# ALGEBRAIC K-THEORY

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Abstract. These are lecture notes for my lecture "Algebraic K-theory" which I taught in the summer term 2025 at LMU Munich.

Contents



## 1. Organization

There will be **no lectures** on 23.06. and 25.06. There will be weekly exercises, starting on May 8. I will upload exercise sheets and this script to the homepage of the course

Course Webpage

weakly and after each lecture, respectively. The examination will be an oral exam at the end of the term.

## 2. Introduction and several motivations

2.1. **History.** We begin with some historical remarks. In its simplest form, algebraic $K$-theory can be viewed as a sequence of functors

$$K_n(-)\colon \mathrm{Rings} \to \mathrm{Ab}, \quad n \in \mathbb{Z}$$

where $K_0(R)$ is the *group completion* of the abelian monoid of isomorphism classes of finitely generated projective $R$-modules $\mathrm{Proj}(R)/\mathrm{iso}$ under direct sum. Concretely:

$$K_0(R) = \mathbb{Z}[\mathrm{Proj}(R)/\mathrm{iso}]/\langle [P] + [Q] = [P \oplus Q] \rangle.$$

This group was introduced by Grothendieck in 1957 (in fact in greater generality as we indicate below). At the same time, Bott proved his famous periodicity theorem for the homotopy groups of the stable unitary group U, and hence also for the classifying space for stable vector bundles BU, and Atiyah and Hirzebruch defined the topological $K$-groups $K^*(X)$ in 1959. In

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1964, Bass defined $K_1(R) := \mathrm{GL}(R)^{\mathrm{ab}}$ and proved what is called the fundamental theorem of algebraic $K$-theory: There is an exact sequence of abelian groups

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t^{\pm 1}]) \xrightarrow{\partial} K_0(R) \to 0$$

and the map $\partial$ is split by the map induced by sending $P$ to $\cdot t\colon P \otimes_R R[t^{\pm 1}] \to P \otimes_R R[t^{\pm 1}]$.[1] Bass used this to define negative $K$-groups inductively: For $n < 0$, he sets

$$K_n(R) = \mathrm{coker}(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t^{\pm 1}])).$$

Moreover, Bass, Milnor, and Murthy established an *excision exact sequence* in $K$-groups starting with $K_1(-)$ and lowering degree: For a map $f\colon A \to B$ of rings carrying an ideal $I \subseteq A$ isomorphically to an ideal $J \subseteq B$, there is a long exact sequence:

$$K_1(A) \to K_1(A/I) \oplus K_1(B) \to K_1(B/J) \xrightarrow{\partial} K_0(A) \to K_0(A/I) \oplus K_0(B) \to K_0(B/J)$$

which in fact can be contined indefinitely to the right using Bass' definition of negative $K$-groups. Swan proved that there is no functorial way to extend this sequence to putative higher $K$-groups to the left. Nevertheless, in 1967, Milnor defines $K_2(R)$ and computes $K_2(\mathbb{Z})$. It is slightly more involved to describe $K_2(R)$ than $K_1(R)$, but it goes as follows. One defines the *Steinberg group* $\mathrm{St}(R)$ of a ring $R$ as the group generated by symbols $e_{i,j}(r)$, where $i \neq j$ are natural numbers and $r \in R$, subject to the standard relations that the elementary matrices $E_{i,j}(r) \in \mathrm{GL}(R)$ satisfy.[2] One obtains a group homomorphism $\mathrm{St}(R) \to E(R) \subseteq \mathrm{GL}(R)$ and Milnor defines $K_2(R) = \ker(\mathrm{St}(R) \to \mathrm{GL}(R))$; one can show that this agrees with the center $C(\mathrm{St}(R))$ of $\mathrm{St}(R)$, and in particular $K_2(R)$ is indeed abelian. Since $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ by Whiteheads lemma, there is an exact sequence

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 0.$$

As described, both $K_1(R)$ and $K_2(R)$ are purely algebraic definitions, and there was good reason to believe that these are "the correct definitions" – mostly, because they participate in certain long exact sequences for quotients by a two-sided ideal. In his thesis in 1968, Matsumoto gave an explicit presentation for $K_2(-)$ for fields, leading Milnor to define a form of higher $K$-groups for fields now known as Milnor $K$-theory, but it seems to have been clear that his is not even the "correct" definition of higher $K$-groups for fields, let alone for more general rings. In particular, it was not at all clear at the tine how to correctly define higher $K$-groups. This was eventually solved by Quillen in 1971. In modern language (and to the best of my knowledge largely inspired by insights of Segal) he observed that it is better to consider $\iota\mathrm{Proj}(R)$ as a symmetric monoidal groupoid and not take its isomorphism classes (which is then an abelian monoid). Symmetric monoidal groupoids are then examples of commutative monoids in anima (aka spaces, $\infty$-groupoids, etc..). The collection of such form an $\infty$-category $\mathrm{CMon}(\mathrm{An})$ which contains a full subcategory $\mathrm{CGrp}(\mathrm{An})$ of grouplike monoids, i.e. those, where every point admits an inverse (or equivalently $\pi_0(-)$ forms an ordinary abelian group). Just like in the case of abelian monoids and groups in sets, the inclusion

[1] Here, we need to note that an automorphism of a finitely generated projective can be extended to an automorphism of a finitely generated free module; this can be represented by a matrix and then represents an element in $K_1(-)$.

[2] We will make this more explicit later in the course.

$\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ admits a left adjoint, the *group completion* $(-)^{\mathrm{gp}}$.[3] Quillen came up with an ad hoc construction, the *Q-construction*, which implements this group completion, and defines the $K$-theory space:

$$K(R) := (\iota\mathrm{Proj}(R))^{\mathrm{gp}}.$$

By comparing universal properties, one finds $\pi_0(K(R)) = K_0(R)$. It is less obvious that $\pi_1(K(R)) = K_1(R)$ and $\pi_2(K(R)) = K_2(R)$, these rely on the *group completion theorem* of Segal and McDuff as we will prove in this course. The following are among the most important first computations about $K$-theory. It is fair to say, that there is not a single "simple" computation of $K(R)$; all computations really invoke or establish deep mathematics.

(1) In 1971, when defining $K(R)$ in general and setting up a number of influential basic results [Qui73b], Quillen also computed $K(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field [Qui72]; the result is that for $n > 0$, $K_{2n}(\mathbb{F}_q) = 0$, and $K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/q^n - 1$. In fact, Quillen constructs a map of spaces
$$K(\mathbb{F}_q) \to \mathrm{fib}(\mathrm{BU} \xrightarrow{\psi^q - 1} \mathrm{BU})$$
where $\psi^q$ is an Adams operation, and shows that this map induces an isomorphism on positive homotopy groups. Moreover, Quillen showed that $K_n(\mathcal{O}_F)$ is a finitely generated abelian group for all $n \geq 0$, where $F$ is a number field with ring of integers $\mathcal{O}_F$ [Qui73a].

