of spectra,
(2) for a (commutative) ring spectrum A, the category $\mathrm{Mod}(A)$ of A-modules in Sp, similarly the category of perfect A-modules $\mathrm{Perf}(A)$,[37]
(3) for a (commutative) ordinary ring R, the derived ∞-category $\mathcal{D}(R)$ of R, similarly the prefect objects $\mathrm{Perf}(R)$ therein.

[37]Here, perfect modules are the smallest stable subcategory containing A which is in addition closed under retracts. This coincides with compact objects, and if A is commutative, also with dualizable objects.

(4) for a (qcqs) scheme X, the derived ∞-category $\mathcal{D}(X)$ of quasi-coherent sheaves and the subcategory of dualizable objects $\operatorname{Perf}(X)$.[38]
(5) If $\mathcal{C}$ is stable, and I is small, then $\operatorname{Fun}(I,\mathcal{C})$ is a stable category.

5.15. **Construction** Let $\mathcal{E}$ be an exact ∞-category. Associated to it is its category of spans $\operatorname{Span}(\mathcal{E}) = \operatorname{Span}(\mathcal{E}, \mathrm{pr}\mathcal{E}, \mathrm{in}\mathcal{E})$ which is informally given as follows: The objects are the objects of $\mathcal{E}$, and morphisms in $\operatorname{Span}(\mathcal{E})$ from X to Y are given by spans

$$\begin{tikzcd} & Z \arrow[dl, two heads] \arrow[dr, hook] & \\ X & & Y \end{tikzcd}$$

and composition is obtained by forming the diagram

$$\begin{tikzcd} & & U \arrow[dl, two heads] \arrow[dr, hook] & & \\ & Z \arrow[dl, two heads] \arrow[dr, hook] & & W \arrow[dl, two heads] \arrow[dr, hook] & \\ X & & Y & & V \end{tikzcd}$$

in which the square is a pullback, and then taking the big span. Formally, this goes as follows. Recall that the twisted arrow category $\operatorname{Tw}(\Delta^n)$ of Δ^n is a category whose objects are pairs $i \leq j$ in $[n]$, and morphisms from $(i \leq j)$ to $(k \leq l)$ correspond to a relation $(i \leq k \leq l \leq j)$, i.e. a factorization of $(i \leq j)$ through $(k \leq l)$. The association $[n] \mapsto \operatorname{Tw}(\Delta^n)$ is then naturally a cosimplicial category, since the association $\mathcal{C} \mapsto \operatorname{Tw}(\mathcal{C})$ is functorial.

One then defines a simplical space $[n] \mapsto \mathrm{Q}_n(\mathcal{E})$ with $\mathrm{Q}_n(\mathcal{E}) \subseteq \operatorname{Fun}(\operatorname{Tw}(\Delta^n), \mathcal{E})$ the full subcategory on diagrams in which the maps $X_{0\leq j} \to X_{0\leq j-1}$ are projections for all j, the map $X_{j\leq n} \to X_{j+1\leq n}$ are inclusions for all j, and for all $i \leq k \leq l \leq j$ the squares

$$\begin{tikzcd} X_{i\leq j} \arrow[r] \arrow[d] & X_{k\leq j} \arrow[d] \\ X_{i\leq l} \arrow[r] & X_{k\leq l} \end{tikzcd}$$

are exact squares in $\mathcal{E}$. Note that $\mathrm{Q}_0(\mathcal{E}) = \mathcal{E}$ and that $\mathrm{Q}_1(\mathcal{E})$ is precisely the category of spans with left leg a projection and right leg an inclusions. Objects of $\mathrm{Q}_n(\mathcal{E})$ are depicted by diagrams as follows:

$$\begin{tikzcd}[column sep=small]
 & & & & X_{0\leq n} \arrow[dl, two heads] \arrow[dr, hook] & & & & \\
 & & & \dots \arrow[dl, two heads] & & \dots \arrow[dr, hook] & & & \\
 & & X_{0\leq 2} \arrow[dl, two heads] \arrow[dr, hook] & \dots & \dots & & X_{n-2\leq n} \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & X_{0\leq 1} \arrow[dl, two heads] \arrow[dr, hook] & & X_{1\leq 2} \arrow[dl, two heads] \arrow[dr, hook] & \dots & X_{n-2\leq n-1} \arrow[dr, hook] & & X_{n-1\leq n} \arrow[dl, two heads] \arrow[dr, hook] & \\
X_{0\leq 0} & & X_{1\leq 1} & & X_{2\leq 2} & \dots & X_{n-1\leq n-1} & & X_{n\leq n}
\end{tikzcd}$$

[38]Or equivalently, those objects whose restriction to any affine open is dualizable, compact, or perfect.

where all squares are exact and morphisms are natural transformations of such diagrams. In this picture, the Segal maps $\rho_i \colon \mathrm{Q}(\mathcal{E})_n \to \mathrm{Q}(\mathcal{E})_1$

$$\mathrm{Q}_n(\mathcal{E}) \to \mathrm{Q}_1(\mathcal{E}) \times_{\mathcal{E}} \cdots \times_{\mathcal{E}} \mathrm{Q}_1(\mathcal{E})$$

simply extract the lower two rows of the above diagram. Since the rest of the diagram can be reconstructed from it by iteratively taking pullbacks, we find that $\mathrm{Q}(\mathcal{E})$ is a Segal object in Cat_∞. Moreover, $\mathrm{Q}(\mathcal{E})$ turns out to be a *complete* Segal object as we show now. To that end, it suffices to prove that the square

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathrm{Q}_3(\mathcal{E}) \arrow[d] \\
\mathcal{E} \times \mathcal{E} \arrow[r] & \mathrm{Q}_1(\mathcal{E}) \times \mathrm{Q}_1(\mathcal{E})
\end{tikzcd}$$

is a pullback. We observe that for all n, the degeneracy $\mathcal{E} \to \mathrm{Q}_n(\mathcal{E})$ is fully faithful (its image consists of constant functors and $\mathrm{Tw}(\Delta^n)$ is contractible). Since pullbacks of fully faithful functors are also fully faithful, it follows that the map from $\mathcal{E}$ to the pullback of the above square is also fully faithful. Its essential image consists of those diagrams in $\mathrm{Q}_3(\mathcal{E})$

$$\begin{tikzcd}[column sep=small]
 & & & X_{0\le 3} \arrow[dl, two heads] \arrow[dr, hook] & & & \\
 & & X_{0\le 2} \arrow[dl, two heads] \arrow[dr, hook] & & X_{1\le 3} \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & X_{0\le 1} \arrow[dl, two heads] \arrow[dr, hook] & & X_{1\le 2} \arrow[dl, two heads] \arrow[dr, hook] & & X_{2\le 3} \arrow[dl, two heads] \arrow[dr, hook] & \\
X_{0\le 0} & & X_{1\le 1} & & X_{2\le 2} & & X_{3\le 3}
\end{tikzcd}$$

in which all maps are equivalences. The pullback itself consists of the diagrams where the maps $X_{0\le 0} \leftarrow X_{0\le 2} \to X_{2\le 2}$ and $X_{1\le 1} \leftarrow X_{1\le 3} \to X_{3\le 3}$ are equivalence. Since all squares are pullbacks and pushouts (and pullbacks and pushouts of equiavlences are equivalences), we deduce that in fact all arrows are equvialences, showing the desired essential surjectivity. Since complete Segal anima are equivalent to ∞-categories via the associated category functor, we arrive at the definition $\mathrm{Span}(\mathcal{E}, \mathrm{pr}\mathcal{E}, \mathrm{in}\mathcal{E}) = \mathrm{asscat}\,\iota\mathrm{Q}(\mathcal{E})$.

5.16. **Definition** Let $(\mathcal{E}, \mathrm{pr}\mathcal{E}, \mathrm{in}\mathcal{E})$ be an exact ∞-category. We define its connective K-theory space as $\Omega_0|\mathrm{Span}(\mathcal{E}, \mathrm{pr}\mathcal{E}, \mathrm{in}\mathcal{E})|$.

5.17. **Remark** We claim that the association $\mathcal{E} \mapsto K(\mathcal{E})$ refines to a functor $K \colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}$. To see this, we note that $\mathcal{E} \mapsto \mathrm{Fun}(\mathrm{Tw}(\Delta^\bullet), \mathcal{E})$ defines a functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \times \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty$. To see that $\mathcal{E} \mapsto \mathrm{Q}(\mathcal{E})$ is also a functor, we then need to check that for an exact functor $\mathcal{E} \to \mathcal{E}'$ and $[n] \in \Delta$, the induced functor $\mathrm{Fun}(\mathrm{Tw}(\Delta^n), \mathcal{E}) \to \mathrm{Fun}(\mathrm{Tw}(\Delta^n), \mathcal{E}')$ restricts to a functor $\mathrm{Q}_n(\mathcal{E}) \to \mathrm{Q}_n(\mathcal{E}')$; this follows from the definition of exact functors and Lemma 5.12. Then we may use that $\Omega|-| \colon \mathrm{sAn} \to \mathrm{sAn}$ is also a functor.

We leave the proofs of the following two lemmata as exercises.

5.18. **Lemma** *Let $\mathcal{E}$ be an exact ∞-category. There is a canonical equivalence $K(\mathcal{E}) \simeq K(\mathcal{E}^{\mathrm{op}})$.*

5.19. **Lemma** *Let $\mathcal{E}\colon I \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ be a filtered diagram of exact ∞-categories. The induced map* $\mathrm{colim}_i K(\mathcal{E}_i) \to K(\mathrm{colim}_i \mathcal{E}_i)$ *is an equivalence.*

5.20. **Lemma** *There is a canonical equivalence* $\mathrm{Span}(\mathcal{E} \oplus \mathcal{E}') \simeq \mathrm{Span}(\mathcal{E}) \times \mathrm{Span}(\mathcal{E}')$.

Proof. Indeed, this is checked readily on for $\mathrm{Q}(\mathcal{E} \oplus \mathcal{E}')$ as the exact structure on $\mathcal{E} \oplus \mathcal{E}'$ is defined componentwise. Then we can use the product of complete Segal anima corresponds to the product of ∞-categories under the associated category functor. □

Therefore, $\mathrm{Span}\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty$ preserves products and hence canonically upgrades to a functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$. As the functor $\Omega|-|$ also preserves products (and takes values in groups), we arrive at the following definition.

5.21. **Definition** The K-anima functor is the functor

$$K\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{CGrp}(\mathrm{An}) \xleftarrow{\simeq} \mathrm{Sp}_{\geq 0}, \quad \mathcal{E} \mapsto \Omega|\mathrm{Span}(\mathcal{E}, \mathrm{pr}\mathcal{E}, \mathrm{in}\mathcal{E})|$$

taking values in commutative groups in anima, or equivalently, connective spectra.

5.22. **Remark** We note that $|\mathrm{Span}(\mathcal{E}, \mathrm{pr}(\mathcal{E}), \mathrm{in}(\mathcal{E}))|$ is canonically equivalent to $|\iota\mathrm{Q}(\mathcal{E})|$, i.e. the colimit over the simplicial anima $\iota\mathrm{Q}(\mathcal{E})$. Moreover, since $\mathrm{Q}(\mathcal{E})$ takes values in additive ∞-categories and ι preserves finite products, we find that $\iota\mathrm{Q}(\mathcal{E})$ is really a simplicial object in commutative monoids in anima.

5.23. **Definition** We define the K-theory functor on stable ∞-categories as the composite $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{ex}} \xrightarrow{K} \mathrm{Sp}_{\geq 0}$. Hence $K(\mathcal{C}) = \Omega|\mathrm{Span}(\mathcal{C})|$ where the spans are unrestricted since in a stable category, every morphism is an inclusion and a projection. We will study several pleasent properties of the K-theory functor on stable ∞-categories soon.

First, let us define the K- and G-theory of rings and schemes as follows:

5.24. **Definition** Let X be a scheme or a ring. We define its naive K-theory $K^{\mathrm{naive}}(X)$ to be $K(\mathrm{Vect}(X))$. For a scheme $(X, \mathcal{O}_X)$ or ring R where $\mathcal{O}_X$ or R is coherent we set $G(X) = K(\mathrm{Coh}(X))$.

5.25. **Remark** For a coherent ring R he have $\mathrm{Coh}(R) = \mathrm{Mod}^{\mathrm{fp}}(R)$ is simply the (abelian) category of finitely presented modules. If R is Noetherian, then $\mathrm{Coh}(R) = \mathrm{Mod}^{\mathrm{fg}}(R)$ is the (abelian) category of finitely generated modules. In full generality, the category of coherent modules over a ring R is an abelian category, but if R is not coherent, we will not be able to say much about it, and so choose not to consider it explicitly. Note that for a coherent ring or scheme with coherent structure sheaf, any finite projective module or vector bundle is coherent and hence there is a canonical map $K^{\mathrm{naive}}(X) \to G(X)$ – this, for instance, is not true for rings which are not coherent: Surely, R is a finitely generated projective, but not coherent.

5.26. **Definition** For qcqs schemes X, we will define $K(X) = K(\mathrm{Perf}(X))$ where $\mathrm{Perf}(X)$ is a stable ∞-category of *perfect* $\mathcal{O}_X$-modules. This will turn out to agree with $K^{\mathrm{naive}}(X)$ in many, but not all cases.

Similarly, we will define $G(X) = K(\mathrm{APerf}^{\mathrm{b}}(X))$ where $\mathrm{APerf}^{\mathrm{b}}(X)$ is the stable ∞-category of bounded and almost perfect (aka pseudocoherent) modules. In the case where X is a coherent ring R, this is the same as the full subcategory of $\mathcal{D}(R)$ on those complexes where all but finitely many homology groups are trivial, and the non-trivial ones are all coherent R-modules. We will come back to these definitions later.

Most of the next results we present are due to Quillen in the context of exact categories.

5.2. $+ = Q$**.** Firstly, we have now competing definitions of $K(R)$ for a ring R. Indeed, we have already defined $K(R)$ as $\iota\mathrm{Proj}(R)^{\mathrm{gp}}$. However, we may also view $\mathrm{Proj}(R)$ as an additive (and hence as a minimal exact) category and take its K-theory in the sense of Definition 5.16. These definitions turn out to agree:

5.27. **Theorem** (Group-completion or "Q=+") *For any additive ∞-category $\mathcal{A}$, the canonical map $(\iota\mathcal{A})^{\mathrm{gp}} \to K(\mathcal{A})$ is an equivalence.*

For the proof of Theorem 5.27, it is useful to first compare $|\mathrm{Span}(\mathcal{A})| = |\mathrm{Q}(\mathcal{A})|$ with a slightly smaller construction, namely the S-construction[39] first prominently used by Waldhausen in his work on the algebraic K-theory of spaces; for a general exact category, it is also induced by a simplicial construction. To explain it, recall that $\mathrm{Ar}(\Delta^n)$ has objects $(i \le j)$ and a morphism from $(i \le j)$ to $(k \le l)$ if $i \le k$ and $j \le l$.

5.28. **Construction** Consider for $[n] \in \Delta$ the full subcategory $\mathrm{S}_n(\mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Ar}(\Delta^n), \mathcal{E})$ consisting of those functors X such that $X_{i\le i} = 0$ for all i and where for all $i \le j \le k \le l$, the square

$$\begin{tikzcd}
X_{i\le k} \arrow[r] \arrow[d] & X_{i\le l} \arrow[d] \\
X_{j\le k} \arrow[r] & X_{j\le l}
\end{tikzcd}$$

is exact. Objects of $\mathrm{S}_n(\mathcal{E})$ are given by diagrams as follows:

$$\begin{tikzcd}
0 \arrow[r] & X_{0\le 1} \arrow[r, hook] \arrow[d, two heads] & X_{0\le 2} \arrow[r, hook] \arrow[d, two heads] & \dots \arrow[r, hook] & X_{0\le n} \arrow[d, two heads] \\
 & 0 \arrow[r] & X_{1\le 2} \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & X_{1\le n} \arrow[d, two heads] \\
 & & 0 \arrow[r] & \dots & \dots \arrow[d, two heads] \\
 & & & 0 \arrow[r] & X_{n-1\le n} \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

in which all horizontal arrows are inclusions, all vertical arrows are projections, and all squares are exact. Note that $[n] \mapsto \mathrm{S}_n(\mathcal{E})$ is a simplicial ∞-category, i.e. that the simplicial structure maps on $\mathrm{Fun}(\mathrm{Ar}(\Delta^\bullet), \mathcal{E})$ restrict to $\mathrm{S}(\mathcal{E})$, and that similarly as for $\mathrm{Q}(\mathcal{E})$, this simplicial category is functorial exact functors $\mathcal{E} \to \mathcal{E}'$.

We also note that extracting the top row

$$0 = X_{0\le 0} \to X_{0\le 1} \to X_{0\le 2} \to \dots \to X_{0\le n}$$

[39]The S stands for Segal.

of a diagram in $\mathrm{S}_n(\mathcal{E})$ describes $\mathrm{S}_n(\mathcal{E})$ as the full subcategory of $\mathrm{Fun}(\Delta^{n-1}, \mathcal{E})$ on those objects all whose structure morphisms are inclusions (but the natural transformations of such diagrams need not be inclusions). Indeed, the remaining part of the a diagram in $\mathrm{S}_n(\mathcal{E})$ is obtained by taking iterated pushouts.

5.29. **Remark** Similarly to Remark 5.22, we have that $\iota\mathrm{S}(\mathcal{E})$ is a simplicial object in $\mathrm{CMon}(\mathrm{An})$.

Our first goal now is to show that there is a canonical equivalence $K(\mathcal{E}) \simeq \Omega|\iota\mathrm{S}(\mathcal{E})|$ and to then analyze the latter. To that end, we need to talk about the following construction:

5.30. **Construction** (Edgewise subdivision) There is a functor $\mathrm{sd}\colon \Delta \to \Delta$ given by $[n] \mapsto [n] \star [n]^{\mathrm{op}} = \{0_l \leq \cdots \leq n_l \leq n_r \leq \cdots \leq 0_r\}$; the subscripts refer to whether the element of $[n]$ is viewed in the left or the right joint factor. Note that $[n]\star[n]^{\mathrm{op}} \cong [2n+1]$ but the former way of writing it makes the functoriality in $[n]$ apparent. Precomposition with sd defines an endo-functor of simplicial anima

$$\mathrm{sd}^*\colon \mathrm{sAn} \to \mathrm{sAn}$$

called the *edgewise subdivision functor*.

