TOPOLOGIE IV – EXERCISE SHEET 8

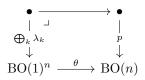
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An explicit splitting principle for BO(n). The sum map $\oplus: O(1)^n \to O(n)$ induces a morphism $\theta: BO(1)^n \to BO(n)$

which by definition classifies the assignment $(\lambda_1, \ldots, \lambda_n) \to \bigoplus_k \lambda_k$. In particular, denoting

• λ_k the universal line bundle over $\mathrm{BO}(\{k\})$ for $1 \le k \le n$

• p the universal rank n vector bundle $BO(n-1) \rightarrow BO(n)$ we obtain a cartesian square



Given $0 \le i \le n$, naturality of the Stiefel–Whitney classes together with the Cartan formula yields

$$\theta^*(w_i) = \theta^*(w_i(p))$$
$$= w_i \Big(\bigoplus_i \lambda_i\Big)$$
$$= \sigma_i (c_1(\lambda_1), \dots, c_1(\lambda_n))$$

in $\mathrm{H}^{i}(\mathrm{BO}(1);\mathbb{F}_{2})^{\otimes n}$, where $\sigma_{i} \in \mathbb{Z}[x_{1},\ldots,x_{n}]$ denotes the *i*-th elementary symmetric polynomial. In particular, the morphism

*:
$$\mathrm{H}^{i}(\mathrm{BO}(n);\mathbb{F}_{2}) \to \mathrm{H}^{i}(\mathrm{BO}(1);\mathbb{F}_{2})^{\otimes n} \simeq \mathbb{F}_{2}[x_{1},\ldots,x_{n}]$$

is injective. The same proof works with integer coefficients for BU and BSp.

A combinatorial identity. For all $n \ge 0$ and any integer $k \le n$, we have:

$$\binom{n+1}{k} \equiv \sum_{k \le 2i \le n} \binom{n-i}{i} \binom{i}{k-i} \pmod{2}$$

Here, we use the convention that $\binom{a}{b} = 0$ if b < 0.

 θ^{i}

Proof. The proof proceeds by induction on n, and uses repeatedly that $\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$ for $a \ge b$.

- The base cases n = 0 and n = 1 are trivial.
- Assuming the result for both n and n-1, then

$$\sum_{k \le 2i \le n+1} \binom{\binom{n+1-i}{i}\binom{i}{k-i}}{\equiv} \sum_{k \le 2i \le n+1} \binom{\binom{n-i}{i} + \binom{n-i}{i-1}}{\binom{k-i}{k-i}} \\ \equiv \sum_{k \le 2i \le n+1} \binom{\binom{n-i}{i}\binom{i}{k-i}}{\binom{i}{k-i}} + \sum_{k \le 2i \le n+1} \binom{\binom{n-i}{i-1}\binom{i-1}{k-i-1}}{\binom{k-1-i}{k-1-i}} \\ \equiv \sum_{k \le 2i \le n} \binom{\binom{n-i}{i}\binom{i}{k-i}}{\binom{k-i}{k-1}} + \sum_{k-1 \le 2i \le n-1} \binom{\binom{n-1-i}{i}\binom{i}{k-1-i}}{\binom{k-1-i}{k-2 \le 2i \le n-1}} \binom{\binom{n-1-i}{i}\binom{i}{k-2-i}}{\binom{n-1-i}{i}\binom{k-2-i}{k-2-i}}$$

$$\equiv \binom{n+1}{k} + \binom{n}{k-1} + \binom{n}{k-2}$$
$$\equiv \binom{n+1}{k} + \binom{n+1}{k-1}$$
$$\equiv \binom{n+2}{k}$$

for all $k \leq n$. The case k = n + 1 is clear, since then the sum has at most one term.

Exercise 1.