(2) In 1974, Borel then computed $K(\mathcal{O}_F) \otimes \mathbb{Q}$: These groups are trivial in even (positive) degrees, and have rank $r_1 + r_2 - 1$ in degree 1, and for degrees larger than 1, the ranks are given by $r_1 + r_2$ in degrees 1 mod 4 and $r_2$ in degrees 3 mod 4. Here, $r_1$ and $r_2$ are the numbers of real and pairs of complex conjugate complex embeddings of $F$, respectively. Borel's proof again uses crucially the group completion theorem to reduce the computation of *homotopy groups* to a computation of *homology groups*, in the case of interest of certain arithmetic groups.

(3) In 1984, Suslin computed $K(k)/n$, where $k$ is an algebraically (or separably) closed field and $n \in k^\times$. In fact, he proves a *rigidity theorem*, that whenever $k \subseteq k'$ is an inclusion of algebraically closed fields and $n \in k^\times$, then the inclusion induced map $K(k)/n \to K(k')/n$ is an equivalence, and the common term is in turn equivalent to $\mathrm{ku}/n$; here ku denotes the (connective) complex $K$-theory spectrum – to see this, by rigidity, it suffices to study the cases $K(\overline{\mathbb{F}}_p)$ and $K(\mathbb{C})$; the former essentially then follows from Quillen's computation, and the latter is another result of Suslin from 1984, confirming a conjecture of Milnor's about the relation of the group homologies of $\mathrm{GL}_n(\mathbb{C})^\delta$ and $\mathrm{GL}_n(\mathbb{C})$ – once with the discrete and once with its analytic topology.

(4) In 1983, Gabber gave a talk explaining the following theorem (his result was then published in 1989 [?]): If $(A, I)$ is a henselian pair[4] and $n \in A^\times$, then the induced map $K(A)/n \to K(A/I)/n$ is an equivalence. Examples of henselian pairs include

[3]Warning: This is not quite as simple as the one for sets described above. Consider for instance the symmetric monoidal groupoid of finite sets with bijections under disjoint union. Its group completion is then the (anima underlying the) sphere spectrum, whose homotopy groups are the famously notoriously mysterious and hard to compute stable homotopy groups of spheres.

[4]That is, $A$ is a commutative ring, $I \subseteq \mathrm{Jac}(A)$ is contained in the Jacobson radical and for every monic polynomial $f \in A[X]$ with factorization $\bar{f} = \bar{g}\bar{h}$ with $\bar{g}, \bar{h} \in A/I[X]$ monic and generating the unit ideal, there exists a lifted factorization $f = gh$ with $g, h \in A[X]$ monic.

the case where $A$ is $I$-adically complete (in particular if $I$ is nilpotent), and the case where $I$ is locally nilpotent.

Let us now turn to some results from different fields which aim to convey the slogan: $K$-theory is everywhere and everywhere interesting.

2.2. **Algebraic geometry.** We begin with the origin of $K$-theory: Grothendieck's goal to understand (and vastly generalize in his typical manner) the theorem of Riemann and Roch. Let us recall the theorem of Riemann–Roch from the 1850's:

So let $\Sigma$ be a compact Riemann surface of genus $g(\Sigma)$. Let $D \in \mathbb{Z}[\Sigma]$ be a divisor on $\Sigma$, that is, a formal finite linear combination of points in $\Sigma$ with coefficients in $\mathbb{Z}$. The degree $\deg(D)$ of a divisor $D$ is the sum of its coefficients. Divisors can be added and form an abelian group $\mathrm{Div}(\Sigma)$. For $f\colon \Sigma \to \mathbb{C} \in \mathcal{M}(\Sigma;\mathbb{C})$ a meromorphic function, one can consider its associated principal divisor $D(f)$ whose coefficient $D(f)_x$ at $x \in \Sigma$ is given by

$$D(f)_x = \begin{cases} n & \text{if } x \text{ is a zero of order } n \\ -n & \text{if } x \text{ is a pole of order } n \\ 0 & \text{otherwise} \end{cases}.$$

In the theory of Riemann surfaces, one is then interested in the $\mathbb{C}$-vector spaces

$$\mathcal{M}(D) = \{f \in \mathcal{M}(\Sigma;\mathbb{C}) \mid D(f)_x \geq -D_x\}$$

and in particular, one would like to compute the dimension of $\mathcal{M}(D)$. Riemann proved the inequality

$$\dim \mathcal{M}(D) \geq \deg(D) + 1 - g(\Sigma).$$

This in particular implies that $\mathcal{M}(D)$ is non-empty if $\deg(D) + 1 - g(\Sigma) \geq 0$. Riemann's inequality was then improved by his student Roch as follows. First, note that $\dim \mathcal{M}(D) = \dim \mathcal{M}(D + D(f))$ and $\deg(D) = \deg(D + D(f))$; it follows that $D \mapsto \dim_{\mathbb{C}} \mathcal{M}(D)$ can be thought of as a function on the divisor class group $\mathrm{Cl}(\Sigma) = \mathrm{Div}(\Sigma)/\mathrm{pDiv}(\Sigma)$. This group is isomorphic to the Picard group $\mathrm{Pic}(\Sigma)$ consisting of holomorphic line bundles (under tensor product) on $\Sigma$. Let $K_\Sigma$ be the canonical bundle on $\Sigma$ (i.e. the holomorphic cotangent bundle) and $D_\Sigma \in \mathrm{Cl}(\Sigma)$ be its associated divisor (up to principal divisors). For any $D \in \mathrm{Cl}(\Sigma)$, set $D^\vee := D_\Sigma - D$. Then the Riemann–Roch (RR) theorem states:

$$\dim \mathcal{M}(D) - \dim \mathcal{M}(D^\vee) = \deg(D) + 1 - g(\Sigma).$$

Let us now go towards Grothendieck's generalization of the Riemann–Roch theorem. We aim to reinterprete several players involved in the above formula. Firstly, when $L$ is the line bundle with associated divisor $D$, then $\mathcal{M}(D)$ canonically identifies with $H^0_{\mathrm{sh}}(\Sigma; L) = \Gamma(\Sigma; L)$, i.e. the holomorphic global sections of the sheaf on $\Sigma$ represented by $L$. Therefore, the left hand side of RR becomes

$$\dim H^0_{\mathrm{sh}}(\Sigma; L) - \dim H^0_{\mathrm{sh}}(\Sigma; L^{-1} \otimes K_\Sigma)$$

which by Serre duality (and the fact that Riemann surfaces are complex curves, i.e. 1-dimensional) is equal to