Let us first indicate the following general result.

5.31. **Lemma** *Let $X \in \mathrm{sAn}$ be a simplicial anima. There is a canonical equivalence $|X| \simeq |\mathrm{sd}^*(X)|$.*

Proof. Indeed, to see this, note that $\mathrm{sd}^*\colon \mathrm{sAn} \to \mathrm{sAn}$ preserves colimits; it is hence uniquely determined by its restriction along the Yoneda embedding $h\colon \Delta \subseteq \mathrm{sAn}$. Exercise: for $[n] \in \Delta \subseteq \mathrm{sAn}$, we have that $\mathrm{sd}^*[n]$ is ordinary nerve of the 1-category $\mathrm{Tw}(\Delta^n)$, thought of as a simplicial set. Hence, we have $\operatorname{colim} \mathrm{sd}^*[n] \simeq |\mathrm{Tw}(\Delta^n)| \simeq *$ for all $[n]$ since $\mathrm{Tw}(\Delta^n)$ has an initial object. As a consequence, $\operatorname{colim} \circ \mathrm{sd}^* \circ h$ identifies with the constant with value $*$ functor $\Delta \to \mathrm{An}$, just like colim itself. □

5.32. **Proposition** *There is a natural equivalence* $\mathrm{sd}^*(\mathrm{S}(\mathcal{E})) \to \mathrm{Q}(\mathcal{E})^{\mathrm{op}}$ *between the (opposite of the*[40] *) Q-construction and the edgewise subdivision of the S-construction.*

Proof. Consider the natural functor $\mathrm{Tw}(\Delta^n) \to \mathrm{Ar}([n] \star [n]^{\mathrm{op}})$ given by sending $(i \leq j)$ to $(i_l \leq j_r)$, where the subscripts denote in which of the join factors the object lies. Restriction along this functor induces a map $\mathrm{Fun}(\mathrm{Ar}([n]\star[n]^{\mathrm{op}}), \mathcal{E}) \to \mathrm{Fun}(\mathrm{Tw}(\Delta^n), \mathcal{E})$ which one checks to restrict to a map $(\mathrm{sd}^*\mathrm{S})_n(\mathcal{E}) \to \mathrm{Q}_n(\mathcal{E})$ for each $[n]$, and in fact to map of simplicial anima $\mathrm{sd}^*\mathrm{S}(\mathcal{E}) \to \mathrm{Q}(\mathcal{E})^{\mathrm{op}}$. This is the desired equivalence as one checks directly. To see more concretely what this does, let us analyze some low dimensional cases. For instance, on 1-simplices, we need to have an equivalence $\mathrm{S}_3(\mathcal{E}) = \mathrm{sd}_1(\mathrm{S}(\mathcal{E})) \to \mathrm{Q}_1(\mathcal{E})$, which we take to be

[40]There is a functor $\mathrm{op}\colon \Delta \to \Delta$ with $\mathrm{op}[n] = [n]$ and for $f\colon [m] \to [n]$, $\mathrm{op}(f)(i) = n - f(m-i)$. Precomposition with op defines a functor $(-)^{\mathrm{op}}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ for any ∞-category $\mathcal{C}$.

the functor which extracts from a diagram

$$\begin{tikzcd}
0 \arrow[r] & X_{0\leq 1} \arrow[r, hook] \arrow[d] & X_{0\leq 2} \arrow[r, hook] \arrow[d, two heads] & X_{0\leq 3} \arrow[d, two heads] \\
 & 0 \arrow[r] & X_{1\leq 2} \arrow[r, hook] \arrow[d, two heads] & X_{1\leq 3} \arrow[d, two heads] \\
 & & 0 \arrow[r] & X_{2\leq 3} \arrow[d] \\
 & & & 0
\end{tikzcd}$$

the span

$$\begin{tikzcd}
 & X_{0\leq 2} \arrow[dl, two heads] \arrow[dr, hook] & \\
X_{1\leq 2} & & X_{0\leq 3}
\end{tikzcd}$$

as an element of $Q_1(\mathcal{E})^{\mathrm{op}}$. Notice that the whole upper S-construction diagram can be reproduced from this span by taking appropriate pullbacks and pushouts. Similarly, a general n-simplex in the subdivided S-construction is sent to zig-zag

$$X_{n\leq n+1} \leftarrow X_{n-1\leq n+1} \to X_{n-1\leq n+2} \leftarrow X_{n-2\leq n+2} \to \cdots \leftarrow X_{0\leq 2n} \to X_{0\leq 2n+1}$$

determining the lower two rows (and hence all of the diagram) in $Q_n(\mathcal{E})^{\mathrm{op}}$. □

5.33. **Corollary** *There is a canonical equivalence $|\iota Q(\mathcal{E})| \simeq |\iota S(\mathcal{E})|$, natural in exact ∞-categories $\mathcal{E}$.*

Proof. This follows from combining Lemma 5.31 and Proposition 5.32 together with the observation that there is a canonical equivalence $|\iota Q(\mathcal{E})| \simeq |\iota(Q(\mathcal{E})^{\mathrm{op}})|$.[41] □

Proof of Theorem 5.27. By Corollary 5.33, it suffices to prove that $\iota\mathcal{A} \to \Omega|S(\mathcal{A})|$ is a group completion. Let us first observe what the Segal maps are in the simplicial object $S(\mathcal{E})$ for a general exact ∞-category $\mathcal{E}$. Since they are induced by the maps $[1] \to [n]$ giving the morphism $(i-1 \leq i)$, we obtain that the associated Segal map extracts from a diagram $X\colon \mathrm{Ar}(\Delta^n) \to \mathcal{E}$ the term $X_{i-1\leq i}$. Hence, if $\mathcal{E}$ is an exact category, we find that the Segal maps

$$S_n(\mathcal{E}) \to \prod_{i=1}^{n} S_1(\mathcal{E}) = \prod_{i=1}^{n} \mathcal{E}$$

takes a diagram X to the tuple $(X_{0\leq 1}, X_{1\leq 2}, \ldots, X_{n-1\leq n})$. If $\mathcal{E} = \mathcal{A}$ has the minimal exact structure, these maps are equivalences since in the diagrams in $S_n(\mathcal{A})$ all squares are exact and the horizontal maps are direct summand inclusions. Recall from Remark 5.29 that $\iota S(\mathcal{A})$ is a simplicial object in CMon(An) and note that

$$\iota S(\mathcal{A})_{|\Delta^{\mathrm{op}}_{\leq 1}} \simeq \mathrm{Bar}(\iota\mathcal{A})_{|\Delta^{\mathrm{op}}_{\leq 1}}$$

[41] In the case at hand, this is equivalent to the statement that for an ∞-category $\mathcal{C}$, there is a canonical equivalence $|\mathcal{C}| \simeq |\mathcal{C}^{\mathrm{op}}|$.

as functors taking values in CMon(An). Moreover, $\mathrm{Bar}(\iota\mathcal{A})$, again viewed as an object of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{CMon}(\mathrm{An}))$, is left Kan extended from its restriction to $\Delta^{\mathrm{op}}_{\leq 1}$ as discussed in Remark 4.20. We hence obtain a canonical map

$$\mathrm{Bar}(\iota\mathcal{A}) \to \iota\mathrm{S}(\mathcal{A})$$

which induces an equivalence on 0- and 1-simplices and where source and target are both Segal objects. Such a map is an equivalence as we have noted in the proof of Remark 4.20. Hence we obtain:

$$K(\mathcal{A}) = \Omega|\iota\mathrm{Q}(\mathcal{A})| \simeq \Omega|\iota\mathrm{S}(\mathcal{A})| = \Omega|\mathrm{Bar}(\iota(\mathcal{A}))| \simeq (\iota\mathcal{A})^{\mathrm{gp}},$$

where the last equivalence was discussed in Remark 4.16. □

5.34. **Corollary** *Let R be a ring. Then $K(R) \simeq K(\mathrm{Proj}(R)) = K^{\mathrm{naive}}(R)$.*

Proof. For a ring R, $\mathrm{Proj}(R)$ is viewed as an additive category when forming $K(\mathrm{Proj}(R))$, so the result follows from Theorem 5.27. □

5.35. **Remark** Note that for X a non-affine scheme, $K^{\mathrm{naive}}(X)$ is *not* the group completion of $\mathrm{Vect}(X)$: Indeed, for non-affine schemes, vector bundles are not in general projective, so the exact structure on $\mathrm{Vect}(X)$ is not the minimal one. Hence, one cannot apply the group completion, or $Q = +$ Theorem 5.27 above.

Another consequence of the above comparison between the Q- and the S-construction is the following.

5.36. **Corollary** *Let $\mathcal{E}$ be an exact ∞-category. Then $K_0(\mathcal{E})$ is generated by equivalence classes of objects $x \in \mathcal{E}$, with relations given by $[x] + [z] = [y]$ for every exact sequence $x \hookrightarrow y \twoheadrightarrow z$ in $\mathcal{E}$.*

Proof. It is a general fact that for a simplicial anima X with $X_0 \simeq *$, one has

$$\pi_1(|X|) \cong \langle \pi_0(X_1) \mid \pi_0(X_2) \rangle$$

where by the symbols on the right hand side we more concretely mean the following: The group $\pi_1(|X|)$ is generated by $\pi_0(X_1)$ and for each point $y \in \pi_0(X_2)$ we have the relation

$$d_1(y) = d_2(y) \star d_0(y)$$

where $\star$ is the multiplication in the free group generated by $\pi_0(X_1)$ and d_i are the face operators of the simplicial anima X. Now let us specialize this to $\mathrm{S}(\mathcal{E})$. By definition, we have

(1) $\mathrm{S}_0(\mathcal{E}) = *$,
(2) $\mathrm{S}_1(\mathcal{E}) = \mathcal{E}$, and
(3) $\mathrm{S}_2(\mathcal{E}) = \mathrm{Ex}(\mathcal{E})$, the ∞-category of exact sequences in $\mathcal{E}$.[42]

The face operators $d_i \colon \mathrm{Ex}(\mathcal{E}) = \mathrm{S}_2(\mathcal{E}) \to \mathrm{S}_1(\mathcal{E}) = \mathcal{E}$ are then concretely given as follows: For an exact sequence $e = [x \hookrightarrow y \twoheadrightarrow z]$ in $\mathrm{S}_2(\mathcal{E})$ we have that $d_0(e) = z$, $d_2(e) = x$ and $d_1(e) = y$, giving the desired result. □

In what follows, we will often make use of the following result.

5.37. **Proposition** (Quillen's theorem A) *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then f is cofinal if and only if for all $d \in \mathcal{D}$, the category $\mathcal{C}_{d/}$ is contractible. Moreover, if f is cofinal, then $|f| \colon |\mathcal{C}| \to |\mathcal{D}|$ is an equivalence.*

[42] As noted before, this is the equivalently given by the full subcategory of $\mathrm{Ar}(\mathcal{E})$ on inclusions.

5.3. **Resolution.** An important tool for invariance properties of K-theory is the notion of resolving functors of exact ∞-categories:

5.38. **Definition** Let $\mathcal{E}$ be an exact ∞-category and $i\colon \mathcal{A} \to \mathcal{E}$ be a full inclusion of an exact subcategory.[43] Then i is called *resolving* if the following conditions hold:

(1) For every exact sequence $x \hookrightarrow y \twoheadrightarrow z$ in $\mathcal{E}$ with $y \in \mathcal{A}$, it follows that $x \in \mathcal{A}$.
(2) For every $z \in \mathcal{E}$, there exists an projection $y \twoheadrightarrow z$ with $y \in \mathcal{A}$.

If i^{op} is resolving, we call i op-resolving.

5.39. **Remark** In the setup of Definition 5.38, i is resolving if and only if it satisfies the following conditions:

(1') For every exact sequence $x \hookrightarrow y \twoheadrightarrow z$ in $\mathcal{E}$ with $y, z \in \mathcal{A}$, it follows that $x \in \mathcal{A}$.
(2') For every $z \in \mathcal{E}$, there exists an exact sequence $x \hookrightarrow y \twoheadrightarrow z$ with $x, y \in \mathcal{A}$.

Indeed, we first note that (1) implies (1'), (1) + (2) implies (2'), and (2') implies (2). In particular, a resolving functor satisfies (1') and (2'). Now assume i satisfies (1') and (2'). We need to show that (1) holds. By (2'), we may pick an exact sequence $x' \hookrightarrow y' \twoheadrightarrow z$ with $x', y' \in \mathcal{A}$. Consider then the pullback square

$$\begin{tikzcd} y \times_z y' \arrow[r] \arrow[d] & y' \arrow[d] \\ y \arrow[r] & z \end{tikzcd}$$

Since the square is a pullback, the induced maps on vertical and horizontal fibres are equivalences. The (common) vertical fibre is given by x', so the left vertical fibre sequence, together with the assumption that $y \in \mathcal{A}$ and that $\mathcal{A}$ is extension closed in $\mathcal{E}$ implies that $y \times_z y' \in \mathcal{A}$. Since $y' \in \mathcal{A}$ as well, it follows from (1') that the (common) horizontal fibre x lies in $\mathcal{A}$ as needed.

Quillen's resolution theorem states:

5.40. **Theorem** (Resolution theorem) *Let $i\colon \mathcal{A} \to \mathcal{E}$ be an (op) resolving functor of exact ∞-categories. Then the induced map $K(\mathcal{A}) \to K(\mathcal{E})$ is an equivalence.*

Proof. We consider the functor $\mathrm{Span}(\mathcal{A}) \to \mathrm{Span}(\mathcal{E})$ and denote by $\mathcal{B}$ its essential image, so that we obtain a factorization

$$\mathrm{Span}(\mathcal{A}) \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathrm{Span}(\mathcal{E}).$$

We will show that g is coinitial and that f is cofinal, both using Quillen's theorem A. To show that f is cofinal we need to show that $\mathcal{B}_{x/}$ is weakly contractible. By construction, $\mathcal{B}_{x/}$ is the full subcategory of $\mathrm{Span}(\mathcal{E})_{x/}$ on those objects whose target lies in $\mathcal{A}$, that is, on objects which are represented by spans $x \twoheadleftarrow y \hookrightarrow a$ with $a \in A$ and where the inclusion and projections are in $\mathcal{E}$. Morphisms from $x \twoheadleftarrow y \hookrightarrow a$ to $x \twoheadleftarrow y' \hookrightarrow a'$ are diagrams of the

[43] Recall from Definition 5.3 that this means that $\mathcal{A} \subseteq \mathcal{E}$ is an additive subcategory closed under extensions, and the a map in $\mathcal{A}$ is an inclusion if and only if it is one in $\mathcal{E}$ and its cofibre lies in $\mathcal{A}$; similarly for projections.

following shape

$$\begin{tikzcd}[column sep=small]
 & & y' \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & y \arrow[dl, two heads] \arrow[dr, hook] & & y'' \arrow[dl, two heads] \arrow[dr, hook] & \\
x & & a & & a'
\end{tikzcd}$$

with middle square exact in $\mathcal{E}$. Let $\mathcal{B}_{x/}^{\mathrm{req}}$ denote the full subcategory of $\mathcal{B}_{x/}$ on those spans where the right pointing map $y \hookrightarrow a$ is an equivalence. One computes that $\mathcal{B}_{x/}^{\mathrm{req}}$ is equivalent to the category whose objects consist of projections $a \to x$ and whose morphisms from $a \twoheadrightarrow x$ to $a' \twoheadrightarrow x$ consist of projections $a' \twoheadrightarrow a$ making the evident triangle commute. In particular, we find that this category admits binary products, given by $(a \twoheadrightarrow x) \times (a' \twoheadrightarrow x) = a \times_x a' \twoheadrightarrow x$, where we again use the axioms of resolving functors and the fact that $\mathcal{A} \subseteq \mathcal{E}$ is closed under extensions to see that $a \times_z a' \in \mathcal{A}$: the fibre of the map $a \times_x a' \to a$ is equivalent to the fibre of $a' \twoheadrightarrow x$ which is in $\mathcal{A}$ by axiom (1) of resolving functors. In particular, it follows from an exercise below that $\mathcal{B}_{x/}^{\mathrm{req}}$ is contractible. Moreover, we claim that $\mathcal{B}_{x/}^{\mathrm{req}} \to \mathcal{B}_{x/}$ admits a right adjoint. To that end, consider a span $x \twoheadleftarrow y \hookrightarrow a$ in $\mathcal{B}_{x/}$. Note that axiom (2) of resolving functors implies that $y \in \mathcal{A}$, so the span $x \twoheadleftarrow y = y$ is an object in $\mathcal{B}_{x/}^{\mathrm{req}}$. We now argue that $(x \twoheadleftarrow y \hookrightarrow a) \mapsto (x \twoheadleftarrow y = y)$ is right adjoint to the inclusions. Indeed, the datum of a diagram

$$\begin{tikzcd}[column sep=small]
 & & y' \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & y \arrow[dl, two heads] \arrow[dr, hook, "\simeq"] & & y'' \arrow[dl, two heads] \arrow[dr, hook] & \\
x & & a & & a'
\end{tikzcd}$$

which describes a morphism in $\mathcal{B}_{x/}$ from from the left small span to the big span (with the left small span being an object of $\mathcal{B}_{x/}^{\mathrm{req}}$), is indeed equivalent to the datum of a projection $a' \twoheadrightarrow a$ whose composite with $a \twoheadrightarrow x$ is identified with $a' \twoheadrightarrow x$. This in turn is precisely the datum of a map in $\mathcal{B}_{x/}^{\mathrm{req}}$ from $x \twoheadleftarrow a = a$ to $x \twoheadleftarrow a' = a'$ as needed. We now deduce from Proposition 5.37 that f is cofinal and hence induces an equivalence $|\mathcal{B}| \to |\mathrm{Span}(\mathcal{E})|$.