$$\dim H^0_{\mathrm{sh}}(\Sigma; L) - \dim H^1_{\mathrm{sh}}(\Sigma; L)^\vee$$

which is the Euler characteristic $\chi(\Sigma; L)$ of $\Sigma$ with coefficients in $L$ (as all cohomology groups here are finite dimensional $\mathbb{C}$-vector spaces). In 1954, Hirzebruch then found the following generalization of RR, the Hirzebruch–Riemann–Roch theorem: For $E \to X$ a holomorphic vector bundle over a compact complex $d$-dimensional manifold $X$, there is the formula

$$\chi(X;E) = \langle \mathrm{ch}(E) \cdot \mathrm{td}(TX), [X] \rangle$$

where $\mathrm{ch}(E)$ is the Chern character of $E$ and $\mathrm{td}(TX)$ is the Todd genus (another characteristic class describable in terms of Chern classes) of the tangent bundle $TX$. Specialized to $L \to \Sigma$ with $\Sigma$ a Riemann surface and $L$ a holomorphic line bundle, the right hand side of the above equality can be computed to be $\deg(D)+1-g(\Sigma)$, so Hirzebruch really generalizes the classical Riemann–Roch theorem to higher dimensional compact complex manifolds.

Grothendieck, among other things, wanted to generalize the above result to a relative setting, where one considers a proper morphism $f\colon X \to Y$ where $Y$ need not be a point. In this situation, how could one generalise left and right hand sides of the equation? Recalling that

$$\chi(X;E) = \sum_{i \geq 0} (-1)^i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{sh}}(X;E)$$

and noting that $H^i(X;E) = R^i p_*(E)$, where $R^i p_*$ is the $i$th right derived functor of the functor $\Gamma(-;E)\colon \mathrm{Sh}(X;\mathrm{Ab}) \to \mathrm{Ab}$, here $p\colon X \to *$ is the map to the point. For a morphism $f\colon X \to Y$, one can still consider the values $R^i f_*(E)$ of the right derived functors $R^i f_*$ of $f_*\colon \mathrm{Sh}(X;\mathrm{Ab}) \to \mathrm{Sh}(Y;\mathrm{Ab})$ and one would like to form

$$\sum_{i \geq 0} (-1)^i \cdot R^i f_*(E).$$

But how are we supposed to interpret this alternating sum? You see that for the above formula for $\chi(X;E)$ to make sense, we have used $\dim_{\mathbb{C}}(-)$ to obtain natural numbers, and then know what it means to take an alternating sum. Grothendieck's insight here was to simply *define* an abelian group in which the above formula involving alternating sums of higher pushforward sheaves makes sense. Indeed, he defined[5] $K_0(X)$ to be the group completion of the abelian monoid of isomorphism classes of of coherent sheaves on $X$, modulo the relation $[\mathcal{F}_1] = [\mathcal{F}_0] + [\mathcal{F}_2]$ if there is a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0.$$

Now, if $f\colon X \to Y$ is proper, the higher pushforward functors $R^i f_*(-)$ preserve coherent sheaves and hence $f$ induces a morphism $f_*\colon K_0(X) \to K_0(Y)$, with

$$f_*(\mathcal{F}) = \sum_{i \geq 0} (-1)^i R^i f_*(\mathcal{F}) \in K_0(Y)$$

which is now perfectly well-defined and precisely the putative candidate for the left hand side of the Grothendieck–Riemann–Roch theorem (and also the reason for working with coherent modules, rather than vector bundles: In general $R^i f_*$ does not preserve vector bundles). But then the next question is how to generalize the right hand side of the RR theorem? Hirezbruch extension already showed that terms like a Chern character and the Todd class

[5] This is in fact not generally what $K_0(X)$ is, rather what one would call $G_0(X)$. $K_0(X)$ is defined with vector bundles rather than coherent sheaves instead. The fact that the two yield the same group has to do with the smoothness of $X$. We write $K_0(X)$ rather than $G_0(X)$ as to not make a big fuzz about the difference at this point.

appear at least in the holomorphic case. To see how Grothendieck treated this, let us briefly talk about algebraic cycles. For $X$ a smooth variety over a field, let the *cycle group* be

$$Z(X) = \mathbb{Z}[\text{ irred. subvarieties of X }].$$

One would like to have an intersection product on cycles, informally taking a pair $(Z_1, Z_2)$ to $Z_1 \cap Z_2$. This turns out to work up to rational equivalence. One therefore defines the *Chow ring*

$$A(X) = Z(X)/\text{rational equivalence}$$

which may perhaps be thought of as the algebraic analog of singular (co)homology (note that $A(X)$ is graded by codimension, and in particular trivial in degrees greater than the dimension of $X$). The association $X \mapsto A(X)$ is contravariantly functorial for flat maps (by taking preimages) and can be given a covariant functoriality for proper maps, essentially by taking the image of a subvariety if the dimension of the image does not drop (multiplied with the degree of the resulting extensions) or taking zero if the dimension drops. Grothendieck then proved the following result about the relation between $A(X)$ and $K_0(X)$: He constructed an explicit isomorphism of rings

$$A(X) \otimes \mathbb{Q} \cong K_0(X) \otimes \mathbb{Q}$$

and obtains a Chern character $\mathrm{ch}(-)$ as the composition

$$K_0(X) \to K_0(X) \otimes \mathbb{Q} \cong A(X) \otimes \mathbb{Q}.$$

This is very much in analogy with the situation in algebraic topology, where one can construct an isomorphism

$$\bigoplus_{n \geq 0} H^{2n}(X; \mathbb{Q}) \simeq \mathrm{KU}^0(X) \otimes \mathbb{Q}$$

inducing in the same manner the topological Chern character.

Denote now by $T_f$ the difference $TX - f^*(TY) \in K_0(X)$; a kind of relative tangent bundle. Note that $f^*(TY)$ is even a vector bundle on $X$ so in particular a coherent sheaf. In order to define the Chern character, Grothendieck really constructed algebraic Chern classes from which he extracts the Chern character, just like Hirzebruch did in the complex case. One can then also define a Todd class $\mathrm{td}(T_f) \in A(X) \otimes \mathbb{Q}$ via these algebraic Chern classes. Indeed, the Todd class can be defined for general coherent sheaves $\mathcal{F}$ on $X$, satisfies $\mathrm{td}(\mathcal{F}) = \mathrm{td}(\mathcal{F}') \cdot \mathrm{td}(\mathcal{F}'')$ for all short exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, and is of the form $1 + x$ in $A(X) \otimes \mathbb{Q}$ where $x$ sits is positive grading (with respect to the codimension grading indicated above). Hence, $\mathrm{td}(\mathcal{F})$ is in fact invertible in the ring $A(X) \otimes \mathbb{Q}$, and so the map $\mathcal{F} \mapsto \mathrm{td}(\mathcal{F})$ extends uniquely to a group homomorphism $\mathrm{td}\colon K(X) \to [A(X) \otimes \mathbb{Q}]^\times$. If $f$ is a smooth and proper map between smooth varieties, then $T_f$ is in fact itself a vector bundle, the tangent bundle along the fibres of $f$, and $TX = T_f \oplus f^*(TY)$.