Next, we show that $g\colon \mathrm{Span}(\mathcal{A}) \to \mathcal{B}$ is coinitial, so pick an object $a \in \mathcal{B}$ and will show that $\mathrm{Span}(\mathcal{A})_{/a}$ is contractible. This category has objects given by spans $x \twoheadleftarrow y \hookrightarrow a$ all whose objects lie in $\mathcal{A}$ and whose inclusions and projections are those in $\mathcal{E}$.[44] A morphism in $\mathrm{Span}(\mathcal{A})_{/a}$ from $x \twoheadleftarrow y \hookrightarrow a$ to $x' \twoheadleftarrow y' \hookrightarrow a$ consists of a diagram

$$\begin{tikzcd}[column sep=small]
 & & y \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & y'' \arrow[dl, two heads, "\in\mathcal{A}"'] \arrow[dr, hook, "\in\mathcal{A}"] & & y' \arrow[dl, two heads] \arrow[dr, hook] & \\
x & & x' & & a
\end{tikzcd}$$

where the label $\in \mathcal{A}$ means that these morphisms are in fact projections and inclusions in $\mathcal{A}$ (i.e have cofibre, respectively fibre also contained in $\mathcal{A}$).

Dually to before, we may consider the full subcategory $\mathrm{Span}(\mathcal{A})_{/a}^{\mathrm{leq}} \subseteq \mathrm{Span}(\mathcal{A})_{/x}$ on those objects whose left pointing map $x \twoheadleftarrow y$ is an equivalence. As before, we note that $\mathrm{Span}(\mathcal{A})_{/a}^{\mathrm{leq}}$

[44] But: from axiom (1) of resolving functors, we find that $y \in \mathcal{A}$, and using the same axiom again, we find that the fibre of $y \to x$ is also in $\mathcal{A}$, i.e. $y \to x$ is in fact a projection in $\mathcal{A}$.

is equivalently described as having objects $y \hookrightarrow a$ being inclusions in $\mathcal{E}$ between objects in $\mathcal{A}$, and morphisms from $y \hookrightarrow a$ to $y' \hookrightarrow a$ are inclusions $y \hookrightarrow y'$ in $\mathcal{A}$ making the triangle commute. This category has an initial object given by $0 \hookrightarrow a$. Moreover, we observe that $\mathrm{Span}(\mathcal{A})^{\mathrm{leq}}_{/x} \to \mathrm{Span}(\mathcal{A})_{/x}$ admits a left adjoint given by $(x \twoheadleftarrow y \hookrightarrow a) \mapsto (y = y \hookrightarrow a)$. Indeed, the datum of a diagram

$$\begin{tikzcd}[column sep=small]
& & y \arrow[dl, two heads] \arrow[dr, hook] & & \\
& y'' \arrow[dl, two heads, "\in\mathcal{A}"'] \arrow[dr, hook, "\in\mathcal{A}"] & & y' \arrow[dl, two heads, "\simeq"] \arrow[dr, hook] & \\
x & & x' & & a
\end{tikzcd}$$

which describes a morphism in $\mathrm{Span}(\mathcal{A})_{/a}$ from the big span to the right small span, is equivalently described by the datum of $y \hookrightarrow y'$ making the composite with $y' \to a$ identified with $y \hookrightarrow a$. Moreover, the map $y \hookrightarrow y'$ is an inclusion in $\mathcal{A}$ since the square appearing above is in particular a pushout, and hence the cofibre of $y \to y'$ indeed lies in $\mathcal{A}$. Again using Proposition 5.37, we conclude that g is coinitial and hence induces an equivalence $|\mathrm{Span}(\mathcal{A})| \to |\mathcal{B}|$, so the theorem is proven. □

Exercise. Show that a non-empty category which admits finite products is contractible. Hint: for $X \in \mathcal{C}$, consider the functor $F = - \times X \colon \mathcal{C} \to \mathcal{C}$. It comes with natural transformations to the identity and the constant functor at X.

5.41. **Definition** Let R be a coherent ring. We denote by $\mathrm{Mod}^{\mathrm{fp}}_{\leq n}(R)$ the full subcategory of $\mathrm{Mod}^{\mathrm{fp}}(R)$ consisting of those finitely presented R-modules which have projective dimension $\leq n$ and by $\mathrm{Mod}^{\mathrm{fp}}_{\mathrm{fpd}}(R) = \bigcup_n \mathrm{Mod}^{\mathrm{fp}}_{\leq n}(R)$ the full subcategory of finitely presented modules with finite projective dimension.

5.42. **Corollary** *Let R be a coherent ring. Then $K(R) \to K(\mathrm{Mod}^{\mathrm{fp}}_{\mathrm{fpd}}(R))$ is an equivalence.*

Proof. We note that $\mathrm{Proj}(R) = \mathrm{Mod}^{\mathrm{fp}}_{\leq 0}(R)$. We now show that for all $n \geq 0$, the functor

$$\mathrm{Mod}^{\mathrm{fp}}_{\leq n}(R) \to \mathrm{Mod}^{\mathrm{fp}}_{\leq n+1}(R)$$

is resolving. First, we note that indeed both categories are exact, as is the canonical inclusion. To prove property (1), given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with $\mathrm{pdim}(M) \leq n$ and $\mathrm{pdim}(M'') \leq n+1$, we need to show that $\mathrm{pdim}(M') \leq n$ as well. For this, consider an arbitrary R-module N. Then there is an exact sequence

$$\mathrm{Ext}^k_R(M, N) \to \mathrm{Ext}^k_R(M', N) \to \mathrm{Ext}^{k+1}(M'', N)$$

so if $k \geq n+1$ the first and the last term vanish, and hence so does the middle term, showing $\mathrm{pdim}(M') \leq n$. To prove property (2), it suffices to recall that any finitely presented module admits by definition a surjection from a finite projective module. The result now follows from Theorem 5.40. □

5.43. **Corollary** *Let R be regular coherent. Then $K(R) \to G(R)$ is an equivalence.*

Proof. By definition of regular coherent rings, we have $\mathrm{Mod}^{\mathrm{fp}}(R) = \mathrm{Mod}^{\mathrm{fp}}_{\mathrm{fpd}}(R)$, so the result follows from the definitions and Corollary 5.42. □

5.44. **Example** Typical examples of regular coherent rings are regular Noetherian rings and valuation rings (or, in the non-local situation more generally Bezout domains); note that a valuation ring is Noetherian if and only if it is a discrete valuation ring (in which case it is in fact a principal ideal domain).

Let us also mention the global versions of the above results. To that end, we make the following definition.

5.45. **Definition** A quasi-compact and quasi-separated scheme X is called *divisorial* if its structure sheaf is coherent and it admits an ample family of line bundles, that is, a family $\{\mathcal{L}_i\}_{i\in I}$ of line bundles $\mathcal{L}_i$ such that for each coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists a finite subset $J \subseteq I$ and for each $j \in J$ a number $n_j > 0$ and a surjection $\bigoplus_{j\in J} \mathcal{L}_i^{\otimes n_i} \to \mathcal{F}$.

5.46. **Example** Quasi-projective schemes over commutative rings are divisorial by a theorem of Serre; in fact for such schemes, $\mathcal{O}(-1)$ is an ample line bundle. Moreover, separated regular Noetherian schemes are also divisorial by a theorem of Illusie (here, really only an ample family exists in general). Subschemes of divisorial schemes are also divisorial.

The analog of Corollary 5.43 for schemes is then:

5.47. **Lemma** *Let X be a divisorial scheme. Then $K^{\mathrm{naive}}(X) \to G(X)$ is an equivalence.*

Proof. Let $\mathrm{Coh}_{\leq n}(X)$ be the full subcategory of $\mathrm{Coh}(X)$ on those coherent modules which have a length n resolution by vector bundles. We claim that the inclusion $\mathrm{Coh}(X)_{\leq n} \to \mathrm{Coh}(X)_{\leq n+1}$ is resolving. To see (1), we need to show that given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

such that $\mathcal{F}$ has a vector bundle resolution of length n and $\mathcal{F}''$ has one of length $n+1$, then $\mathcal{F}'$ also has one of length n. This will be on this week's exercise sheet. Part (2) is again trivial since by definition of divisorial, every coherent sheaf admits a surjection from a vector bundle. Then we note that $\mathrm{Coh}(X)_{\leq 0} = \mathrm{Vect}(X)$, moreover, for a divisorial scheme, we have $\mathrm{Coh}(X) = \mathrm{Coh}(X)_{\leq\infty} = \bigcup_n \mathrm{Coh}(X)_{\leq n}$. The resolution theorem, together with the fact that K-theory commutes with filtered colimits then gives that the map $K(\mathrm{Vect}(X)) \to K(\mathrm{Coh}(X))$ is an equivalence as claimed. $\square$

5.4. **Dévissage.**

5.48. **Theorem** (Dévissage) *Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ a full additive subcategory*[45] *with the following properties:*

(1) *For every exact sequence $a \hookrightarrow b \twoheadrightarrow a'$ in $\mathcal{A}$ with $b \in \mathcal{B}$, it follows that $a, a' \in \mathcal{B}$,*[46]
(2) *every object of $a \in \mathcal{A}$ admits a finite filtration whose filtration quotient lies in $\mathcal{B}$, that is, there is a sequence*

$$0 = a_0 \hookrightarrow a_1 \hookrightarrow a_2 \hookrightarrow \cdots \hookrightarrow a_n = a$$

such that $a_{i+1}/a_i \in \mathcal{B}$ for all i.

Then the inclusion $\mathcal{B} \to \mathcal{A}$ induces an equivalence $K(\mathcal{B}) \to K(\mathcal{A})$.

[45] That is, $0 \in \mathcal{B}$ and $\mathcal{B}$ is closed under finite sums in $\mathcal{A}$.

[46] That is, $\mathcal{B}$ is closed under subobjects and quotients in $\mathcal{A}$.

Proof. First, let us note that $\mathcal{B}$ is itself abelian and the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is exact; recall also that abelian categories are viewed as exact categories as in Example 5.9, see Exercise Sheet 8. Moreover, we claim that $\mathrm{Span}(\mathcal{B}) \subseteq \mathrm{Span}(\mathcal{A})$ is fully faithful. Indeed, this follows since inclusions in $\mathcal{A}$ between objects in $\mathcal{B}$ are already inclusions in $\mathcal{B}$, and likewise for projections, by assumption (1). We prove that the map $\mathrm{Span}(\mathcal{B}) \to \mathrm{Span}(\mathcal{A})$ is coinitial. So let $a \in \mathrm{Span}(\mathcal{A})$ and let us aim to prove that $\mathrm{Span}(\mathcal{B})_{/a}$ is contractible. By assumption there exists a sequence

$$0 = a_0 \hookrightarrow a_1 \hookrightarrow a_2 \hookrightarrow \cdots \hookrightarrow a_n = a$$

giving rise to a sequence of functors

$$\mathrm{Span}(\mathcal{B})_{/0} \to \mathrm{Span}(\mathcal{B})_{/a_1} \to \mathrm{Span}(\mathcal{B})_{/a_2} \to \cdots \to \mathrm{Span}(\mathcal{B})_{/a}.$$

Since $0 \in \mathcal{B}$, we deduce from the fully faithfulness of $\mathrm{Span}(\mathcal{B}) \to \mathrm{Span}(\mathcal{A})$ that $\mathrm{Span}(\mathcal{B})_{/0}$ is contractible. It hence suffices to show that all the functors appearing above are weak equivalences. All of these are functors of the kind

$$\mathrm{Span}(\mathcal{B})_{/x} \to \mathrm{Span}(\mathcal{B})_{/y}$$

for some $x \hookrightarrow y$ whose cokernel lies in $\mathcal{B}$. Let us therefore show, that any such functor is a weak equivalence. For this, consider the full subcategory $\mathrm{Span}(\mathcal{B})^x_{/y}$ of $\mathrm{Span}(\mathcal{B})_{/y}$ consisting of those spans $b \twoheadleftarrow a \hookrightarrow y$ in $\mathcal{A}$ with $b \in \mathcal{B}$ and giving rise to a commutative diagram

$$\begin{tikzcd} p \arrow[r, hook, dashed] \arrow[d, hook] & x \arrow[d, hook] \\ a \arrow[r, hook] & y \end{tikzcd}$$

where $p = \ker(a \twoheadrightarrow b)$. Note that the existence of that dashed arrow is indeed a property, thanks to $x \hookrightarrow y$ being a monomorphism in $\mathcal{A}$. We obtain a canonical factorization of the functor under investigation as

$$\mathrm{Span}(\mathcal{B})_{/x} \xrightarrow{c} \mathrm{Span}(\mathcal{B})^x_{/y} \xrightarrow{i} \mathrm{Span}(\mathcal{B})_{/y}.$$

We claim that the first functor c admits a right adjoint r given by the following construction: It takes a span $b \twoheadleftarrow a \hookrightarrow y$ to the span $(a \cap x)/p \twoheadleftarrow a \cap x \to x$. Moreover, the functor i admits a left adjoint l given by taking a span $b \twoheadleftarrow a \to y$ to the span $a/(p \cap x) \twoheadleftarrow a \hookrightarrow y$ where $p = \ker(a \to b)$. Since $p \cap x \to a \to y$ evidently factors through $x \to y$, to see that this is well-defined, it remains to show that $a/p \cap x \in \mathcal{B}$. To that end, consider the composite $a \to b \oplus y \to b \oplus y/x$. Its kernel is given by the intersection of $p = \ker(a \to b)$ and $\ker(a \to y \to y/x) = a \cap x$, and hence is given by $p \cap x$. Hence we have a monomorphism $a/p \cap x \to b \oplus y/x$ so that the assumption $y/x \in \mathcal{B}$ and condition (1) on $\mathcal{B}$ implies that $a/p \cap x \in \mathcal{B}$ as desired. Now, one computes readily that $rc = \mathrm{id}$. Moreover, cr is the functor given by

$$(b \twoheadleftarrow a \hookrightarrow y) \mapsto ((a \cap x)/p \twoheadleftarrow a \cap x \hookrightarrow y)$$

There is a natural transformation between cr and the identity depicted objectwise by the following diagram:

$$\begin{tikzcd} & & a \cap x \arrow[dl, two heads] \arrow[dr, hook] & & \\ & (a \cap x)/p \arrow[dl, equal] \arrow[dr] & & a \arrow[dl, two heads] \arrow[dr, hook] & \\ (a \cap x)/p & & b & & y \end{tikzcd}$$

I recommend writing out explicitly that this really is a map in $\mathrm{Span}(\mathcal{B})_{/y}$ from $cr(-)$ to $\mathrm{id}(-)$.

Similarly, we have $li = \mathrm{id}$ and $il(b \twoheadleftarrow a \hookrightarrow y) = (a/(p\cap x) \twoheadleftarrow a \hookrightarrow y)$. The following diagram then witnesses a natural map between id and li:

$$\begin{tikzcd}
 & & a \arrow[dl, two heads] \arrow[dr, equal] & & \\
 & a/(p\cap x) \arrow[dl, two heads] \arrow[dr, equal] & & a \arrow[dl, two heads] \arrow[dr, hook] & \\
b = a/p & & a/(p\cap x) & & y
\end{tikzcd}$$

Hence, both functors under investigation induce equivalences upon geometric realization as needed. $\square$

5.49. **Corollary** *Let R be a coherent ring and I a two-sided nilpotent ideal. Then the canonical map $G(R) \to G(R/I)$ is an equivalence.*

Proof. Consider the full inclusion of abelian categories $\mathrm{Mod}^{\mathrm{fp}}(R/I) \subseteq \mathrm{Mod}^{\mathrm{fp}}(R)$, with image consisting of those R-modules on which I acts trivially (note that the restriction functor indeed sends finitely presented modules to finitely presented modules if we assume I is itself finitely generated. Since I is the filtered colimit over its finitely generated sub R-modules and K-theory commutes with filtered colimits, we may restrict to the case where I is finitely generated). Note that the composite $\mathrm{Mod}^{\mathrm{fp}}(R/I) \to \mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}^{\mathrm{fp}}(R/I)$ is the identity, so it suffices to prove that the former functor induces an equivalence on K-theory. We show that this follows from Theorem 5.48: It satisfies assumption (1) of Theorem 5.48 evidently. For assumption (2), let $n \geq 0$ be such that $I^n = 0$, consider a finitely presented R-module M and consider its filtration given by

$$0 = MI^n \subseteq MI^{n-1} \subseteq \cdots \subseteq MI \subseteq M$$

and note that its associated graded MI^k/MI^{k+1} is tautologically an R/I-module. Hence, the above functor induces the claimed equivalence $G(R/I) \to G(R)$. $\square$

5.50. **Corollary** *Let $\mathcal{A}$ be an abelian category such that every object has finite filtration with associated graded consisting of simple objects in $\mathcal{A}$. Then there is a canonical equivalence*

$$K(\mathcal{A}) \simeq \bigoplus_{T_s \in S} K(\mathrm{End}_{\mathcal{A}}(T_s))$$

where S is the set of isomorphism classes of simple objects T_s.

Proof. Let $\mathcal{B} \subseteq \mathcal{A}$ be the subcategory of semi-simple objects, i.e. finite sums of simple objects. Then the inclusion $\mathcal{B} \subseteq \mathcal{A}$ satisfies the assumptions of Theorem 5.48, so it suffices to show that $K(\mathcal{B})$ is as described in the theorem. For a finite subset $I \subseteq S$, let $\mathcal{B}_I$ be the subcategory of $\mathcal{B}$ given by finite sums of the simple objects T_i with $i \in I$. Then $\mathcal{B} = \mathrm{colim}_{I \subseteq S}\, \mathcal{B}_I$ and the colimit is filtered. Moreover, the canonical map $\prod_{i\in I} \mathcal{B}_{\{i\}} \to \mathcal{B}_I$, given by taking the sums is an equivalence of categories: It is essentially surjective by construction, and fully faithfulness follows from the fact that $\mathrm{Hom}(T_i, T_j) = 0$ if $i \neq j$. Finally, on Exercise Sheet 8 we show that $\mathcal{B}_{\{s\}} \simeq \mathrm{Proj}(\mathrm{End}_{\mathcal{A}}(T_s))$, giving the desired result. $\square$

To give another application of the dévissage theorem, we consider the following situation. Let R be a coherent ring, $x \in R$ a central element[47]. Denote by $\mathrm{Mod}^{\mathrm{fp}}_x(R)$ the kernel of the localisation functor $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}^{\mathrm{fp}}(R[x^{-1}])$, that is, the full subcategory of $\mathrm{Mod}^{\mathrm{fp}}(R)$

[47]This is merely to avoid talking about general localisations of noncommutative rings.

consisting of finitely presented R-modules M with $M \otimes_R R[x^{-1}] = 0$. There is an evident functor $\mathrm{Mod}^{\mathrm{fp}}(R/x) \to \mathrm{Mod}^{\mathrm{fp}}_x(R)$ induced by the restriction of scalars along $R \to R/x$.

5.51. **Corollary** *In the situation above, the map $G(R/x) \to K(\mathrm{Mod}^{\mathrm{fp}}_x(R))$ is an equivalence.*

Proof. (1) of Theorem 5.48 is readily checked by exactness of the functor $- \otimes_R R[x^{-1}]$. For (2), note that since M is finitely presented, there exists $N \geq 1$ such that $x^N \cdot M = 0$. Then we may filter M by the submodules $M_k = \{m \in M \mid x^k m = 0\}$, giving rise to a finite filtration having the property that $x(M_{k+1}/M_k) = 0$, i.e. so that the associated graded is indeed a module over R/x. □

5.52. **Example** The typical example is $\mathbb{Z}$ and $p \in \mathbb{Z}$: In this case we find that $\mathrm{Proj}(\mathbb{F}_p)$ as a subcategory of p-primary torsion abelian groups induces an equivalence in K-theory.

Again, we describe briefly the geometric analog. For this let X be a scheme with coherent structure sheaf and $Z \subseteq X$ a closed subscheme. We denote by $\mathrm{Coh}_Z(X)$ the coherent modules $\mathcal{F}$ which are supported at Z, i.e. which satisfy $\mathcal{F}_{|X\setminus Z} = 0$. There is again an evident functor $i_*\colon \mathrm{Coh}(Z) \to \mathrm{Coh}(X)$ which extends a sheaf on Z by 0 to a sheaf on X.

5.53. **Corollary** *In the situation above, the map $G(Z) \to K(\mathrm{Coh}_Z(X))$ is an equivalence.*

Proof. We use here the fact from algebraic geometry that $i_*\colon \mathrm{Coh}(Z) \to \mathrm{Coh}(X)$ is well-defined and indeed has image contained in $\mathrm{Coh}_Z(X)$. Similarly as in the situation of rings, the exactness of the functor $j^*\colon \mathrm{Coh}(X) \to \mathrm{Coh}(X \setminus Z)$ gives property (1) of Theorem 5.48 and property (2) is also proven similarly as in the case of rings. □

To make efficient use of the last two examples, we briefly mention the following result but will only prove its stable version later. To formulate it, we say that a full subcategory $\mathcal{B} \subseteq \mathcal{A}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if it is closed under sums, subobjects, quotients, and extensions. In that case, $\mathcal{B}$ is itself abelian and the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is exact. Moreover, there exists an abelian category $\mathcal{A}/\mathcal{B}$ together with an exact functor $p\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ having the following universal property: For any abelian category $\mathcal{A}'$, the restriction functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}/\mathcal{B}, \mathcal{A}') \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{A}')$$

is fully faithful and has essential image those exact functors which vanish on $\mathcal{B}$. Moreover, $\mathcal{B} = \ker(p)$. The abelian category $\mathcal{A}/\mathcal{B}$ is called the quotient abelian category. It can be constructed as follows: The objects are the same as that of $\mathcal{A}$ and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(a, b) = \operatorname*{colim}_{\substack{a' \subseteq a, \tilde{b} \subseteq b \\ \text{s.t. } a/a', \tilde{b}' \in \mathcal{B}}} \mathrm{Hom}_{\mathcal{A}}(a', b/\tilde{b}).$$

Exercise: Show that these are the homsets in an abelian category and that it satisfies the desired universal property.

Let us consider an example.

5.54. **Example** Let R be a coherent ring and S a central multiplicatively closed subset and denote by $R \to R[\frac{1}{S}]$ the localisation map. Then $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}^{\mathrm{fp}}(R[\frac{1}{S}])$ is the abelian quotient category by the full subcategory of S-torsion R-modules, i.e. the kernel of the functor $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}(R[\frac{1}{S}])$.

Similarly, if X is a scheme with coherent structure sheaf, $i\colon Z \hookrightarrow X$ a closed subscheme with open complement U, then $i_*\colon \mathrm{Coh}(Z) \subseteq \mathrm{Coh}(X)$ is the inclusion of a Serre subcategory and the canonical functor $\mathrm{Coh}(X)/\mathrm{Coh}(Z) \to \mathrm{Coh}(U)$ is an equivalence.

5.55. **Theorem** (Abelian Localization) *Let $\mathcal{B} \subseteq \mathcal{A}$ be the inclusion of Serre subcategory. Then there is a fibre sequence*

$$K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})$$

where $\mathcal{A}/\mathcal{B}$ is the quotient abelian category.

As an application, of the above localization theorem together with the dévissage theorem, one can obtain the following:

5.56. **Theorem** *Let D be a Dedekind domain and T a multiplicative subset of D. Then there is a fibre sequence*

$$\bigoplus_{\mathfrak{p} \in S} K(\mathbb{F}_{\mathfrak{p}}) \to K(D) \to K(D[\tfrac{1}{T}])$$

where S ranges through the non-zero prime ideals $\mathfrak{p}$ of D such that $\mathfrak{p} \cap T \neq \emptyset$.

Proof. As indicated above, the localisation functor $\mathrm{Mod}^{\mathrm{fp}}(D) \to \mathrm{Mod}^{\mathrm{fp}}(D[\frac{1}{T}])$ is the abelian quotient by the full subcategory of $\mathrm{Mod}^{\mathrm{fp}}(D)$ on the T-torsion submodules. We claim that this category identifies with the category $\mathrm{colim}_{S' \subseteq S} \prod_{\mathfrak{p} in S'} \mathrm{Mod}^{\mathrm{fp}}_{\mathfrak{p}}(D)$, where S' is a finite subset of S and $\mathrm{Mod}^{\mathrm{fp}}_{\mathfrak{p}}(D)$ denotes the $\mathfrak{p}$-torsion submodules. This category is equipped with a functor $\mathrm{Vect}(\mathbb{F}_{\mathfrak{p}}) \to \mathrm{Mod}^{\mathrm{fp}}(D)$ which satisfies the assumption of Theorem 5.48. Since K-theory commutes with filtered colimits, the K-theory of the T-torsion modules is indeed as claimed. To finish the proof, it then suffices to note that D and $D[\frac{1}{T}]$ are Dedekind domains, so that Corollary 5.43 gives that the canonical maps $K(D) \to K(\mathrm{Mod}^{\mathrm{fp}}(D))$ and $K(D[\frac{1}{T}]) \to K(\mathrm{Mod}^{\mathrm{fp}}(D[\frac{1}{T}])$ are equivalences. □

5.57. **Remark** Later, we might see that more generally, the following holds true: Let D be a Dedekind domain and S a set of non-zero prime ideals of R. Then there is a fibre sequence

$$\bigoplus_{\mathfrak{p} \in S} K(\mathbb{F}_{\mathfrak{p}}) \to K(D) \to K(D_S)$$

where D_S denotes the localization of D *away from* S and $\mathbb{F}_{\mathfrak{p}} = D/\mathfrak{p}$ is the residue field at $\mathfrak{p}$.

Here, we recall that D_S can be described as follows. It is given by evaluating the structure sheaf $\mathcal{O}_D$ on the set $U = \mathrm{Spec}(D) \setminus S$. In case S is not finite, U is not open. In this case, by $\mathcal{O}_D(U)$ we denote the colimit over $\mathcal{O}_D(V)$ where V ranges through the open subsets of $\mathrm{Spec}(D)$ which contain U. Concretely, D_S may be identified with the subring of the fraction field K of D consisting of the elements $x \in K$ such that the $\mathfrak{p}$-adic valuation $\nu_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \in U$. Recall here that the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$ is a discrete valuation ring, so K obtains a $\mathfrak{p}$-adic valuation. In general, D_S is not obtained from D by inverting a set of elements in D. However, $D \to D_S$ is flat and $D_S \otimes_D D_S \to D_S$ is an isomorphism, so it shares many properties of a localization at a set of elements.

We have the following special cases of the construction D_S.

(1) $S = \mathrm{Spec}(D) \setminus \{0\}$. In this case $D_S = K$ is the fraction field.
(2) Given a multiplicative subset $T \subseteq D$, consider the set $S = \{\mathfrak{p} \mid \mathfrak{p} \cap T \neq \emptyset\}$. Then $D_S = D[\frac{1}{T}]$.
(3) If $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_n^{r_n} = (x)$, then $D_S = D[\frac{1}{x}]$.

In particular, Theorem 5.56 is indeed a special case of the above. However, since $D \to D_S$ is not a localisation by a set of elements, I do not want to claim here that $\mathrm{Mod}^{\mathrm{fp}}(D) \to \mathrm{Mod}^{\mathrm{fp}}(D_S)$ is a quotient abelian category, though it might well be.

5.58. **Example** As a special case of Theorem 5.56, we obtain an exact sequence

$$\bigoplus_{p\in\mathcal{P}} K_2(\mathbb{F}_p) \to K_2(\mathbb{Z}) \to K_2(\mathbb{Q}) \to \bigoplus_{p\in\mathcal{P}} K_1(\mathbb{F}_p) \to K_1(\mathbb{Z}) \to K_1(\mathbb{Q}) \to \bigoplus_{p\in\mathcal{P}} K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q})$$

Now recall that from Matsumoto's theorem, we have $K_2(\mathbb{F}_p) = 0$ and that Milnor proved that $K_2(\mathbb{Z}) = \mathbb{Z}/2$. Moreover, we have $K_1(\mathbb{Z}) = \mathbb{Z}^\times$, $K_1(\mathbb{Q}) = \mathbb{Q}^\times$, and $K_1(\mathbb{F}_p) = \mathbb{F}_p^\times = \mathbb{Z}/(p-1)$. It follows that there are short exact sequences

$$0 \to \mathbb{Z}/2 \to K_2(\mathbb{Q}) \to \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/(p-1) \to 0$$

and

$$0 \to \mathbb{Z}/2 \to \mathbb{Q}^\times \to \bigoplus_{p\in\mathcal{P}} \mathbb{Z} \to 0$$

the latter of which splits since the quotient is free. The former sequence in fact also splits: We have indicated earlier that $K(\mathbb{Z}) \to K(\mathbb{R}) \to \mathrm{ko}$ induces an isomorphism on π_2, see Remark 3.70.

5.59. **Example** In fact, for F any number field, we have a fibre sequence

$$\bigoplus_{\mathfrak{p}\in S} K(\mathbb{F}_\mathfrak{p}) \to K(\mathcal{O}_F) \to K(F)$$

where all $\mathbb{F}_\mathfrak{p}$ are finite fields and hence again have vanishing $K_2(-)$; moreover as we have noted earlier (Theorem 3.58) that $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$ and hence injects into $K_1(F) = F^\times$. Consequently, we deduce that $K_2(F) \neq 0$ for all number fields. Nevertheless, it turns out that $K_2(\overline{\mathbb{Q}}) = 0$. Writing $\overline{\mathbb{Q}}$ as a filtered colimit of number fields, it is the transition maps on $K_2(-)$ that make the colimit trivial. Recall also that in the introduction to these lectures, we have stated Suslin's theorem that $K(\overline{\mathbb{Q}})/n = \mathrm{ku}/n$; Exercise: See how these two results are compatible. Does one imply the other?

Finally, let us keep the promise made in Remark 3.73.

5.60. **Example** Let F be a finite field. Then $F[t]$ is a PID with fraction field $F(t)$. The localisation sequence from Theorem 5.56 then reads as follows:

$$\bigoplus_{f\in I} K(F[t]/(f)) \to K(F[t]) \to K(F(t))$$

where I denotes the set of equivalence classes of irreducible elements in $F[t]$. Again, $F[t]/(f)$ is a finite field and hence has vanishing $K_2(-)$. Moreover, we will argue later that the canonical map $K(F) \to K(F[t])$ is an equivalence, so it also has vanishing K_2. Consequently, we obtain an exact sequence

$$0 \to K_2(F(t)) \to \bigoplus_{f\in I} [F[t]/(f)]^\times \to F^\times \hookrightarrow F(t)^\times$$

the latter of which is injective. In particular, as promised in Remark 3.73, $K_2(F(t))$ is a big sum of the finite abelian groups $[F[t]/(f)]^\times$.

5.5. **K-theory of stable ∞-categories.** Next, we will be interested in understanding more about the relation between stable and exact categories (and their K-theory). To begin, we need the following general result:

5.61. **Lemma** *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ of small stable ∞-categories has cofibres and fibres.*[48].

Proof. Given a functor $f\colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories, define its kernel $\ker(f)$ to be the full subcategory of $\mathcal{C}$ on objects which are sent to 0. This evidently is a fibre of f. To construct a cofibre of f, one finds that we may replace f by the inclusion of its essential image, and hence restrict to the case where f is fully faithful. In this case, define $\mathcal{D}/\mathcal{C}$ as the localisation of $\mathcal{D}$ at all morphisms whose fibre lies in $\mathcal{C}$. Then one can show that $\mathcal{D}/\mathcal{C}$ is stable, the localisation functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact and

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is indeed a cofibre sequence in $\mathrm{Cat}_\infty^{\mathrm{st}}$. In addition, one has the following computation of mapping spectra in $\mathcal{D}/\mathcal{C}$:

$$\operatorname*{colim}_{\beta\colon z\to y\in\mathcal{C}_{/y}} \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(\beta)) \to \mathrm{map}_{\mathcal{D}/\mathcal{C}}([x],[y])$$

is an equivalence; see e.g. [CDH⁺25, Appendix A] for a thorough discussion. The stable ∞-category $\mathcal{D}/\mathcal{C}$ is called the *Verdier quotient* of $\mathcal{D}$ by the full stable subcategory $\mathcal{C}$. $\square$

To that end, we first record the following result.

5.62. **Proposition** *The inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ admits a left adjoint, $\mathcal{E} \mapsto \mathrm{St}(\mathcal{E})$. The unit of the adjunction $\mathcal{E} \to \mathrm{St}(\mathcal{E})$ is fully faithful and exhibits $\mathcal{E}$ as a full exact subcategory of $\mathrm{St}(\mathcal{E})$.*

Proof. There are several constructions. We give the following. First, consider $\mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{An})$, the category of product preserving presheaves on $\mathcal{E}$. Since $\mathcal{E}$ is additive, this is equivalent to the category $\mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$, and since representable functors preserve products, the Yoneda embedding gives a fully faithful functor $\mathcal{E} \to \mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$. Since $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ preserves products, we obtain fully faithful functors

$$\mathcal{E} \to \mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}) \to \mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}).$$

where the latter category is stable (exercise). Denote by $\mathrm{St}^{\mathrm{add}}(\mathcal{E})$ the smallest stable subcategory of $\mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$ which contains the Yoneda embedding $\mathcal{E} \subseteq \mathrm{Fun}^\pi(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$, and let $h\colon \mathcal{E} \to \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ denote the induced embedding. Then $\mathrm{St}^{\mathrm{add}}(\mathcal{E})$ is generated under finite colimits from $\mathcal{E}$, so that for any stable ∞-category $\mathcal{D}$, we have

$$h^*\colon \mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}^{\mathrm{add}}(\mathcal{E}), \mathcal{D}) \to \mathrm{Fun}^\pi(\mathcal{E}, \mathcal{D})$$

is an equivalence; In other words, $\mathcal{E} \to \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ is a left adjoint of the forgetful functor $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\pi}$ from stable to additive ∞-categories; note that we have not made use of the exact structure on $\mathcal{E}$ at this point. Let us do this now, and consider an exact sequence $e = [x \hookrightarrow y \twoheadrightarrow z]$ in $\mathcal{E}$. Then there is a canonical morphism in $\mathrm{St}^{\mathrm{add}}(\mathcal{E})$ given by

$$\mathrm{cofib}[h(x) \to h(y)] \to h(z).$$

[48]In fact, it has all limits and colimits, but we shall not use this now.

Denote the cofibre of this morphism in $\mathrm{St}^{\mathrm{add}}(\mathcal{E})$ by $E(e)$ and let $\mathrm{Ac}(\mathcal{E}) \subseteq \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ be the smallest stable subcategory of $\mathrm{St}^{\mathrm{add}}(\mathcal{E})$ containing $\mathrm{Ac}(\mathcal{E})$. Then define $\mathrm{St}(\mathcal{E})$ as the Verdier quotient $\mathrm{St}^{\mathrm{add}}(\mathcal{E})/\mathrm{Ac}(\mathcal{E})$. By the universal property of Verdier quotients, we find that

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}(\mathcal{E}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}^{\mathrm{add}}(\mathcal{E}), \mathcal{D}) \to \mathrm{Fun}^{\pi}(\mathcal{E}, \mathcal{D})$$

is fully faithful and has image those additive functors $\mathcal{E} \to \mathcal{D}$ which send exact sequences to fibre sequences; indeed, an exact functor $\mathrm{St}^{\mathrm{add}}(\mathcal{E}) \to \mathcal{D}$ lies in the image of the first functor if and only if it vanishes on objects of the form $E(e)$. This in turn is equivalent to the statement that its restriction to $\mathcal{E}$ along h sends exact sequences to fibre sequences as claimed. Exercise: Such functors are precisely the exact functors from $\mathcal{E}$ to $\mathcal{D}$.