Now, Grothendieck's version of the HRR theorem, proved around 1957,[6] states that for $f\colon X \to Y$ a proper morphism between smooth varieties over a field, and any coherent sheaf $\mathcal{F}$ on $X$, there is the equality

$$\mathrm{ch}(f_*(\mathcal{F})) = f_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(T_f)).$$

[6] Grothendieck presented his version of the HRR theorem at the Arbeitstagung in Bonn in 1957 which was organized by Hirzebruch.

In other words, it shows $\mathrm{td}(T_f)$ is a correction term accounting for the non-commutativity of the diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f_*"'] & A(X)\otimes\mathbb{Q} \arrow[r, dashed] \arrow[d, "f_*"] & H^{2*}(X(\mathbb{C});\mathbb{Q}) \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}"] & A(Y)\otimes\mathbb{Q} \arrow[r, dashed] & H^{2*}(Y(\mathbb{C});\mathbb{Q}) \end{tikzcd}$$

i.e. $\mathrm{td}(T_f)$ measures the failure of the Chern character map to be compatible with the proper pushforward (here the right hand dashed maps exist if $X$ is defined over $\mathbb{C}$).[7] Written differently, the diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{td}(TX)"] \arrow[d, "f_*"'] & A(X)\otimes\mathbb{Q} \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}(-)\cdot\mathrm{td}(TY)"'] & A(Y)\otimes\mathbb{Q} \end{tikzcd}$$

commutes (this uses the usual projection formula for $f_*$ and $f^*$: $f_*(a\cdot f^*(b)) = f_*(a)\cdot b$) in the Chow ring). Let us indicate that this indeed recovers the earlier results: When $Y$ is a point, we have $K_0(Y)=\mathbb{Z}$ via the dimension, and the lower horizontal map is simply the inclusion $\mathbb{Z}\subseteq\mathbb{Q}$. Hence the LHS of Grothendieck's version indeed becomes $\chi(X;\mathcal{F})$. Moreover, $T_f = TX$. Now, if in addition the base field is $\mathbb{C}$ and $\mathcal{F}=E$ is a holomorphic vector bundle, under the map $A(X)\otimes\mathbb{Q}\to H^{2*}(X(\mathbb{C});\mathbb{Q})$, and the symbols $\mathrm{ch}(E)$ and $\mathrm{td}(T_f)$ give precisely the terms appearing in Hirzebruch's version of the Riemann–Roch theorem (i.e. Grothendieck's Chern character is mapped to Hirzebruch's Chern character, and similarly for the Todd class). Finally, in this situation, the map $f_*\colon A(X)\otimes\mathbb{Q}\to\mathbb{Q}$ equals the composite $A(X)\otimes\mathbb{Q}\to H^{2*}(X(\mathbb{C});\mathbb{Q})\to\mathbb{Q}$, where the latter map is the evaluation on the fundamental class $[X]$, so we finally arrive at Hirzebruch's formula.

There are more interesting things to say about $K_0(X)\otimes\mathbb{Q}$: It turns out that $K_0(X)$ is canonically a $\lambda$-ring and hence carries Adams operations $\psi^k$ for all integers $k$. Rationally, any $\lambda$-ring $\Lambda$ decomposes into "common Eigenspaces" for these Adams operations; that is, into the sum (over $i\in\mathbb{Z}$) of its subspaces $(\Lambda\otimes\mathbb{Q})_{(i)}$ where $\psi^k$ acts via $k^i$ for all $k$. Under Grothendieck's isomorphism, these recover the fact that the Chow ring is graded by codimension. Now in fact, we will see that there is a full spectrum $K(X)$ all of whose homotopy groups are interesting. Rationally, they again decompose into the common Eigenspaces for Adams operations. In 1986, Bloch developed a higher version of Chow groups and extended Grothendieck's comparison between $A(X)$ and $K_0(X)$ to (rational) higher Chow and $K$-groups. These higher Chow groups define what is called (rational) *motivic cohomology*, so that we learn that rational motivic cohomology and rational algebraic $K$-theory determine each other: There is rational motivic cohomology $H^n_{\mathrm{mot}}(X;\mathbb{Q}(i))$ with Tate twist coefficients $\mathbb{Q}(i)$; This then identifies with $(K_{2i-n}(X)\otimes\mathbb{Q})_{(i)}$, the weight $i$ part of $K$-theory, characterized by the property that for all $k$, the Adams operation $\psi^k$ acts by multiplication by $k^i$. In particular, $K_0(X)\otimes\mathbb{Q}\simeq\oplus_i H^{2i}_{\mathrm{mot}}(X;\mathbb{Q}(i))\cong A(X)\otimes\mathbb{Q}$. If time permits, will discuss this

[7] This is a very interesting map: The Hodge conjecture is a conjecture about its image and assuming it, a further conjecture of Bloch and Beilinson implies that the map is an isomorphism if and only if $H^{p,q}(X)=0$ for $p\neq q$.

$\lambda$-ring structure on $K_0(X)$ and possibly on $K_*(X)$ later in the course.

Another interesting interaction between algebraic $K$-theory and algebraic geometry is the sensitivity of $K$-theory to regularity or smoothness. An instance of this is the following result: Let $X$ be a regular Noetherian scheme. Then the map $\mathbb{A}^1_X \to X$ induces an equivalence $K(X) \simeq K(\mathbb{A}^1_X)$ and $K_n(X) = 0$ for $n < 0$ (we will prove this later in this course). The following two prominent conjectures aim to convey that $K$-theory is an invariant very sensitive to singularities. Indeed, from the Bass–Milnor–Murthy excision sequence, it was already known that singular curves can have non-trivial $K_{-1}$ (e.g. the nodal curve) but need not have non-trivial $K_{-1}$ (e.g. the cuspidal curve). Moreover, in these cases $K_{-1}$ is free abelian, and somewhat determined by "topology" and there are no non-trivial lower negative $K$-groups. Weibel then conjectured that this is generally so:

2.1. **Conjecture** (Weibel) Let $X$ be a regular Noetherian scheme of Krull dimension $d$. Then $K_n(X) = 0$ for $n < -d$ and $K_{-d}(X)$ can be described "topologically".[8]

When $X$ is a variety over a field $k$ with $\mathrm{char}(k) = 0$, it was shown by Cortinas–Haesemeyer–Schlichting–Weibel [CHSW08] and by Geisser–Hesselholt [GH10] and Krishna [Kri09] for varieties over of field satisfying a strong form of resolution of singularities. Weibel's conjecture was fully resolved in work of Kerz–Strunk–Tamme [KST18] and has been extended to a regular schemes of valuative dimension $d$ in the non-Noetherian situation [KST24].

2.2. **Conjecture** (Vorst) Let $k$ be a field and $A$ a $k$ algebra essentially of finite type of Krull dimension $n$. If $K(A) \to K(A[X_1, \dots, X_{n+1}])$ is an equivalence, then $A$ is regular.

Vorst showed this for $\dim(A) = 1$ (1979), when $\mathrm{char}(k) = 0$, it was shown by Cortinas–Haesemeyer–Weibel (2008) [CHW08], and for perfect fields $k$ with $\mathrm{char}(k) = p > 0$, it was shown by Geisser–Hesselholt [GH12]. A generalisation of their result, without assuming resolution of singularities was recently proven by Kerz–Strunk–Tamme [KST21].

2.3. **Number theory.** Algebraic $K$-theory also has a wonderful relation to number theory. For instance, to (special values of) $\zeta$-functions. Most of what I write here is from [Kah05] which gives a very nice overview of the relations between $K$-theory and number theory. Recall the Riemann $\zeta$-function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

for $s \in \mathbb{C}$. This is the special case of the arithmetic $\zeta$-function associated to schemes $X$ which are of finite type over $\mathrm{Spec}(\mathbb{Z})$, the Riemann $\zeta$-function being the case of $X = \mathrm{Spec}(\mathbb{Z})$ itself:

$$\zeta_X(s) = \prod_{\substack{x \in X \\ \text{closed point}}} \frac{1}{1 - |\kappa(x)|^{-s}}$$

where $\kappa(x)$ is the residue field of $X$ at the closed point $x$. The function $\zeta_X(-)$ converges for $\mathrm{Re}(s) > \dim(X)$ and is conjectured to have a meromorphic continuation to the whole complex

[8] Precisely, $K_{-d}(X)$ is conjecturally given by $H^d_{\mathrm{cdh}}(X; \mathbb{Z})$ is given by a sheaf cohomology group, where the topology is a certain *completely decomposed* topology on schemes introduced by Voevodsky in his work on motivic cohomology.

plane; this is known at least when $\mathrm{Re}(s) > \dim(X) - \frac{1}{2}$. Soulé conjectures (in particular) the following.

2.3. **Conjecture** (Soulé) Let $X$ be regular and of finite type over $\mathbb{Z}$ of dimension $d$ and let $n \in \mathbb{Z}$. Then

$$\begin{aligned}\mathrm{ord}_{s=n}\zeta_X(s) &= \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}}(K_i(X)\otimes\mathbb{Q})_{(d-n)} \\ &= \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}} H^{2(d-n)-i}_{\mathrm{mot}}(X;\mathbb{Q}(d-n)).\end{aligned}$$

where the subscript denotes the weight $(d-n)$ part of the Adams operation decomposition, that is, where all Adams operations $\psi^k$ act by multiplication by $k^{d-n}$, and hence the appropriate rational motivic cohomology group by what we have indicated above.[9]

In particular, this assumes that the meromorphic continuation exists, and that the dimensions appearing on the right hand side are all finite and almost surely zero. See **??** below for further conjectures about finiteness of $K$-groups.

Soulé's conjecture is known to hold for $n > \dim(X)$ – in that case both sides vanish. For $n = \dim(X) - 1$, it implies the famous conjecture of Birch and Swinnerton-Dyer (which asserts that the rank of the $K$-points of an elliptic curve $E$ agrees with the order of the zero of $L(E,s)$ at $s = 1$, where $L(E,s)$ is the Hasse-Weil $L$-function of $E$).

In order to appreciate the next conjecture, it is worthwhile to spell out Soulés conjecture for $\mathrm{Spec}(\mathcal{O}_F)$ for number fields $F$: Here, it turns out that the Adams Eigenspaces are explicitly known: For $i \geq 1$ we have $(K_*(\mathcal{O}_F)\otimes\mathbb{Q})_{(i)} = K_{2i-1}(\mathcal{O}_F)$; and $(K_0(\mathcal{O}_F)\otimes\mathbb{Q})_{(0)} = K_0(\mathcal{O}_F)\otimes\mathbb{Q}$. Hence we obtain for

$$\mathrm{ord}_{s=n}\zeta_F(s) = \begin{cases} 0 & \text{for } n \geq 2 \\ -1 & \text{for } n = 1 \\ \dim_{\mathbb{Q}}(K_{1-2n}(\mathcal{O}_F)\otimes\mathbb{Q}) & \text{for } n \leq 0 \end{cases}$$

As indicated above, the ranks of the $K$-groups have been computed by Borel, so the right hand side is known explicitly. Moreover, the $\zeta$-function (in particular of a number field) satisfies a functional equation, relating $\zeta_F(s)$ with $\zeta_F(1-s)$ (involving so-called $\Gamma$-factors, cosinus, and sinus functions). From this functional equation, and the fact that $\zeta_F(s)$ indeed has a simple pole at $s = 1$, one can show that Soulés conjecture is true for $\mathrm{Spec}(\mathcal{O}_F)$.

Lichtenbaum then conjectured the following about the special values of the $\zeta$-function at non-positive integers, i.e. the coefficient of the leading term for a Taylor expansion around a non-positive integer. Concretely, this special value at $-n$ can be computed as

$$\mathrm{sv}(\zeta_F)(-n) = \lim_{s\to -n}(s+n)^{\mathrm{ord}_{s=-n}\zeta_F(s)}\cdot\zeta_F(s)$$

and Lichtenbaum conjectures:

[9]This might actually not be what Soulé conjectures; He has a more general version for arbitrary (possibly non-regular) schemes of finite type over $\mathbb{Z}$, where one replaces $K_i(X)$ by $G_i(X)$, i.e. the $K$-theory of coherent sheaves, not of vector bundles. Consequently, he works with Adams operations on $G$-theory, which rationally identify with Borel–Moore motivic *homology* rather than motivic cohomology: $(G_i(X)\otimes\mathbb{Q})_{(j)} \cong H^{\mathrm{BM,mot}}_{2j+i}(X;\mathbb{Q}(j))$. I am then alluding to a possible Poincaré duality statement in motivic cohomology for regular schemes of dimension $d$ of finite type over $\mathbb{Z}$ which might be incorrect in general – but is correct for $X = \mathrm{Spec}(\mathcal{O}_F)$ for a number ring $F$, this is the case we then use:.