It follows that $\mathcal{E} \mapsto \mathrm{St}(\mathcal{E})$ is a left adjoint of the inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}$. I will not prove the remaining claims in the lectures, but see e.g. [SW25, Thm. 3.13] or the following remark. □

5.63. **Remark** An alternative route to $\mathrm{St}(\mathcal{E})$ goes as follows, see [NW25] for details: Consider the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$ of left exact $\mathrm{Sp}_{\geq 0}$-valued presheaves on $\mathcal{E}$. Left exact here means that exact squares in $\mathcal{E}$ are sent to pullback squares in $\mathrm{Sp}_{\geq 0}$. Since exact squares in $\mathcal{E}$ are pushout squares, it follows that the representable presheaves are left exact, so there is a tautological Yoneda embedding $\mathcal{E} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$. Now, one can show that this category is *prestable*, that is, it is pointed, admits finite colimits, the suspension functor is fully faithful, and all morphisms $x \to \Sigma y$ in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$ admit a fibre $f \to x$ making the fibre sequence $f \to x \to \Sigma y$ also a cofibre sequence. Define then $\tilde{\mathrm{St}}(\mathcal{E}) = \mathrm{colim}_\Sigma \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$; this is often called the Spanier–Whitehead stabilization of a prestable ∞-category. In particular, this is stable, and by the fully faithfulness of Σ, the functor $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}) \to \tilde{\mathrm{St}}(\mathcal{E})$ is again fully faithful. One also writes $\tilde{\mathrm{St}}(\mathcal{E})_{\geq 0} := \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$. One then also obtains that for all stable $\mathcal{D}$, the functors

$$\mathrm{Fun}^{\mathrm{ex}}(\tilde{\mathrm{St}}(\mathcal{E}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{rex}}(\tilde{\mathrm{St}}(\mathcal{E})_{\geq 0}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{D})$$

are equivalences. In particular, $\mathrm{St}(\mathcal{E}) \simeq \tilde{\mathrm{St}}(\mathcal{E})$. Under this equivalence, $\mathrm{St}(\mathcal{E})_{\geq 0}$ is the smallest subcategory of $\mathrm{St}(\mathcal{E})$ closed under finite colimits and containing the Yoneda image $\mathcal{E}$.

5.64. **Remark** One can characterize the essential image of $\mathrm{St}^{\mathrm{add}}(-)\colon \mathrm{Cat}_\infty^{\pi} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ as those stable ∞-categories that admit a *bounded weight structure* and the essential image of $\mathrm{St}(-)\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ as those which admit a *bounded heart structure.* We will not really need any of these notions at this point, and will hence refrain from explaining this structure; see [Sau23, SW25] for definitions and discussions.

5.65. **Remark** It follows that an exact ∞-category $\mathcal{E}$ actually has two canonical K-theories, namely that of $\mathcal{E}$ and that of its stabilization $\mathrm{St}(\mathcal{E})$. What we know so far is that there is a canonical comparison functor $K(\mathcal{E}) \to K(\mathrm{St}(\mathcal{E}))$ induced by the exact functor $\mathcal{E} \to \mathrm{St}(\mathcal{E})$ which is the unit of the adjunction from Proposition 5.62. We will argue that this comparison map is an equivalence in Theorem 5.68.

The notion of resolving functors of exact ∞-categories also has implications for the stabilization:

5.66. **Proposition** *Let $i\colon \mathcal{A} \to \mathcal{E}$ be an (op) resolving functor of exact ∞-categories. Then $\mathrm{St}(\mathcal{A}) \to \mathrm{St}(\mathcal{E})$ is an equivalence.*

Proof. We will only indicate why the result is true, for one formal proof, see [Sau23, Lemma 2.7] or Remark 5.67. By Yoneda and adjunction, it suffices to show that for all stable $\mathcal{D}$, the map

$$i^*\colon \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E},\mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A},\mathcal{D})$$

is an equivalence. If i is resolving, there is at most one extension of an exact functor $F\colon \mathcal{A} \to \mathcal{D}$ to an exact functor $\hat{F}\colon \mathcal{E} \to \mathcal{D}$: Indeed, given $z \in \mathcal{E}$, there exists an exact sequence $x \hookrightarrow y \twoheadrightarrow z$ with $x, y \in \mathcal{A}$, so we must have

$$\hat{F}z = \mathrm{cofib}(\hat{F}x \to \hat{F}y) = \mathrm{cofib}(Fx \to Fy).$$

We might try to define $\hat{F}$ via this formula[49], and once we see that this yields a well-defined and exact functor, this construction defines an inverse of i^*. Let us at least indicate why this the putative value of $\hat{F}$ on an object $z \in \mathcal{E}$ does not depend on the choice of an exact sequence resolving z. To that end, let $x' \hookrightarrow y' \twoheadrightarrow z$ be another exact sequence. Then let us consider the diagram

$$\begin{array}{ccccc} w & \longrightarrow & x & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ y \times_z y' & \twoheadrightarrow & y & \twoheadrightarrow & z \end{array}$$

where the right hand square is exact in $\mathcal{E}$ and the left hand square is defined to be a pullback. Observe then that the map $y \times_z y' \to y$ has fibre equivalent to x' and is hence a projection in $\mathcal{A}$. Since exact functors preserve pullbacks along projections, we deduce that the square

$$\begin{array}{ccc} F(w) & \longrightarrow & F(x) \\ \downarrow & & \downarrow \\ F(y \times_z y') & \longrightarrow & F(y) \end{array}$$

is a pullback and hence a pushout as $\mathcal{D}$ is stable. As a result, the induced map on vertical cofibres is an equivalence. The same argument applies to the square with $x \to y$ replaced by $x' \to y'$, showing the "independence" of $\hat{F}$ on the choice of a resolution. Let us also show that the "resulting" functor $\hat{F}$ is exact. To that end, consider an exact sequence $x \hookrightarrow y \twoheadrightarrow z$ in $\mathcal{E}$ and pick $x' \hookrightarrow y' \twoheadrightarrow z$ with $x', y' \in \mathcal{A}$. We obtain a diagram

$$\begin{array}{ccccc} x' & = & x' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ x \times_y y' & \hookrightarrow & y' & \longrightarrow & z \\ \downarrow & & \downarrow & & \| \\ x & \hookrightarrow & y & \twoheadrightarrow & z \end{array}$$

in which each row and column is an exact sequence and all terms appearing except for x, y are objects of $\mathcal{A}$. Using the "uniqueness" of $\hat{F}$, we may compute $\hat{F}$ of the lower sequence by applying F to the upper part of the diagram and then passing to vertical cofibres. But since F is exact, it sends the upper horizontal sequences to cofibre sequences, and so the claim follows from the fact that cofibres of cofibre sequences are again cofibre sequences. □

5.67. **Remark** Let $\mathcal{A} \subseteq \mathcal{E}$ be an exact full subcategory. This inclusion is called left-special if for every projection $x \twoheadrightarrow a$ with $x \in \mathcal{E}$ and $a \in \mathcal{A}$, there exists a map $b \to x$ with $b \in \mathcal{A}$ such that the composite $b \to x \to a$ is a projection in $\mathcal{A}$. In [SW25, Theorem 1.2] it is shown that

[49] It turns out that this formula is in hindsight the formula for the left Kan extension – the problem is only that it is not a priori clear that the left Kan extension exists if $\mathcal{D}$ is stable (and e.g. only has finite colimits).

left-special inclusions $\mathcal{A} \subseteq \mathcal{E}$ induce fully faithful functors $\mathrm{St}(\mathcal{A}) \subseteq \mathrm{St}(\mathcal{E})$ on stabilizations. A resolving functor is left special: given a projection $x \to a$ with $a \in \mathcal{A}$, we can pick $b \twoheadrightarrow x$ with $b \in B$. Then $b \to x \to a$ is again a projection in $\mathcal{E}$, and part of the definition of a resolving functor says that if $c \to b \to a$ is the associated exact sequence, then $c \in \mathcal{A}$, so $b \to a$ is indeed a projection in $\mathcal{A}$. We deduce that if $i\colon \mathcal{A} \to \mathcal{E}$ is resolving, then $\mathrm{St}(\mathcal{A}) \to \mathrm{St}(\mathcal{E})$ is fully faithful. But it is also essentially surjective: Indeed, for this, we only need to show that $\mathcal{E}$ lies in the essential image, since it generates $\mathrm{St}(\mathcal{E})$ as a stable category. But by definition of resolving functors, for each $x \in \mathcal{E}$ there is an exact sequence $a \to b \to x$ with $a, b \in \mathcal{A}$. By construction, we then see that there is a fibre sequence in $\mathrm{St}(\mathcal{E})$ given by $h(a) \to h(b) \to h(x)$, and the former two lie in $\mathrm{St}(\mathcal{A})$, showing that every object in $\mathcal{E}$ lies in the essential image of the inclusion $\mathrm{St}(\mathcal{A}) \subseteq \mathrm{St}(\mathcal{E})$. This gives another proof of Proposition 5.66.

5.68. **Theorem** (Stable Comparison) *For any exact ∞-category $\mathcal{E}$, the canonical map $K(\mathcal{E}) \to K(\mathrm{St}(\mathcal{E}))$ is an equivalence.*

Proof. Let us use the following notions for an object $x \in \mathrm{St}(\mathcal{E})$, see [SW25, Def. 5.1].

- We say that x admits a length 0 resolution by $\mathcal{E}$ if x is in the essential image of the Yoneda embedding $h\colon \mathcal{E} \subseteq \mathrm{St}(\mathcal{E})$.
- For $n \geq 0$, we say that x admits a length $n+1$ resolution by $\mathcal{E}$ if there exists a cofibre sequence in $\mathrm{St}(\mathcal{E})$ as follows $z \to h(y) \to x$ where z admits a length n resolution by $\mathcal{E}$.

We note that objects which admit a length n-resolution are contained in $\mathrm{St}(\mathcal{E})_{\geq 0}$. We write $\mathrm{St}(\mathcal{E})_n$ for the full subcategory of $\mathrm{St}(\mathcal{E})_{(\geq 0)}$ on objects which admit a length n resolution by $\mathcal{E}$.

We will show the following results about $\mathrm{St}(\mathcal{E})_n$ on the exercise sheet.

(a) We have $\mathrm{St}(\mathcal{E})_n \subseteq \mathrm{St}(\mathcal{E})_{n+1}$, and
(a) $\mathrm{St}(\mathcal{E})_n$ is closed under extensions in $\mathrm{St}(\mathcal{E})$.

In particular, it follows that $\mathrm{St}(\mathcal{E})_n$ is canonically an exact ∞-category: The inclusions/projections are those maps in $\mathrm{St}(\mathcal{E})$ whose cofibre/fibre are again contained in $\mathrm{St}(\mathcal{E})_n$. Next, we show:

(1) We have $\mathrm{St}(\mathcal{E})_{\geq 0} = \bigcup_n \mathrm{St}(\mathcal{E})_n$.
(2) The inclusion functor $\mathrm{St}(\mathcal{E})_n \subseteq \mathrm{St}(\mathcal{E})_{n+1}$ is resolving.
(3) The inclusion $\Omega^n(\mathrm{St}(\mathcal{E})_{\geq 0}) \subseteq \Omega^{n+1}\mathrm{St}(\mathcal{E})_{\geq 0}$ is op-resolving.

For the statement of (3), we use the fact that $\mathrm{St}(\mathcal{E})_{\geq 0}$ is prestable, in particular, that its suspension functor is fully faithful: This indeed implies $\Omega^n\mathrm{St}(\mathcal{E})_{\geq 0} = \Omega^n\Omega\Sigma(\mathrm{St}(\mathcal{E})_{\geq 0}) \subseteq \Omega^{n+1}\mathrm{St}(\mathcal{E})_{\geq 0}$. Now, to see (1), it suffices to know that the collection of objects which admit a finite length resolution by $\mathcal{E}$ is closed under finite colimits in $\mathrm{St}(\mathcal{E})_{\geq 0}$ and contains $\mathcal{E}$. The latter is clear as $\mathcal{E} = \mathrm{St}(\mathcal{E})_0$. For the former, it suffices to show closure under direct sums and under cofibres. For direct sums this follows immediately by induction, and for cofibres we argue as follows. Suppose $x \to y \to z$ is a cofibre sequence in $\mathrm{St}(\mathcal{E})_{\geq 0}$ such that x and y have finite length resolutions, we may assume that both x and y have a length n resolution. Pick $h(b) \to y$ such that its fibre f has a resolution of length $n-1$ and consider the diagram

$$\begin{tikzcd}
c \arrow[r, hook] \arrow[d] & h(b) \arrow[r] \arrow[d] & z \arrow[d, equal] \\
x \arrow[r] & y \arrow[r] & z
\end{tikzcd}$$

Then c is an extension of x and f, both of which have a length n resolution, so it follows from the closure under extensions that c also has a length n resolution. The upper cofibre sequence then shows that z has a resolution of length $n+1$.

To see (2), we need to show that for every object $x \in \mathrm{St}(\mathcal{E})_{n+1}$, there exists a surjection in $\mathrm{St}(\mathcal{E})_{n+1}$ of the form $a \to x$ with $a \in \mathrm{St}(\mathcal{E})_n$. But by definition, there is a map $h(y) \to x$ whose fibre is in $\mathrm{St}(\mathcal{E})_n \subseteq \mathrm{St}(\mathcal{E})_{n+1}$, and $h(y) \in \mathrm{St}(\mathcal{E})_0 \subseteq \mathrm{St}(\mathcal{E})_n$. Moreover, we need to show that for every exact sequence $x \to y \to z$ in $\mathrm{St}(\mathcal{E})_{n+1}$ with $y \in \mathrm{St}(\mathcal{E})_n$, then also $x \in \mathrm{St}(\mathcal{E})_n$. This will also appear on the exercise sheet.

For now, we leave (3) unproven (we will give a different argument for its implications to this proof later). Note however, that by the fact that $\mathrm{St}(\mathcal{E})$ is the Spanier–Whitehead stabilization of $\mathrm{St}(\mathcal{E})_{\geq 0}$, we find that for all $x \in \mathrm{St}(\mathcal{E})$, there exists $k \geq 0$ such that $\Sigma^k x \in \mathrm{St}(\mathcal{E})_{\geq 0}$; i.e. $\mathrm{St}(\mathcal{E}) = \bigcup_n \Omega^n(\mathrm{St}(\mathcal{E})_{\geq 0})$. Now we obtain the following. The maps

$$K(\mathcal{E}) \simeq K(\mathrm{St}(\mathcal{E})_0) \xrightarrow{\simeq} K(\mathrm{St}(\mathcal{E})_n)$$

are equivalences for all $n \geq 0$ by the resolution theorem together with part (2) above. Using that K-theory commutes with filtered colimits, we find that $K(\mathcal{E}) \to K(\mathrm{St}(\mathcal{E})_{\geq 0})$ is an equivalence. Similarly, we then also obtain from (3) and the what we have observed above, that the maps

$$K(\mathrm{St}(\mathcal{E})_{\geq 0}) \to \operatorname*{colim}_n K(\Omega^n \mathrm{St}(\mathcal{E})_{\geq 0}) \to K(\mathrm{St}(\mathcal{E}))$$

are equivalences. This is the statement of which we also give a separate proof, see **??**. $\square$

Having now settled the relation between K-theory of exact ∞-categories and their stable ∞-categories, we now stick to properties of K-theory of stable ∞-categories. The first relevant result is the stable analog of the abelian localization Theorem 5.55. To formulate it, we make the following definition.

5.69. **Definition** A bifibre sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

of stable ∞-categories is called a Verdier sequence. Such a Verdier sequence is called left/right split, if both functors admit left/right adjoints. It is called split if both functors admit both left and right adjoints.

5.70. **Remark** That $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a Verdier sequence is hence equivalent to the condition that $\mathcal{C} \to \mathcal{D}$ is fully faithful and that the resulting functor $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is an equivalence. In particular, the functor $\mathcal{D} \to \mathcal{E}$ in a Verdier sequence is essentially surjective.

Left/right split Verdier sequences are also called semi-orthogonal sequences in the literature (or equivalently, t-structures on $\mathcal{D}$ whose connective and coconnective parts are stable subcategories). Split Verdier sequences are then also called orthogonal decompositions and are also equivalent to stable recollements (the notion of recollements exists in the non-stable context, but simplifies stably to split Verdier sequences).

This implies that a right split Verdier sequence is equivalently given by a functor $p \colon \mathcal{D} \to \mathcal{E}$ which admits a fully faithful right adjoint r, and denoting by $\mathcal{C}$ the kernel of p, the sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

turns out to be a right split Verdier sequence; see Proposition 5.76 for the argument that $\mathcal{C} \to \mathcal{D}$ has a right adjoint in this situation. The same results hold true for left split and split Verdier sequences. See e.g. [CDH+25, Appendix A] for a discussion of these things.

5.71. **Theorem** (Localization theorem) $K(-)\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ *is localizing, i.e. it sends Verdier sequences to fibre sequences of spectra.*

The proof of the localization theorem will proceed in two steps; first we prove the following *additivity theorem.*

5.72. **Theorem** (Additivity theorem) $K(-)\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ *is additive, i.e. sends split Verdier sequences to fibre sequences of spectra.*

Proof. We wish to show that for a split Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the resulting sequence

$$K(\mathcal{C}) \to K(\mathcal{D}) \to K(\mathcal{E})$$

is a fibre sequence of spectra. Note that since $\mathcal{D} \to \mathcal{E}$ is a localisation, any adjoint is fully faithful. That is, the composite $\mathcal{E} \to \mathcal{D} \to \mathcal{E}$ is an equivalence. It follows that the map $K(\mathcal{D}) \to K(\mathcal{E})$ also admits a splitting. Moreover, all K-theory spectra are connective, by definition of K-theory. Since Ω^∞ is conservative on connective spectra, it follows that it is sufficient to show that the sequence of underlying anima

$$K(\mathcal{C}) \to K(\mathcal{D}) \to K(\mathcal{E})$$

is a fibre sequence in anima.