2.4. **Conjecture** (Lichtenbaum) Let $F$ is a number field with ring of integers $\mathcal{O}_F$ and $n \geq 0$. Then

$$\mathrm{sv}(\zeta_K)(-n) = \pm \frac{|K_{2n}(\mathcal{O}_F)|}{|K_{2n+1}(\mathcal{O}_F)_{\mathrm{tors}}|} \cdot R_{n+1}(F)$$

where $R_{n+1}(F)$ is a transcendental number called the *Borel regulator*.[10]

For $n = 0$, by $K_{2n}(\mathcal{O}_F)$ we really mean the reduced $K_0$-group, obtained by modding out the subgroup generated by $\mathcal{O}_F$ itself, which is isomorphic to the Class group $\mathrm{Cl}(\mathcal{O}_F)$ or equivalently the Picard group $\mathrm{Pic}(\mathcal{O}_F)$. Moreover, $K_1(\mathcal{O}_F)$ is isomorphic to $\mathcal{O}_F^\times$ which is a finitely generated group of rank $r_1 + r_2 - 1$ by Dirichlet's unit theorem and the torsion elements are precisely the roots of unity $\mu(F)$ of $F$ and therefore a cyclic group. The case $n = 0$ is therefore closely related to the class number formula discussed in a number theory course, see e.g. [Neu92, Korollar 5.11]. In loc. cit., the class number formula however relates the special value of the $\zeta$-function at the simple pole $s = 1$ with something like the right hand side in the above equation; Using the functional equation for the $\zeta$-function, this determines the special value at $s = 0$, and in fact, in this formulation, the formula simplifies a bit (for instance powers of 2, $\pi$, and the discriminant of $F$ appear on the right hand side of the class number formula precisely because of the contribution coming from the functional equation); In particular, Lichtenbaum's conjecture is known for $n = 0$.

One reason to expect relations between special vaues of $\zeta$-functions and quotients of orders of $K$-groups to hold is that $K$-groups of number rings like $\mathcal{O}_F$ tend to be describable in terms of étale cohomology groups, and relations between special values of $\zeta_X$ and étale cohomology appear for instance in work of Wiles and Mazur-Wiles. As a consequence, Lichtenbaum's conjecture is also known if $F$ is an abelian extension of $\mathbb{Q}$, and it is also known for totally real number fields.

Let us also talk about the Kummer–Vandiver conjecture.

2.5. **Conjecture** (Kummer, Vandiver) If $p$ is a prime number, then $p$ does not divide the class number of the maximal real subfield $\mathbb{Q}(\zeta_p)^+$ of $\mathbb{Q}(\zeta_p)$.

Let us mention that the class number of number field $F$ is $|\mathrm{Pic}(\mathcal{O}_F)|$ and $\mathrm{Pic}(\mathcal{O}_F) \cong \widetilde{K}_0(\mathcal{O}_F)$. The class number $h$ of $\mathbb{Q}(\zeta_p)$ is known to be the product $h_1 h_2$ of the class number $h_1$ of $\mathbb{Q}(\zeta_p)^+$ and a second number $h_2$; this second number $h_2$ is quite well understood, can be computed in terms of Bernoulli numbers and is typically quite large. It really is the other factor $h_1$ in the class number of $\mathbb{Q}(\zeta_p)$ that is the mysterious one. Recall also that a prime is called *regular* if it does not divide the class number of $\mathbb{Q}(\zeta_p)$; the first irregular prime is 37. The above conjecture is therefore true for all regular primes (it is conjectured that $\sim 60\%$ of all primes are regular) and it has been verified for all primes $p < 2^{31}$.

2.6. **Theorem** (Kurihara [Kur92]) *The Kummer–Vandiver conjecture is equivalent to the statement that $K_{4n}(\mathbb{Z}) = 0$ for all $n > 0$.*

Moreover, we know the following things about $K(\mathbb{Z})$:

(1) $K_{2k+1}(\mathbb{Z})$ is known explicitly for all $k \geq 0$.

[10]The sign in this conjecture can be made fully explicit: It is $(-1)^{\frac{n+1}{2}r_1 + r_2}$ if $n$ is odd and $(-1)^{\frac{n}{2}r_1}$ if $n$ is even.

(2) The orders of $K_{4k+2}(\mathbb{Z})$ are known explicitly for all $k \geq 0$; they are conjectured to be cyclic; this is implied by the Kummer–Vandiver conjecture, but a priori a weaker assertion.
(3) $K_4(\mathbb{Z}) = 0$ (Rognes [Rog00]) and $K_8(\mathbb{Z}) = 0$ (Kupers [Kup17]); but as of now, we do not know $K_{12}(\mathbb{Z})$.

Finally, we mention Clausen's $K$-theoretic approach to Artin maps [Cla17]. To that end, in class field theory, there appears for a global field $F$ with ring of adèles $\mathbb{A}_F$, the Artin map for $F$: It is a homomorphism

$$\mathbb{A}_F^\times / F^\times \to \mathrm{Gal}(F)^{\mathrm{ab}}$$

where $\mathrm{Gal}(F)$ denotes the absolute Galois group of $F$. Similarly, there is an Artin map for a local field $F$, taking the form

$$F^\times \to \mathrm{Gal}(F)^{\mathrm{ab}}$$

as well as an Artin map for a finite field $F$, taking the form

$$\mathbb{Z} \to \mathrm{Gal}(F)^{\mathrm{ab}}.$$

The final map seems simple to define: It merely sends $1 \in \mathbb{Z}$ to the Frobenius of the finite field $F$. However, these Artin maps obey a certain functoriality in $F$, which uniquely characterises them. The fact that such a compatible, functorial set of Artin maps exists in a non-trivial result. Clausen constructs these maps via the following $K$-theoretic construction. Associated to a field $F$, (in fact more generally) he defines a category $\mathrm{LC}_F = \mathrm{Fun}_{\mathbb{Z}}(\mathrm{Perf}(F), \mathrm{Perf}(\mathrm{LCA}))$ of "$\mathrm{Perf}(F)$-modules in the derived category of (second countable) locally compact abelian groups" – whatever that is, it is something of which one can take $K$-theory, and considers $K(\mathrm{LC}_F)$. He shows that there are maps from the sources of all the above Artin maps to $\pi_1(K(\mathrm{LC}_F))$. On the other hand, he constructs another invariant which he calls Selmer $K$-homology: $dK^{\mathrm{Sel}}(F)$, which is more complicated to define at this point, as it uses on the one hand more sophisticated homotopy theory (some height one chromatic Anderson duality) as well as another invariant, called topological cyclic homology which is closely related to algebraic $K$-theory. He proves that $\pi_1(dK^{\mathrm{Sel}}(F)) \cong \mathrm{Gal}(F)^{\mathrm{ab}}$. Moreover, he constructs a natural map

$$K(\mathrm{LC}_F) \to dK^{\mathrm{Sel}}(F)$$

and this map induces all the above Artin maps on $\pi_1$, with its functoriality, at once. The above, I think, serves as good motivation that one also wants to study the $K$-theory of suitable *categories*, not "only" that of rings or schemes.