Now, we show that $p\colon \mathcal{D} \to \mathcal{E}$ is a bicartesian fibration. Indeed, first, we recall that a morphism $d \to d'$ in $\mathcal{D}$ is p-cocartesian if and only if for all $z \in \mathcal{D}$, the square

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{D}}(d', z) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}}(d, z) \arrow[d] \\
\mathrm{map}_{\mathcal{E}}(pd', pz) \arrow[r] & \mathrm{map}_{\mathcal{E}}(pd, pz)
\end{tikzcd}$$

is a pullback. Since p admits a left adjoint, say l, the lower map identifies with $\mathrm{map}_{\mathcal{D}}(lpd', z) \to \mathrm{map}_{\mathcal{D}}(lpd, z)$ and under this identification, the square becomes the one obtained from

$$\begin{tikzcd}
lpd \arrow[r] \arrow[d] & lpd' \arrow[d] \\
d \arrow[r] & d'
\end{tikzcd}$$

by applying the functor $\mathrm{map}_{\mathcal{D}}(-, z)$; here, the vertical maps are the counit of the adjunction (l, p). By Yoneda, we deduce that $d \to d'$ is p-cocartesian if and only if this square is a pushout in $\mathcal{D}$. In particular, it follows that the collection of p-cocartesian edges is closed under pullbacks. Now let $pd \to e$ be a morphism in $\mathcal{E}$ and consider the pushout square

$$\begin{tikzcd}
lpd \arrow[r] \arrow[d] & le \arrow[d] \\
d \arrow[r] & d'
\end{tikzcd}$$

First, we note that $pd \to pd'$ is the pushout of $plpd \to ple$, as p preserves pushouts. But since p is a localization, l is fully faithful and hence $pl = \mathrm{id}_{\mathcal{E}}$. Hence, $d \to d'$ is a lift of $pd \to e$. Moreover, we claim that $d \to d'$ is p-cocartesian which follows from the fact that under the equivalence $pd \simeq e$, the left vertical map in the above pushout is the counit $lpd \to d$. Now, since p also admits a (fully faithful) right adjoint, applying the same arguments to p^{op}, we find that p is a bicartesian fibration. We will now show that this implies that the induced functor

$$\mathrm{Span}(p)\colon \mathrm{Span}(\mathcal{D}) \to \mathrm{Span}(\mathcal{E})$$

is a cocartesian fibration. The canonical equivalence $\mathrm{Span}(-) \simeq \mathrm{Span}(-)^{\mathrm{op}}$ then implies that it is also a cartesian fibration. Indeed, we claim that a morphism in $\mathrm{Span}(\mathcal{D})$ is $\mathrm{Span}(p)$-cocartesian if and only if it is given by a span

$$y \xleftarrow{f} x \xrightarrow{g} z$$

where f is p-cartesian and g is p-cocartesian. We show that such spans are $\mathrm{Span}(p)$-cocartesian; this is enough for our purposes. To that end, suppose given diagram

$$\begin{tikzcd} & & py' \arrow[dl] \arrow[dr] & & \\ & py \arrow[dl] \arrow[dr] & & py'' \arrow[dl] \arrow[dr] & \\ px & & pz & & pz' \end{tikzcd}$$

representing a 2-simplex in $\mathrm{Span}(\mathcal{E})$, in particular, the square appearing in the middle is a pullback, and the solid part of the diagram

$$\begin{tikzcd} & & y' \arrow[dl, dashed] \arrow[dr, dashed] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & y \arrow[dl, "f"'] \arrow[dr, "g"] & & y'' \arrow[dl, dashed] \arrow[dr, dashed] & \\ x & & z & & z' \end{tikzcd}$$

to see that the span appearing in the lower left part of this diagram is $\mathrm{Span}(p)$-cocartesian, we need to show that there are essentially unique dashed arrows making the middle square a pullback, and such that p applied to this diagram is the upper solid diagram. Now, since f is p-cartesian, there exists an essentially unique $y' \to y$ lying over $py' \to py$ making the composite with f the given map $y' \to x$. Since the collection of p-cocartesian morphisms are closed under pullbacks, we know that the map $y' \to y''$ (if it exists) is p-cocartesian over $py' \to py''$, so let us simply choose such a p-cocartesian lift. Then the dashed arrow $y'' \to z$ again exists essentially uniquely with the property that its composition with $y' \to y''$ is the given map $y' \to z'$ and that its image under p is the given map $py'' \to pz$. Similarly, there is an essentially unique map $y'' \to z$ whose image under p is the given map $py'' \to pz$ and whose composite with $y' \to y''$ is the composite $y' \to y \to z$. It remains to note that the so constructed square is a pushout. To that end, consider the map of squares induced by the counit of the adjunction (l, p):

$$\begin{tikzcd} & y' \arrow[rr] \arrow[dd] & & y'' \arrow[dd] \\ lpy' \arrow[ur] \arrow[rr] \arrow[dd] & & lpy'' \arrow[ur] \arrow[dd] & \\ & y \arrow[rr] & & z \\ lpy \arrow[ur] \arrow[rr] & & lpz \arrow[ur] & \end{tikzcd}$$

The front face is a pullback since l preserves pullbacks, likewise the bottom and top horizontal faces are pullbacks since $y' \to y''$ and $y \to z$ are p-cocartesian. It follows that the back face is also a pushout as desired.

Next, we need to show that given a morphism α in $\mathrm{Span}(\mathcal{E})$ from $p(y)$ to e, then there exists a $\mathrm{Span}(p)$-cocartesian morphism in $\mathrm{Span}(\mathcal{D})$ form y to z lying over α. To that end, write α as a span

$$p(y) \xleftarrow{\bar{f}} e' \xrightarrow{\bar{g}} e$$

Since p is a cartesian fibration, we can find a p-cartesian morphism $f\colon x \to y$ lying over $\bar{f}$. Since p is cocartesian, we can then find a p-cocartesian morphism $g\colon x \to z$ lying over $\bar{g}$.

We then obtain a cartesian diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Span}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Span}(\mathcal{D}) \arrow[d] \\ * \arrow[r, "0"] & \mathrm{Span}(\mathcal{E}) \end{tikzcd}$$

in which the right vertical map is a bicartesian fibration. To see that the square is cartesian, we can use the Segal space model for the span categories given by the Q-construction. We then appeal to the exercise Sheet on which we show that bicartesian fibrations, like $\mathrm{Span}(\mathcal{D}) \to \mathrm{Span}(\mathcal{E})$ are realization fibrations, i.e. that the square

$$\begin{tikzcd} |\mathrm{Span}(\mathcal{C})| \arrow[r] \arrow[d] & |\mathrm{Span}(\mathcal{D})| \arrow[d] \\ * \arrow[r] & |\mathrm{Span}(\mathcal{E})| \end{tikzcd}$$

is again cartesian. The claim then follows from the fact that $K(-) = \Omega|\mathrm{Span}(-)|$ and that Ω preserves pullbacks. $\square$

5.73. **Remark** We note that we have in fact shown that the functor $|\mathrm{Span}(-)|\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}$ is additive.

5.74. **Remark** We also note that the adjoints in a split Verdier sequence induce maps on K-theory. Therefore, for a split Verdier sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{E}$$

there is a canonical equivalence

$$K(\mathcal{D}) \simeq K(\mathcal{C}) \oplus K(\mathcal{E}).$$

The prototypical example of a split Verdier sequence is the sequence

$$\mathcal{C} \to \mathrm{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C}$$

where the first functor sends an object x to the morphism $x \to 0$. The left functor, sending $x \in \mathcal{C}$ to the unique arrow $x \to 0$, admits a splitting by taking the source s of a morphism. We deduce that the source-and-target functor $(s,t)\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ induces an equivalence

$$(s,t)_*\colon K(\mathrm{Ar}(\mathcal{C})) \xrightarrow{\simeq} K(\mathcal{C}) \oplus K(\mathcal{C}).$$

It follows that the endofunctor of $\mathrm{Ar}(\mathcal{C})$ sending a morphism $f\colon x \to y$ to the zero morphism $0\colon x \to y$ induces the identity on K-theory. Therefore, the effect of $\mathrm{cofib}\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$, sending $f\colon x \to y$ to $\mathrm{cofib}(f)$, on K-theory, is the same as that of the functor taking f to $y \oplus \Sigma y = \mathrm{cofib}(0\colon x \to y)$. This gives another proof of the result we have seen earlier that for any exact sequence $x \to y \to z$ in $\mathcal{C}$, there is the equality $[y] = [x] + [z] \in K_0(\mathcal{C})$. The same arguments apply for $|\mathrm{Span}(-)|$ in place of $K(-)$, as we have only used the additivity theorem.

5.75. **Remark** In fact, the sequence $\mathcal{C} \to \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ is not only the prototypical split Verdier sequence; one can show that it is the universal one in the following sense. Let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a split Verdier sequence. Then there exists a (cartesian) square

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathrm{Ar}(\mathcal{C}) \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{C} \end{tikzcd}$$

whose induced map on vertical fibres is the identity functor of $\mathcal{C}$; that is, any split Verdier sequnce is pulled back from the universal split Verdier sequence $\mathcal{C} \to \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$.

5.76. **Proposition** *The functor* $|\mathrm{Span}(-)|\colon \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{Sp}_{\geq 0}$ *sends left or right split Verdier sequences to fibre sequences. In particular,* $K(-)$ *takes left or right split Verdier sequences to fibre sequences.*

Proof. We prove the right split case. The left split case follows similarly or by passing to opposite categories. So let as suppose that

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "i"] & \mathcal{D} \arrow[l, shift left, "s"] \arrow[r, shift left, "p"] & \mathcal{E} \arrow[l, shift left, "r"] \end{tikzcd}$$

be a right split Verdier sequence, that is r and s are right adjoint to p and i. We first explain the relation between r and s. To that end, consider the unit map $\mathrm{id} \to rp$ as endofunctor of $\mathcal{D}$. The triangle equalities imply that this map is an equivalence after applying p. Therefore, its fibre takes values in the kernel of p, and hence in the image of i. Since i is fully faithful, we may think of $\mathrm{fib}(\mathrm{id} \to rp)$ as a functor $\mathcal{D} \to \mathcal{C}$ which we claim to be right adjoint to i, and hence equivalent to s. Indeed, we need to show that there is a canonical equivalence

$$\mathrm{map}_{\mathcal{D}}(ix, y) \simeq \mathrm{map}_{\mathcal{C}}(ix, \mathrm{fib}(y \to rpy)).$$

but this follows immediately from the fibre sequence $\mathrm{fib}(y \to rpy) \to y \to rpy$ and the fact that $\mathrm{map}_{\mathcal{D}}(ix, rpy) = \mathrm{map}_{\mathcal{E}}(pix, py) = 0$ since $pi = 0$. Consequently, we deduce that there is an exact sequence of functors $is \to \mathrm{id}_{\mathcal{D}} \to rp$. Since $|\mathrm{Span}(-)|$ takes values in $\mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$ (recall that $\pi_0|\mathrm{Span}(-)| \cong \{*\}$), we may consider the maps

$$\begin{tikzcd} |\mathrm{Span}(\mathcal{C})| \oplus |\mathrm{Span}(\mathcal{E})| \arrow[r, shift left, "(i,r)"] & |\mathrm{Span}(\mathcal{D})| \arrow[l, shift left, "(s,p)"] \end{tikzcd}$$

and show that both composites are equivalent to the respective identities. The composite $(s,p) \circ (i,s)$ is an equivalence since $si = \mathrm{id}_{\mathcal{C}}$, $pr = \mathrm{id}_{\mathcal{E}}$, and $pi = 0 = sr$. The other composite is, by construction, the map induced on $|\mathrm{Span}(-)|$ by the functor $\mathcal{D} \to \mathcal{D}$ sending d to $is(d) \oplus pr(d)$. This functor is the composite of

$$\mathcal{D} \to \mathrm{Ar}(\mathcal{D}) \xrightarrow{\mathrm{cofib}} \mathcal{D}$$

where the first functor takes d to the arrow $0\colon \Omega pr(d) \to is(d)$. Now, there is also the functor $\mathcal{D} \to \mathrm{Ar}(\mathcal{D})$ sending d to $\partial\colon \Omega pr(d) \to is(d)$, the boundary map in the fibre sequence $is(d) \to d \to pr(d)$. The composite with the cofibre functor is then by construction the identity of $\mathcal{D}$. Now, in Remark 5.74 we have argued that both of the two functors $\mathcal{D} \to \mathrm{Ar}(\mathcal{D})$ just described induce the same map on $|\mathrm{Span}(-)|$, since the argument there only used the additivity theorem which holds for $|\mathrm{Span}(-)|$. It follows that the composite of these two functors with $\mathrm{cofib}\colon \mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$ also induce the same map, showing that the map under investigation is equivalent to the identity of $|\mathrm{Span}(\mathcal{D})$ as needed. □

Proof of Localization theorem. We recall what we need to prove: Given a bifibre sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ in $\mathrm{Cat}_\infty^{\mathrm{st}}$, we need to show that the induced sequence $K(\mathcal{C}) \to K(\mathcal{D}) \to K(\mathcal{E})$ is a fibre sequence of spectra. Since $\mathcal{D} \to \mathcal{E}$ is essentially surjective, we deduce from Corollary 5.36 that $K_0(\mathcal{D}) \to K_0(\mathcal{E})$ is surjective. As in the proof of the additivity theorem, it hence suffices to show that $K(\mathcal{C}) \to K(\mathcal{D}) \to K(\mathcal{E})$ is a fibre sequence in anima. Since the loop functor preserves fibre sequences, this is implied by the statement that

$$|\mathrm{Span}(\mathcal{C})| \to |\mathrm{Span}(\mathcal{D})| \to |\mathrm{Span}(\mathcal{E})|$$

is a fibre sequence; this is the statement we will establish. To do so, we will argue as follows

(1) We show that there exists an $\mathcal{X}$ and a fibre sequence $|\mathrm{Span}(\mathcal{C})| \to |\mathrm{Span}(\mathcal{D})| \to \mathcal{X}$,[50]
(2) and we show that there is a canonical equivalence $\mathcal{X} \to |\mathrm{Span}(\mathcal{E})|$.

To begin, for a category I, denote by $\mathrm{Fun}^{\mathcal{C}}(I, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(I, \mathcal{D})$ consisting of those functors F such that $\mathrm{cofib}(Fi \to Fj) \in \mathcal{C}$ for all morphisms $i \to j$ in I. Since $\mathcal{C}$ is a full stable subcategory of $\mathcal{D}$, $\mathrm{Fun}^{\mathcal{C}}(I, \mathcal{D})$ is a full stable subcategory of $\mathrm{Fun}(I, \mathcal{D})$. Consider then the simplicial anima X given by $[n] \mapsto |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}))|$ and Y the simplicial stable category given by $[n] \mapsto \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D})$ so that $X = |\mathrm{Span}(Y)|$; note that this is indeed well-defined since for all $f\colon [n] \to [m]$, the induced map $\mathrm{Fun}([m], \mathcal{D}) \to \mathrm{Fun}([n], \mathcal{D})$ restricts to a map $\mathrm{Fun}^{\mathcal{C}}([m], \mathcal{D}) \to \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D})$. As discussed in Exercise 2 Sheet 4, we have a pullback diagram

$$\begin{tikzcd} \mathrm{const}(X_1) \arrow[r] \arrow[d, "d_0"] & \mathrm{dec}(X) \arrow[d, "d_0"] \\ \mathrm{const}(X_0) \arrow[r] & X \end{tikzcd}$$

Now note that $Y_0 = \mathcal{D}$ and that there is a pullback diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & \mathrm{Ar}(\mathcal{D}) \arrow[d, "\mathrm{cofib}"] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

i.e. X_1 consists of arrows $d \to d'$ whose cofibre lies in c. Under this identification, the map $d_0\colon X_1 \to X_0$ is the map sending $d \to d'$ to d'. Since cofib differs from the target projection by an automorphism of $\mathrm{Ar}(\mathcal{D})$, we find from Remark 5.74 that there is a split Verdier sequence

$$\mathcal{D} \to Y_1 \to \mathcal{C}$$

[50]This in fact holds true more generally for any full stable subcategory inclusion $\mathcal{C} \to \mathcal{D}$.

where the first map takes d to the arrow id_d and a splitting of $Y_1 \to \mathcal{C}$ is given by sending c to $\Omega c \to 0$. By the Additivity Theorem 5.72 and Remark 5.73, we see that

$$|\mathrm{Span}(\mathcal{D})| \to |\mathrm{Span}(Y_1)| \to |\mathrm{Span}(\mathcal{C})|$$

is a split fibre sequence and hence that the two maps $\mathcal{D} \times \mathcal{C} \to Y_1$ just described induce an equivalence on K-theory. As a result, we find that $d_0 \colon X_1 \to X_0$ identifies with the map $|\mathrm{Span}(\mathcal{D})| \oplus |\mathrm{Span}(\mathcal{C})| \to |\mathrm{Span}(\mathcal{D})|$ given by the projection on the first factor. Since the constant functor preserve pullbacks, we obtain a pullback diagram

$$\begin{tikzcd}
\mathrm{const}|\mathrm{Span}(\mathcal{C})| \arrow[r] \arrow[d] & \mathrm{dec}(X) \arrow[d, "d_0"] \\
0 \arrow[r] & X
\end{tikzcd}$$

Now we will show that the map $\mathrm{dec}(X) \to X$ is a cartesian transformation. It then follows from Exercise 1 Sheet 4 that the above square remains cartesian after applying geometric realization, and using that $|\mathrm{dec}(X)| \simeq X_0 \simeq |\mathrm{Span}(\mathcal{D})|$, we obtain

$$\begin{tikzcd}
|\mathrm{Span}(\mathcal{C})| \arrow[r] \arrow[d] & |\mathrm{Span}(\mathcal{D})| \arrow[d] \\
0 \arrow[r] & \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}))|
\end{tikzcd}$$

so the term in the right lower corner serves property (1) of the object $\mathcal{X}$ we alluded to in the beginning of the proof. To see that the transformation is indeed cartesian, note that $\mathrm{dec}(Y) \to Y$ is concretely, in simplicial level n, given by the functor that forgets the first morphism of sequence of composable morphisms. This has a fully faithful right adjoint given by the map sending $x_0 \to \cdots \to x_n$ to $x_0 = x_0 \to \cdots \to x_n$, and hence participates in a right split Verdier sequence (see Remark 5.70)

$$\mathcal{C} \to \mathrm{dec}(Y)_n = \mathrm{Fun}^{\mathcal{C}}([1+n],\mathcal{D}) \xrightarrow{d_0} Y_n = \mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}).$$