2.4. **Algebraic and geometric topology.** Algebraic K-theory also appears prominently in algebraic and geometric topology. In first instance, the relevant rings to consider are given by $\mathbb{Z}\pi_1(X)$ for $X$ a space. For instance, suppose $X$ is a compact anima (historically, one would say a finitely dominated space: I.e. one that is a retract up to homotopy of a finite CW complex). Associated to such a space is an element $o(X)$ in $K_0(\mathbb{Z}\pi_1(X))$ often called the $K$-theory Euler characteristic of $X$. Indeed, under the map $K_0(\mathbb{Z}\pi_1(X)) \to K_0(\mathbb{Z}) \cong \mathbb{Z}$, $o(X)$ is sent to $\chi(X)$, the homological Euler characteristic of $X$. Define $\widetilde{K}_0(\mathbb{Z}\pi_1(X)) = \mathrm{coker}[K_0(\mathbb{Z}) \to K_0(\mathbb{Z}\pi_1(X))]$ and denote by $\tilde{o}(X)$ the image of $o(X)$ under the canonical projection. If $X$ is a finite anima (i.e. can be represented by a finite CW complex), then $\tilde{o}(X)$

vanishes, one therefore refers to $\widetilde{o}(X)$ as the *finiteness obstruction* of $X$. Wall then proved the following result.

2.7. **Theorem** (Wall) *The anima $X$ is finite (i.e. can be represented by a finite CW complex) if and only if $\tilde{o}(X) = 0 \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$. Moreover, for every finitely presented group $\pi$, every element $\tilde{o} \in \widetilde{K}_0(\mathbb{Z}\pi)$ appears as the finiteness obstruction of a finitely dominated space $X$.*

Wall came to this theorem from surgery theory: In practice he was often to show the existence of a finitely dominated space (i.e. a compact anima) and would like to have in fact constructed a closed manifold. But the anima of a closed manifold is always finite[11], not only compact, and so he naturally came to study what the difference between finite and compact anima are.

Moving more towards differential topology, consider a closed smooth manifold $M$.[12] Let $W$ be an $h$-cobordism from $M$ to $M'$, that is, $W$ is a cobordism with one boundary piece identified with $M$ (the other end we simply call $M'$), such that both inclusions $M \to W$ and $M' \to W$ are homotopy equivalences. Associated to this, one can associate the Whitehead torsion $\tau(W, M) \in K_1(\mathbb{Z}\pi_1(M))/\langle \pm g \rangle) = Wh(\pi_1(M))$. The following is known as the $s$-cobordism theorem:

2.8. **Theorem** (Smale, Barden, Mazur, Stallings) *Let $M$ be a closed manifold of dimension $\geq 5$. Then the association $(W, M, M') \mapsto \tau(W, M)$ induces a bijection between isomorphism classes of $h$-cobordisms $W$ over $M$ and $Wh(\pi_1(M))$.*

Since the cylinder $M \times [0, 1]$ is an $h$-cobordism with trivial Whitehead torsion, the $s$-cobordism theorem implies that an $h$-cobordism $(W, M, M')$ with trivial Whitehead torsion $\tau(W, M)$ is in fact diffeomorphic to the cylinder, and in particular, there exists a diffeomorphism $M \cong M'$.

We note that for any group $\pi$, there is a comparison map, called the *assembly map*

$$B\pi \otimes K(\mathbb{Z}) \to K(\mathbb{Z}\pi)$$

and that the groups $\widetilde{K_0}(\mathbb{Z}\pi)$ and $Wh(\pi)$ identify with the cokernel of the map induced by the assembly map on $\pi_0$ and $\pi_1$.

Waldhausen has realised that one should consider the variant where $\mathbb{Z}$ is replaced by the sphere spectrum $\mathbb{S}$ and where one uses the group in anima $\Omega X$ rather than its $\pi_0$ (which is $\pi_1(X)$). Doing so, one still obtains two maps, the latter of which is the assembly map and the former of which is induced by the unit of the ring spectrum $K(\mathbb{S})$:

$$X \otimes \mathbb{S} \to X \otimes K(\mathbb{S}) \to K(\mathbb{S}[\Omega X])$$

The cofibre of the composite is called the smooth Whitehead spectrum $\mathrm{Wh}^{\mathrm{sm}}(X)$ of $X$, and the cofibre of the composite is called the topological Whitehead spectrum $\mathrm{Wh}^{\mathrm{top}}(X)$ of $X$. One can then show that $\pi_0$ and $\pi_1$ of the two versions of Whitehead spectra agree, and that their common $\pi_0$ and $\pi_1$ are given by $\widetilde{K}_0(\mathbb{Z}\pi_1(X))$ and $Wh(\pi_1(X))$, respectively (perhaps we will learn some of the ingredients that go into these computations this term, but perhaps also not).

He then indicated a proof of what is now called the stable parametrized $s$-cobordism theorem, the details of which were published in joint work of Waldhausen with Jahren and Rognes.

[11] In fact, a compact ANR is a finite anima by a result of West. Topological manifolds are ANRs, so that compact manifolds are always finite anima.

[12] In fact, all I am about to say holds for topological manifolds as well.

To state it, one has to accept that there can be built a space $\mathcal{H}(M)$ of $h$-cobordisms over $M$ (whose points evidently are $h$-cobordisms over $M$) which comes with stabilisation maps $\mathcal{H}(M) \to \mathcal{H}(M \times [0,1]) \to ..$ whose colimit $\mathcal{H}^s(M)$ is the stable $h$-cobordism space. The stable parametrized $s$-cobordism theorem then states:

2.9. **Theorem** *For a compact smooth/topological manifold of dimension $\geq 5$, there is a canonical equivalence $\mathcal{H}^s(M) \simeq \Omega \mathrm{Wh}^{\mathrm{sm/top}}(M)$.*

The consequence that their $\pi_0$ agree then recovers the $s$-cobordism theorem described above. Moreover, the space $\Omega\mathcal{H}(M)$ itself is described as the stable pseudoisotopy or concordance space $C^s(M)$, and hence contains very interesting information on $M$-parametrized families and hence about certain automorphism groups of $M$, as Igusa proved that the maps $C(M) \to C(M \times [0,1]) \to \cdots \to C^s(M)$ are at least (roughly) $\dim(M)/3$-connected. Here,

$$C(M) = \{f \colon M \times [0,1] \xrightarrow{\cong} M \times [0,1] \mid f|_{M\times\{0\}\cup\partial M \times [0,1]} = \mathrm{id}\}.$$