Indeed, note that the kernel of d_0 consists of diagram $b_0 \to b_1 \to \cdots \to b_{1+n}$ with $b_i = 0$ for all $i \geq 1$, so only b_0 remains as part of the datum, and b_0 lies in $\mathcal{C}$ since by definition of $\mathrm{Fun}^{\mathcal{C}}([1+n],\mathcal{D})$ the cofibre of $b_0 \to b_1$, which is Σb_0 lies in $\mathcal{C}$ and $\mathcal{C}$ is a stable subcategory. Now for $f \colon [m] \to [n]$, we get a commutative diagram

$$\begin{tikzcd}
|\mathrm{Span}(\mathcal{C})| \arrow[r] \arrow[d, equal] & \mathrm{dec}(X)_n \arrow[r, "d_0"] \arrow[d, "\mathrm{dec}(f)"] & X_n \arrow[d, "f"] \\
|\mathrm{Span}(\mathcal{C})| \arrow[r] & \mathrm{dec}(X)_m \arrow[r, "d_0"] & X_m
\end{tikzcd}$$

whose horizontal sequences are fibre sequences by Proposition 5.76 and whose left vertical map is readily checked to be induced by the identity functor of $\mathcal{C}$. Since all objects appearing in the right had square are groups, this suffices to deduce that the right square is cartesian. Hence, $d_0 \colon \mathrm{dec}(X) \to X$ is a cartesian transformation and we obtain a fibre sequence

$$|\mathrm{Span}(\mathcal{C})| \to |\mathrm{Span}(\mathcal{D})| \to \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}))| = \mathcal{X}$$

as explained above. It remains to construct an appropriate equivalence $\mathcal{X} \simeq |\mathrm{Span}(\mathcal{E})|$. To that end, we note that the map $\mathcal{D} \to \mathcal{E}$ induces a functor $\mathrm{Fun}([-],\mathcal{D}) \to \mathrm{Fun}([-],\mathcal{E})$ which restricts to a map $\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D}) \to \mathrm{Fun}(|[-]|,\mathcal{E})$. But since for all $n \in \Delta$, we have $|[n]| \simeq *$, this provides a map $\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D}) \to \mathrm{const}(\mathcal{E})$. The theorem is proven once we show that the induced map

$$\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}))| \to |\mathrm{Span}(\mathcal{E})|$$

is an equivalence. We recall that

$$|\mathrm{Span}(-)| = \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \iota\mathrm{Q}_k(-).$$

and therefore need to analyze $\mathrm{Q}_k(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D})$. To that end, denote by $\mathcal{T}_k$ the subcategory of $\mathrm{Tw}([k])$ given by

$$(0\le 0) \leftarrow (0 \le 1) \to (1 \le 1) \leftarrow (1 \le 2) \to \cdots \leftarrow (k-1 \le k) \to (k \le k)$$

and note that $\mathrm{Q}_k(-) \simeq \mathrm{Fun}(\mathcal{T}_k, -)$ and that $\mathcal{T}_k$ is a finite poset. As a consequence, we find a canonical equivalence

$$\mathrm{Q}_k(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Q}_k(\mathcal{C})}([n],\mathrm{Q}_k(\mathcal{D}))$$

where we note that $\mathrm{Q}_k(\mathcal{C}) \to \mathrm{Q}_k(\mathcal{D})$ is again a full inclusion of a stable subcategory. Now the fact that $\mathcal{T}_n$ is a finite poset (in particular a *strongly finite category* as in [CDH+23, Def. 6.5.1]) implies that the sequence

$$\mathrm{Q}_k(\mathcal{C}) \to \mathrm{Q}_k(\mathcal{D}) \to \mathrm{Q}_k(\mathcal{E})$$

is again a Verdier sequence (it is clear that it is a fibre sequence, but not clear that it is a cofibre sequence and this is where the finiteness of $\mathcal{T}_k$ enters), see e.g. [CDH+23, Prop. 6.5.6]. Therefore, if we can show that for all Verdier sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the map

$$\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \iota\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}) \to \iota\mathcal{E}$$

is an equivalence, we deduce

$$\begin{aligned}
\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D})| &= \operatorname*{colim}_{([n],[k])\in(\Delta^{\mathrm{op}})^2} \iota\mathrm{Q}_k(\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D})) \\
&\simeq \operatorname*{colim}_{([n],[k])\in(\Delta^{\mathrm{op}})^2} \iota\mathrm{Fun}^{\mathrm{Q}_k(\mathcal{C})}([n],\mathrm{Q}_k(\mathcal{D})) \\
&\simeq \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \iota\mathrm{Q}_k(\mathcal{E}) \\
&\simeq |\mathrm{Span}(\mathcal{E})|
\end{aligned}$$

as desired. It hence now remains to show that for all Verdier sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the map $|\iota\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})| \to \iota\mathcal{E}$ is an equivalence. To do that, we note that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}([n],\mathcal{D}) \arrow[d] \\
\mathrm{Fun}([n],\iota\mathcal{E}) \arrow[r] & \mathrm{Fun}([n],\mathcal{E})
\end{tikzcd}$$

natural in $[n] \in \Delta^{\mathrm{op}}$; this in fact uses that $\mathcal{C}$ is closed under retracts in $\mathcal{D}$. Since ι preserves pullback squares, we deduce that there is a canonical equivalence

$$\iota\mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D}) \simeq \iota\mathrm{Fun}([n],\mathcal{D}\times_{\mathcal{E}}\iota\mathcal{E})$$

natural in $[n]$. Hence, the simplicial anima $[n] \mapsto \mathrm{Fun}^{\mathcal{C}}([n],\mathcal{D})$ is the complete Segal anima associated to the ∞-category $\mathcal{D}\times_{\mathcal{E}}\iota\mathcal{E}$. The task is then to show that the functor $\mathcal{D}\times_{\mathcal{E}}\iota\mathcal{E} \to \iota\mathcal{E}$

is an equivalence after geometric realization, or equivalently, is a localization. At this point, we use a result of Cisinski's [Cis19, Cor. 7.6.9]which implies that certain well-behaved localization functors $\mathcal{X} \to \mathcal{Y}$ have the property that the map $\mathcal{X} \times_{\mathcal{Y}} \iota\mathcal{Y} \to \iota\mathcal{Y}$ is again a localization. Moreover, the Verdier projection $\mathcal{D} \to \mathcal{E}$ is such a well-behaved localization. This finishes the proof of the theorem. □

5.77. **Theorem** (Cofinality) *If $\mathcal{C}_0 \to \mathcal{C}$ is a dense inclusion of stable ∞-categories, then $K(\mathcal{C}_0) \to K(\mathcal{C})$ induces an isomorphism on positive homotopy groups and an injection on π_0.*

Proof. The π_0-part of the statement was proven in Exercise 4 Sheet 8. Now, similarly as in the proof of the localization theorem, we claim that the sequence

$$|\mathrm{Span}(\mathcal{C}_0)| \to |\mathrm{Span}(\mathcal{C})| \to \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} |\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}_0}([n],\mathcal{C}))|$$

indeed, we mentioned explicitly that this part of the proof of the localization theorem only uses that $\mathcal{C}_0 \subseteq \mathcal{C}$ is a full stable subcategory, not that it is closed under retracts (which kernels of functors always are). Moreover, we have

$$|\mathrm{Span}(\mathrm{Fun}^{\mathcal{C}_0}([n],\mathcal{C}))| = \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \iota \mathrm{Q}_k\mathrm{Fun}^{\mathcal{C}_0}([n],\mathcal{C}) \simeq \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \iota\mathrm{Fun}^{\mathrm{Q}_k(\mathcal{C}_0)}([n],\mathrm{Q}_k(\mathcal{C})).$$

Now for a general full inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ of stable categories $\iota\mathrm{Fun}^{\mathcal{C}_0}([-],\mathcal{C})$ is the complete Segal anima associated to the subcategory $\mathcal{C}\langle\mathcal{C}_0\rangle$ of $\mathcal{C}$ consisting of all maps whose cofibre lies in $\mathcal{C}_0$. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is closed under retracts, this is equivalent to the condition that the map becomes an equivalence in $\mathcal{C}/\mathcal{C}_0$, this is what we have used in the proof of the localization theorem, but if this is not the case, the we can pick $x \in \mathcal{C} \setminus \mathcal{C}_0$ and consider the map $0 \to x$. Its cofibre is x which is not contained in $\mathcal{C}_0$, but it becomes an equivalence in $\mathcal{C}/\mathcal{C}_0 = 0$. An adaption of Cisinski's argument shows that $|\mathcal{C}\langle\mathcal{C}_0\rangle|$ is discrete with π_0 equal to $K_0(\mathcal{C})/K_0(\mathcal{C}_0)$. Hence we need to compute

$$\operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} K_0(\mathrm{Q}_k(\mathcal{C}))/K_0(\mathrm{Q}_k(\mathcal{C}_0))$$

which, similarly to an earlier argument we made can be identified with the edge-wise subdivision of the Bar construction of the group $A = K_0(\mathcal{C})/K_0(\mathcal{C}_0)$, and is hence equivalent to BA. We therefore find a fibre sequence

$$|\mathrm{Span}(\mathcal{C}_0)| \to |\mathrm{Span}(\mathcal{C})| \to BA$$

and the theorem is proven. □

5.78. **Remark** Suppose $\mathcal{E}_0 \subseteq \mathcal{E}$ is an extension closed and dense subcategory of an exact ∞-category $\mathcal{E}$. Then the inclusion is also left special, see Remark 5.67 for the definition: Given a projection $x \to a$ with $a \in \mathcal{E}_0$ and fibre $z \in \mathcal{E}$, we may choose $z' \in \mathcal{E}$ such that $z \oplus z' \in \mathcal{E}_0$. Then we have a fibre sequence

$$z \oplus z' \to x \oplus z' \to a$$

so that $x \oplus z' \in \mathcal{E}_0$ as $\mathcal{E}_0$ is closed under extensions in $\mathcal{E}$, and the projection $x \oplus z' \to x$ gives the desired property of left-special inclusions.

As a consequence, the induced map $\mathrm{St}(\mathcal{E}_0) \to \mathrm{St}(\mathcal{E})$ is fully faithful and we claim that it is again dense. For this, we observe that in general for an inclusion $\mathcal{A} \subseteq \mathrm{B}$ of a stable subcategory is dense if and only if $\mathrm{B}/\mathcal{A} = 0$ (Exercise). But since $\mathrm{St}(-)$ is a left adjoint, we find that $\mathrm{St}(\mathcal{E})/\mathrm{St}(\mathcal{E}_0) \simeq \mathrm{St}(\mathcal{E}/\mathcal{E}_0)$ and one checks directly that 0 is a quotient of the exact inclusion $\mathcal{E}_0 \to \mathcal{E}$: Any exact functor $F\colon \mathcal{E} \to \mathcal{E}'$ which vanishes on $\mathcal{E}_0$ in fact vanishes. As a result of the stable comparison theorem and the above cofinality theorem, we deduce that $K(\mathcal{E}_0) \to K(\mathcal{E})$ also induces an isomorphism on positive homotopy groups.

Let us discuss some examples of Verdier sequences to which the localization theorem may be applied.

5.79. **Example** First, we record that a sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a Verdier sequence of small stable ∞-categories if and only if

(1) $\mathcal{D} \to \mathcal{E}$ is essentially surjective,
(2) the right adjoint of the functor $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is fully faithful,
(3) the kernel of the functor $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ is compactly generated.

Now if $R \in \mathrm{CAlg}(\mathrm{Sp})$ ans $S \subseteq \pi_*(R)$ is a subset consisting of homogenous elements, then

$$\mathrm{Mod}(R) \to \mathrm{Mod}(R[S^{-1}])$$

has a fully faithful right adjoint: Indeed, the right adjoint is given by the restriction of scalars functor. Since this preserves colimits, the claim reduces to the statement that the counit map $R[S^{-1}] \otimes_R R[S^{-1}] \to R[S^{-1}]$ is an equivalence. This holds true since the action of S is invertible on $R[S^{-1}]$, and in fact holds more generally in case $R \in \mathrm{Alg}(\mathrm{Sp})$ and S satisfies the left/right Ore condition. Moreover, the kernel of $\mathrm{Mod}(R) \to \mathrm{Mod}(R[S^{-1}])$ turns out to be generated by the collection of compact R-modules $\mathrm{cofib}(s)$ where s ranges through the elements of S. Denoting by $\mathrm{Perf}'(R[S^{-1}])$ the essential image of $\mathrm{Perf}(R) \to \mathrm{Perf}(R[S^{-1}])$ (which turns out to be a dense subcategory of $\mathrm{Perf}(R[S^{-1}])$) we obtain a Verdier sequence

$$\mathrm{Perf}(R \text{ on } S) \to \mathrm{Perf}(R) \to \mathrm{Perf}'(R[S^{-1}])$$

where the first term is simply defined to be the kernel of the latter map. See [CDH+25, Appendix A.4] for a thorough discussion of examples of these kind. As a result of the cofinality theorem, we find that the sequence

$$K(\mathrm{Perf}(R \text{ on } S)) \to K(\mathrm{Perf}(R)) \to K(\mathrm{Perf}(R[S^{-1}])$$

is a fibre sequence of anima.

5.80. **Remark** We record also that if R is an ordinary ring, then $\mathrm{Perf}(R)$ is equivalent to $\mathrm{St}(\mathrm{Proj}(R))$. As a consequence of the stable comparison theorem and the the $+ = Q$ theorem, we deduce that $K(\mathrm{Perf}(R)) \simeq K^{\mathrm{naive}}(R) \simeq K(R)$.

To make efficient use of the above fibre sequence in K-theory associated to a localisation $R \to R[S^{-1}]$, we need a better understanding of $K(\mathrm{Perf}(R \text{ on } S))$. This is where the notion of t-structures enters, which we now briefly discuss.

5.81. **Definition** A t-structure on a stable ∞-category $\mathcal{C}$ consists of a pair of subcategories $\mathcal{C}_{\geq 0}$ (the connective objects) and $\mathcal{C}_{\leq 0}$ (the coconnective objects) such that

(1) $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are closed under extensions in $\mathcal{C}$; $\mathcal{C}_{\geq 0}$ is closed under colimits and $\mathcal{C}_{\leq 0}$ is closed under limits.
(2) For every object $X \in \mathcal{C}$ there is a fibre sequence $Y \to X \to Z$ with $\Omega Y \in \mathcal{C}_{\geq 0}$ and $Z \in \mathcal{C}_{\leq 0}$.
(3) We have $\pi_0 \mathrm{map}_C(A, B) = 0$ whenever $\Omega A \in \mathcal{C}_{\geq 0}$ and $B \in \mathcal{C}_{\leq 0}$.

For $n \in \mathbb{Z}$, we write $\mathcal{C}_{\geq n}$ for the full subcategory of objects X which satisfy $\Omega^n X \in \mathcal{C}_{\geq 0}$, and similarly, $\mathcal{C}_{\leq n}$ for the full subcategory of objects X which satisfy $\Omega^n X \in \mathcal{C}_{\leq 0}$. We write

$\mathcal{C}^\heartsuit = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ and call it the *heart* of the t-structure. A t-structure is called *bounded* if for every object X, there exists an $n \geq 0$ such that $X \in \mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq -n}$.

5.82. **Remark** Given a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on a stable ∞-category $\mathcal{C}$, it follows that the mapping *anima* $\mathrm{Map}_{\mathcal{C}}(A, B) = \{*\}$ for all $A \in \mathcal{C}_{\geq 1}$ and $B \in \mathcal{C}_{\leq 0}$. Indeed, this is equivalent to showing that for all $n \geq 0$, we have $\pi_n(\mathrm{Map}_{\mathcal{C}}(A, B)) = 0$, and we have $\pi_n(\mathrm{Map}_{\mathcal{C}}(A, B)) = \pi_0(\mathrm{Map}(A, \Omega^n B)$ which indeed vanishes as $\Omega^n B \in \mathcal{C}_{\leq 0}$ since B is and $\mathcal{C}_{\leq 0}$ is closed under limits in $\mathcal{C}$. As a consequence, we compute that for $B \in \mathcal{C}_{\leq 0}$ and $X \in \mathcal{C}$ with fibre sequence as in Definition 5.81 Item 3 we have a fibre sequence

$$\mathrm{Map}_{\mathcal{C}}(Z, B) \to \mathrm{Map}_{\mathcal{C}}(X, B) \to \mathrm{Map}_{\mathcal{C}}(Y, B)$$

and as just argued, $\mathrm{Map}_{\mathcal{C}}(Y, B)$ is contractible. This shows that the association $X \mapsto Z$ refines to a left adjoint $\tau_{\leq 0} \colon \mathcal{C} \to \mathcal{C}_{\leq 0}$ of the inclusion. Similarly, $X \mapsto Y$ refines to a right adjoint $\tau_{\geq 0} \colon \mathcal{C} \to \mathcal{C}_{\geq 0}$ of the inclusion.

It follows from the above mapping anima computations that $\mathcal{C}^\heartsuit$ is in fact a 1-category, that is for $X, Y \in \mathcal{C}^\heartsuit$, we have that all components of the anima $\mathrm{Map}_{\mathcal{C}}(X, Y)$ are contractible. Moreover, $\mathcal{C}^\heartsuit$ turns out to be an abelian category (Exercise: construct kernels and cokernels in $\mathcal{C}^\heartsuit$ and show that $\mathcal{C}^\heartsuit$ is indeed abelian).

5.83. **Example** The category Sp of spectra has a t-structure with $\mathrm{Sp}_{\geq 0} = \{X \in \mathrm{Sp} \mid \pi_n(X) = 0 \text{ for } n < 0\}$ and $\mathrm{Sp}_{\leq 0} = \{X \in \mathrm{Sp} \mid \pi_n(X) = 0 \text{ for } n > 0\}$. Indeed, the same argument as in anima show that the inclusions $\mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0} \subseteq \mathrm{Sp}$ have the required adjoints, and it is readily checked that this implies the t-structure. This t-structure is not bounded: $\pi_*(\mathbb{S})$ is non-trivial in infinitely many degrees (we will learn about this in Topology V next term).

5.84. **Example** Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be a ring spectrum. There exists a t-structure on $\mathrm{Mod}(R)$ such that $\mathrm{Mod}(R)_{\leq 0} = \{M \in \mathrm{Mod}(R) \mid \pi_n(M) = 0 \text{ for } n > 0\}$; here we do not give the forgetful functor $\mathrm{Mod}(R) \to \mathrm{Sp}$ a name (but use it implicitly to form the homotopy groups of R-modules). Moreover, $\mathrm{Mod}(R)_{\geq 0}$ is the smallest colimit closed subcategory of $\mathrm{Mod}(R)$ containing R. If the underlying spectrum of R is connective, then $\mathrm{Mod}(R)_{\geq 0}$ are precisely the R-modules whose underlying spectrum is connective. If R is coconnective, then $R \in \mathrm{Mod}(R)^\heartsuit$, and vice versa.

5.85. **Proposition** *Let R be an ordinary ring. Then the t-structure on* $\mathrm{Mod}(R)$ *restricts to a t-structure on* $\mathrm{Perf}(R)$ *if and only if R is regular coherent. This t-structure is moreover bounded.*

Proof. The fact that the t-structure (if it exists) is bounded follows from the fact that a perfect R-module only has finitely many non-trivial homotopy groups. Now, for the t-structure to restrict, we show that $\tau_{\leq 0} \colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ preserves perfect modules and show that this implies that the left orthogonal to $\mathrm{Perf}(R)_{\leq 0} = \mathrm{Perf}(R) \cap \mathrm{Mod}(R)_{\leq 0}$ is indeed $\mathrm{Perf}(R)_{\geq 1} = \mathrm{Perf}(R) \cap \mathrm{Mod}(R)_{\geq 1}$. The non-trivial part is to see that if $M \in \mathrm{Perf}(R)$ is left orthogonal to $\mathrm{Perf}(R)_{\leq 0}$, then $M \in \mathrm{Mod}(R)_{\geq 1}$, i.e. that M is also left orthogonal to $\mathrm{Mod}(R)_{\leq 0}$. So let $N \in \mathrm{Mod}(R)_{\leq 0}$. Since $\mathrm{Mod}(R) = \mathrm{Ind}(\mathrm{Perf}(R))$, we can write $N = \mathrm{colim}_{i \in I} N_i$ where N_i is perfect for all $i \in I$ and I is filtered. Then we find that $N = \tau_{\leq 0} N = \mathrm{colim}_{i \in I} \tau_{\leq 0} N_i$, since $\tau_{\leq 0}$ is a left adjoint and filtered colimits commute with homotopy groups and hence the displayed colimit is already coconnective. By assumption $\tau_{\leq 0} N_i$ lies in $\mathrm{Perf}(R)_{\leq 0}$. Since M is perfect, we find

$$\mathrm{Map}_R(M,N) = \mathrm{Map}_R(M, \operatorname*{colim}_{i\in I} \tau_{\leq 0} N_i) = \operatorname*{colim}_{i\in I} \mathrm{Map}_R(M, \tau_{\leq 0} N_i) = 0$$

as needed. Next we argue that $\tau_{\leq 0}\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ restricts to $\mathrm{Perf}(R)$ if and only if for all $M \in \mathrm{Perf}(R)_{\geq 0}$, i.e. perfect R-modules which are connective, we have $\pi_0(M) \in \mathrm{Perf}(R)$. Again, for the non-trivial direction, assume that M is perfect. Then it has only finitely many non-trivial homotopy groups, and the assumption implies that its lowest homotopy group, say $\pi_k(M)$ is perfect. But then there is a fibre sequence $\tau_{\geq k+1}M \to M \to \pi_k(M)$, showing that also $\tau_{\geq k+1}M$ is perfect, and hence that $\pi_{k+1}(M)$ is perfect by a repetition of the same argument as form M. Inductively, we deduce that all homotopy groups of M are perfect, and since $\mathrm{Perf}(R)$ is a stable subcategory of $\mathrm{Mod}(R)$, this implies that $\tau_{\leq 0}(M)$ is also perfect.

Finally, we show that R is regular coherent if and only if for all connective and perfect R-modules M, we have $\pi_0(M)$ is perfect, i.e. the t-structure restricts to $\mathrm{Perf}(R)$ by what we have just argued. If R is regular coherent, then any finitely presented module admits a finite resolution by finite projective R-modules. In particular, any finitely presented R-module is perfect, and $\pi_0(M)$ is finitely presented if M is perfect and connective. For the converse direction, we first show that R is coherent, that is, that all finitely generated ideals I of R are finitely presented. To that end choose a finite generating set F of I and consider the map $f\colon R^F \to R$ with image I. Then $\mathrm{cofib}(f)$ is perfect and connective and moreover $\pi_0\mathrm{cofib}(f) = R/I$. Therefore, R/I is perfect (by assumption) and hence we learn from the fibre sequence $I \to R \to R/I$ that I is also perfect and hence finitely presented as needed. It remains to show that any finitely presented module admits a finite resolution by finite projective R-modules (i.e. is perfect). For this, it suffices to show that submodules of finite free R-modules are perfect: Pick a finite presentation $F \to F' \to M \to 0$, then F' and kernel $F' \to M$ are perfect, and hence so if M. This is shown by an induction over the rank of the finite free module, finishing the proof of the proposition. □

The following theorem due to Barwick [Bar15] is of great importance in algebraic K-theory.

5.86. **Theorem** (Theorem of the heart) *Let $\mathcal{C}$ be a stable ∞-category with a bounded t-structure. Then the map $\mathrm{St}(\mathcal{C}^\heartsuit) \to \mathcal{C}$ induces an equivalence $K(\mathcal{C}^\heartsuit) \to K(\mathcal{C})$.*

Let us discuss some applications of these results. First, we observe:

5.87. **Proposition** *Let R be a regular coherent ring and $S \subseteq \pi_*(R)$ a multiplicative subset satisfying the Ore condition (e.g. S is central or even R is commutative). Then the t-structure on $\mathrm{Perf}(R)$ restricts to a bounded t-structure on $\mathrm{Perf}(R$ on $S)$ whose heart consists of the ordinary abelian category of discrete finitely presented R-modules which vanish upon applying the exact functor $- \otimes_R R[S^{-1}]$.*

Proof. Since the localization $R[S^{-1}]$ is flat, we see that the extension of scalars functor $\mathrm{Perf}(R) \to \mathrm{Perf}(R[S^{-1}])$ is t-exact, that is, it preserves (co)connective objects and commutes with the truncation functors. This implies that the truncation functors on $\mathrm{Perf}(R)$ preserve $\mathrm{Perf}(R$ on $S)$ as claimed. The description of the heart is immediate. □

As a consequence, we get the special case of Quillen's localization sequence Theorem 5.55 for the inclusion $\mathrm{Mod}^{\mathrm{fp}}(R \text{ on } S) \subseteq \mathrm{Mod}^{\mathrm{fp}}(R)$ whose Serre quotient is $\mathrm{Mod}^{\mathrm{fp}}(R[S^{-1}])$.

5.88. **Corollary** *Let R be regular coherent and $S \subseteq \pi_*(R)$ a multiplicative subset satisfying the Ore condition. Then we have a fibre sequence*

$$K(\mathrm{Mod}^{\mathrm{fp}}(R \textit{ on } S)) \to K(R) \to K(R[S^{-1}]).$$

As we discussed, the fibre sequence for Dedekind domains Theorem 5.56 is a consequence of this fibre sequence, by another application of the dévissage theorem to the fibre term $K(\mathrm{Mod}^{\mathrm{fp}}(R \text{ on } S))$.

Let us do another application of the theorem of the heart concerning the Blumberg–Mandell fibre sequence.

5.89. **Corollary** *There is a fibre sequence*

$$K(\mathbb{Z}) \to K(\mathrm{ku}) \to K(\mathrm{KU})$$

where ku *and* KU *are the connective and periodic complex K-theory spectra.*

Proof. First, we recall that $\pi_*(\mathrm{ku}) = \mathbb{Z}[\beta]$ with $|\beta| = 2$ and that $\mathrm{KU} = \mathrm{ku}[\frac{1}{\beta}]$. Hence, we obtain a fibre sequence

$$K(\mathrm{Perf}(\mathrm{ku} \text{ on } \beta)) \to K(\mathrm{ku}) \to K(\mathrm{KU}).$$

Now we claim that the standard t-structure on $\mathrm{Mod}(\mathrm{ku})$ restricts to a bounded t-structure on $\mathrm{Perf}(\mathrm{ku} \text{ on } \beta)$. To see this, we first show that the t-structure restricts to $\mathrm{Perf}(\mathrm{ku})$, and to do that, we again have only to show that $\pi_0(M)$ is a perfect ku-module for every perfect and connective ku-module M. Since M is perfect, $\pi_0(M)$ is a finitely presented $\pi_0(\mathrm{ku}) = \mathbb{Z}$-module, and so it suffices to argue that $\mathbb{Z}$ is a perfect ku-module: Then also $\mathbb{Z}/n = \mathrm{cofib}(\cdot n\colon \mathbb{Z} \to \mathbb{Z})$ is perfect, and so all finitely generated abelian groups (which are just finite sums of $\mathbb{Z}$'s and $\mathbb{Z}/n$'s) are also perfect. But then we see that $\mathbb{Z} = \mathrm{cofib}(\beta\colon \Sigma^2\mathrm{ku} \to \mathrm{ku})$ which is therefore perfect. Now we claim that this t-structure restricts to a bounded t-structure on $\mathrm{Perf}(\mathrm{ku} \text{ on } \beta)$ whose heart consists of the perfect ku-modules with homotopy concentrated in degree 0; this category is just the category $\mathrm{Mod}^{\mathrm{fg}}(\mathbb{Z})$, so the regularity of $\mathbb{Z}$ and Corollary 5.43 give the result. ☐

Finally, we want to show that K-theory is polynomially homotopy invariant on regular rings. Here, we will combine two computations:

5.90. **Corollary** *Let R be a regular coherent ring. Then there is a fibre sequence*

$$K(R) \to K(R[t]) \to K(R[t^{\pm 1}])$$

where the first map is induced by the restriction of scalars functor $\mathrm{Perf}(R) \to \mathrm{Perf}(R[t])$.

Proof. As a special case of Corollary 5.88 above, we have a fibre sequence

$$K(\mathrm{Mod}^{\mathrm{fp}}(R[t] \text{ on } t)) \to K(R[t]) \to K(R[t^{\pm 1}]).$$

Moreover, Corollary 5.51 gives an equivalence $G(R[t]/t) \to K(\mathrm{Mod}^{\mathrm{fp}}(R[t] \text{ on } t))$. But $R[t]/t = R$ and since R is regular coherent, we also have that $K(R) \to G(R)$ is an equivalence by Corollary 5.43. Unravelling the functors that induce these equivalences, we indeed find that $K(R) \to K(R[t])$ is induced by the restriction of scalars functor as claimed. ☐

5.91. **Remark** We also note that the map $K_0(R[t]) \to K_0(R[t^{\pm 1}])$ is surjective, so the above fibre sequence is one in spectra, not only in connective spectra: Since R is regular coherent, so is $R[t]$ and since $K_0(R[t^{\pm 1}])$ is generated by the classes of finite projectives, it suffices to see that such modules can be lifted to finitely presented $R[t]$-modules. To that end, write the finite projective as cokernel of a map between free modules. Such a map is represented by a matrix with finitely many entries in $R[t^{\pm 1}]$. Hence, multiplying by a suitable power of t (which is an isomorphism over $R[t^{\pm 1}]$), we can arrange a representing matrix to take values in $R[t]$ and hence obtain a finitely presented module lifting the given finite projective one.

Next, we will consider the following situation. Note that there are evident ring maps $R[t] \to R[t^{\pm 1}] \leftarrow R[t^{-1}]$. We may then consider the pullback diagram

$$\begin{tikzcd} \mathrm{Perf}(\mathbb{P}^1_R) \arrow[r] \arrow[d] & \mathrm{Perf}(R[t]) \arrow[d] \\ \mathrm{Perf}(R[t^{-1}]) \arrow[r] & \mathrm{Perf}(R[t^{\pm 1}]) \end{tikzcd}$$

where the left upper term is just a name for the pullback in $\mathrm{Cat}^{\mathrm{st}}_\infty$. If R is commutative, however, then this term is indeed perfect complexes on the scheme $\mathbb{P}^1_R$. Now, the right vertical map in this square is a Verdier projection as also used in Corollary 5.90. It is a general fact that therefore also the pulled back functor $\mathrm{Perf}(\mathbb{P}^1_R) \to \mathrm{Perf}(R)$ is a Verdier projection. It follows that the square of K-theory spectra

$$\begin{tikzcd} K(\mathrm{Perf}(\mathbb{P}^1_R)) \arrow[r] \arrow[d] & K(R[t]) \arrow[d] \\ K(R[t^{-1}]) \arrow[r] & K(R[t^{\pm 1}]) \end{tikzcd}$$

is a pullback square. We therefore want to investigate $K(\mathbb{P}^1_R)$ in more detail.

5.92. **Proposition** *There is a right split Verdier sequence*

$$\mathrm{Perf}(R) \to \mathrm{Perf}(\mathbb{P}^1_R) \to \mathrm{Perf}(R)$$

where the first functor is induced by the canonical functor $p\colon \mathrm{Perf}(R) \to \mathrm{Perf}(\mathbb{P}^1_R)$*, and the right adjoint to the projection* $\mathrm{Perf}(\mathbb{P}^1_R) \to \mathrm{Perf}(R)$ *is given by the canonical functor* p *composed with the functor* $- \otimes_{\mathbb{Z}} \mathcal{O}(-1)$*, where* $\mathcal{O}(-1)$ *is the triple* $(\mathbb{Z}[t^{-1}], \mathbb{Z}[t], \cdot t\colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}]) \in \mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}})$*.*

Proof. Let us denote by $\mathcal{O}$ the object $(\mathbb{Z}[t^{-1}], \mathbb{Z}[t], \mathrm{id}\colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}])$; this is the tensor unit in the symmetric monoidal stable ∞-category $\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}})$.[51] First, we need to argue that $\mathrm{Perf}(R) \to \mathrm{Perf}(\mathbb{P}^1_R)$ is fully faithful. For this, it suffices to show fully faithfulness on the generator $R \in \mathrm{Perf}(R)$. But then we see that the endomorphism spectrum in $\mathrm{Perf}(\mathbb{P}^1_R)$ of the image of R is given by the pullback of the cospan $R[t^{-1}] \to R[t^{\pm 1}] \leftarrow R[t]$, which is R as needed. Second, we need to show that $\langle \mathcal{O}, \mathcal{O}(-1) \rangle = \mathrm{Perf}(\mathbb{P}^1_R)$ and lastly, that $\mathrm{map}_{\mathrm{Perf}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{O}(-1)) = 0$, but again, this mapping spectrum is given by the pullback of the cospan $R[t^{-1}] \to R[t^{\pm 1}] \xleftarrow{\cdot t} R[t]$ which indeed vanishes as needed. Finally, we use that $\mathcal{O}(-1)$ is invertible in $\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}})$, so $- \otimes_{\mathbb{Z}} \mathcal{O}(-1)$ is an equivalence, and hence the right adjoint we argue about is indeed fully faithful. $\square$

5.93. **Corollary** *The two functors* p *and* $p(-) \otimes_{\mathbb{Z}} \mathcal{O}(-1)$ *induce an equivalence* $K(R) \oplus K(R) \to K(\mathbb{P}^1_R)$*. Under this equivalence, the canonical map* $K(\mathbb{P}^1_R) \to K(R[t])$ *becomes the composite of the diagonal* $K(R) \oplus K(R) \to K(R)$ *with the canonical map* $K(R) \to K(R[t])$*.*

Proof. The first claim follows immediately from the above analysis and the second claim from the fact that the image of $\mathcal{O}(-1)$ along $\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}}) \to \mathrm{Perf}(\mathbb{Z}[t])$ is just $\mathbb{Z}[t]$. $\square$

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[51] The functors used to define $\mathrm{Perf}(\mathbb{P}^1_R)$ are symmetric monoidal if R is commutative, so the pullback carries a canonical induced symmetric monoidal structure.

From this, we deduce that the square

$$\begin{tikzcd} K(\mathrm{Perf}(\mathbb{P}^1_R)) \arrow[r] \arrow[d] & K(R[t]) \arrow[d] \\ K(R[t^{-1}]) \arrow[r] & K(R[t^{\pm 1}]) \end{tikzcd}$$

admits a map from and to the constant square at $K(R)$, and is hence equivalent to this constant square plus a square of the form

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & NK^+(R) \arrow[d] \\ NK^-(R) \arrow[r] & \bar{K}(R[t^{\pm 1}]) \end{tikzcd}$$

where $NK^+(R) = \mathrm{cofib}[K(R) \to K(R[t])]$ and $NK^-(R) = \mathrm{cofib}[K(R) \to K(R[t^{-1}])]$ are the cofibres of the canonical maps. The above argument then shows that the resulting maps $K(R) \to NK^{\pm}(R)$ are the zero map. We deduce the fundamental theorem of K-theory:

5.94. **Corollary** *For any ring, we have $K(R[t^{\pm 1}]) \simeq K(R) \oplus \Sigma K(R) \oplus NK^+(R) \oplus NK^-(R)$.*

Moreover:

5.95. **Corollary** *Let R be regular coherent. Then $NK(R) = 0$, or in other words, the map $K(R) \to K(R[t])$ is an equivalence, and there is a canonical equivalence $K(R[t^{\pm 1}]) \simeq K(R) \oplus \Sigma K(R)$.*

Proof. In this case, we deduce from Corollary 5.90 that the cofibre of the lower horizontal map in the above square is canonically equivalent to $\Sigma K(R)$. But by the fundamental theorem, the cofibre is $\Sigma K(R) \oplus NK^+(R)$ and being more careful with the precise maps, one deduces that $NK^+(R) = 0$ as claimed. $\square$

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