As a consequence there is a (roughly) $\dim(M)/3$-connected map

$$C(M) \to \Omega^2 \mathrm{Wh}(M).$$

When $M = D^d$ is a disk, source and target are only interesting in the smooth case, and one obtains

$$C(D^d) \to \Omega \mathrm{fib}(\mathbb{S} \to \mathbb{K}(\mathbb{S}))$$

Rationally, it turns out that $\mathbb{S} \to \mathbb{Z}$ is an equivalence as we will learn in the Topology IV course. $K$-theory behaves so well in this situation that this implies $K(\mathbb{S}) \to K(\mathbb{Z})$ is also a rational equivalence, and the latter is famously calculated by Borel. Finally, there is a fibre sequence

$$\mathrm{Diff}_\partial(D^{d+1}) \to C(D^d) \to \mathrm{Diff}_\partial(D^d)$$

and it is known by work of Randal-Williams and Berglund–Madsen, that $\mathrm{Diff}_\partial(D^{2d})$ is rationally (roughly) $2d$-connected. Therefore, the map

$$\mathrm{Diff}_\partial(D^{2d+1})_\mathbb{Q} \to C(D^{2d})_\mathbb{Q} \to \Omega^2 K(\mathbb{Z})_\mathbb{Q}$$

is (roughly) $2d/3$-connected. Nowadays, much more is known about the rational homotopy groups of $\mathrm{Diff}_\partial(D^d)$, mainly due to work of Krannich [Kra22], Krannich–Randal-Williams [KRW21] and Kupers–Randal-Williams [KRW25].

2.5. **K-theory of group rings.** As the finiteness obstruction and the Whitehead torsion are of great geometric relevance, it makes good sense to study the group in which they live in detail. As those are controlled by the assembly map, it therefore makes sense to study the assembly map

$$BG \otimes K(R) \to K(RG)$$

for a ring $R$ and a group $G$. The following conjecture is due to Farrell and Jones:

2.10. **Conjecture** (Farrell–Jones, I) Let $R$ be a regular Noetherian ring and $G$ be a torsion free group. Then the assembly map

$$BG \otimes K(R) \to K(RG)$$

is an equivalence.

In fact, this is just the special case of a conjecture for all rings and all groups:

2.11. **Conjecture** Let $R$ be a ring and $G$ be a group. Then the assembly map

$$\operatorname*{colim}_{H\in \mathrm{Orb}_{\mathrm{VCyc}}(G)} K(RH) \to K(RG)$$

is an equivalence.

Here $\mathrm{Orb}_{\mathrm{VCyc}}(G)$ denotes the full subcategory of the category of $G$-sets on $G$-sets of the form $G/H$ where $H \subseteq G$ is a virtually cyclic group, that is, one which contains a cyclic group of finite index. They come in two families, the ones which admit a surjection onto $\mathbb{Z}$ (with finite kernel), or the ones which admit a surjection onto $D_\infty = \mathbb{Z}/2 \star \mathbb{Z}/2$ (again with finite kernel). When $G$ is torsion free and virtually cyclic, it must therefore be isomorphic to $\mathbb{Z}$. For a regular ring $R$, it is a consequence of the fundamental theorem from the very beginning of this introduction, that the first version of the Farrell–Jones conjecture holds for $R$. It can then be shown that for $R$ regular and $G$ torsion free, the more sophisticated conjecture is really equivalent to the easier one. There is no counterexample known to the sophisticated Farrell–Jones conjecture, and it is known for a large class of groups. In particular, typically, for torsion free groups, the finiteness obstruction and the Whitehead torsion vanish, simply because the groups in which they live are the trivial groups.

Similarly, When $M$ is an aspherical manifold, it follows from the Farrell–Jones conjecture that $\mathrm{Wh}^{\mathrm{top}}(M)_{\mathbb{Q}} \simeq 0$, hence one concludes information about the rational homotopy groups of the (stable) concordance space of $M$.

There are further interesting consequences of the Farrell–Jones conjecture that are more about the representation theory of non-finite groups: First, the comparison map

$$\operatorname*{colim}_{H\in \mathrm{Orb}_{\mathrm{Fin}}(G)} K(RH) \to \operatorname*{colim}_{H\in \mathrm{Orb}_{\mathrm{VCyc}}(G)} K(RH)$$

reducing to the orbits with *finite* stabilizers on the source, induces an isomorphism on negative homotopy groups; essentially one has to show that the result holds for virtually cyclic groups $G$ in which case the target becomes $K(RG)$. Using the classification of virtually cyclic groups, this then follows from known long exact sequences in the algebraic $K$-theory of group rings of amalgamated products and semidirect products over $\mathbb{Z}$.

2.12. **Conjecture** Let $G$ be any group. Then $K_{-n}(\mathbb{Z}G) = 0$ for $n \geq 2$ and there is an isomorphism

$$\operatorname*{colim}_{H\in \mathrm{Orb}_{\mathrm{Fin}}(G)} K_{-1}(\mathbb{Z}H) \to K_{-1}(\mathbb{Z}G).$$

The vanishing result is rather famously known for finite groups and is also true for virtually cyclic groups. The result therefore follows from the Farrell–Jones conjecture and the above comparison isomorphism in negative degrees. It should be noted that $K_{-1}(\mathbb{Z}G)$ for finite $G$ is also classically studied in representation theory.

When the orders of finite all finite subgroups of $G$ are invertible in a regular ring $R$, as is the case for $R = \mathbb{Q}$ and any group $G$, then one obtains:

2.13. **Conjecture** Let $G$ be any group. Then $K_{-n}(\mathbb{Q}G) = 0$ for $n \geq 1$ and there is an isomorphism

$$\operatorname*{colim}_{H\in \mathrm{Orb}_{\mathrm{Fin}}(G)} K_0(\mathbb{Q}H) \to K_0(\mathbb{Q}G).$$

This is a reminiscent of Artin induction for finite groups; It should be noted that $K_0(\mathbb{C}G) = R_{\mathbb{C}}(G)$ is the complex representation ring.

For finite groups $G$, a theorem of Swan asserts that $\widetilde{K}_0(\mathbb{Z}G)$ is itself finite. But by Artin–Wedderburn, $\mathbb{Q}G$ is a product of matrix algebras over division rings $D$; since $K_0(-)$ commutes with finite products, swallows matrix algebras, and $K_0(D) = \mathbb{Z}$ (every module over a division ring is free), we see that $K_0(\mathbb{Q}G)$ is torsion free. In particular, the map $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$ is trivial. It was an open question whether this remains true for general groups, but this turns out not be true, a counterexample was provided by Lehner.

Understanding, for finite groups $G$, the groups $K_0(\mathbb{Z}G)$ is very complicated and not too much beyond the finiteness result mentioned above is known in general. The situation becomes surprisingly different for $Wh(G)$ for finite groups $G$, where Oliver has a program of determining the groups algorithmically.

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