1) Recall that

$$\mathrm{H}^*(\mathbb{P}^\infty(\mathbb{R});\mathbb{F}_2)\simeq\mathbb{F}_2[w_1]$$

and use the Cartan formula to deduce that

$$\operatorname{Sq}^{i}\left(w_{1}^{k}\right) = \binom{k}{i} w_{1}^{n+i}$$

for $i \leq k$. Since the map induced by the inclusion $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^\infty(\mathbb{R})$ for $n \geq 1$ on \mathbb{F}_2 -cohomology is the quotient

$$\mathbb{F}_2[w_1] \to \mathbb{F}_2[w_1] / (w_1^{n+1})$$

one computes

$$v_i(\mathbb{P}^n(\mathbb{R})) \cdot w_1^{n-i} = \operatorname{Sq}^i(w_i^{n-i})$$
$$= \binom{n-i}{i} w_1^n$$

for $0 \le i \le n$. Here the binomial coefficient is to be understood as 0 when 2i > n. Therefore

$$v_i\left(\mathbb{P}^n(\mathbb{R})\right) = \begin{cases} \binom{n-i}{i} w_1^i & \text{if } 2i \le n\\ 0 & \text{otherwise} \end{cases}$$

for $i \ge 0$. The total Stiefel–Whitney class is then computed using Wu's third formula:

$$w(\mathbb{P}^{n}(\mathbb{R})) = \operatorname{Sq}\left(v(\mathbb{P}^{n}(\mathbb{R}))\right)$$
$$= \sum_{2i \leq n} \sum_{j \leq i} {\binom{n-i}{i}} \operatorname{Sq}^{j}(w_{1}^{i})$$
$$= \sum_{2i \leq n} \sum_{j \leq i} {\binom{n-i}{i}} {\binom{i}{j}} w_{1}^{i+j}$$
$$= \sum_{k=0}^{n} \sum_{k \leq 2i \leq n} {\binom{n-i}{i}} {\binom{i}{k-i}} w_{1}^{k}$$
$$= \sum_{k=0}^{n} {\binom{n+1}{k}} w_{1}^{k}$$

and finally

$$w\big(\mathbb{P}^n(\mathbb{R})\big) = (1+w_1)^{n+1}$$

since $w_1^{n+1} = 0$.

2) For $n \geq 1$, all odd Wu classes of $\mathbb{P}^n(\mathbb{C})$ are obviously trivial and

$$v_{2i}(\mathbb{P}^n(\mathbb{C})) = \begin{cases} \binom{n-i}{i} w_2^i & \text{if } 2i \le n \\ 0 & \text{otherwise} \end{cases}$$

for $i \geq 0$. Also

$$w\big(\mathbb{P}^n(\mathbb{C})\big) = (1+w_2)^{n+1}$$

The proofs are similar to 1).

3) The same computation as in 1) yields

$$v_{4i}\left(\mathbb{P}^{n}(\mathbb{H})\right) = \begin{cases} \binom{n-i}{i} w_{4}^{i} & \text{if } 2i \leq n \\ 0 & \text{otherwise} \end{cases}$$

for $n \ge 1$ and $i \ge 0$, and

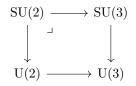
$$w(\mathbb{P}^n(\mathbb{H})) = (1+w_4)^{n+1}$$

6) Let $Q \simeq SU(3)/SO(3)$. Since both SU(3) and SO(3) are connected compact Lie groups of dimension 8 and 3 respectively, Q is a connected 5-dimensional compact manifold sitting in a fiber sequence

$$SO(3) \longrightarrow SU(3) \longrightarrow Q$$

Make the following observations.

• Applying B(-) to the following cartesian square



and taking horizontal fibers gives an identification $SU(3)/SU(2) \simeq U(3)/U(2) \simeq S^5$. In particular, we get a fiber sequence

$$S^3 \simeq \mathrm{SU}(2) \longrightarrow \mathrm{SU}(3) \longrightarrow S^5$$

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and $\pi_i(\mathrm{SU}(3)) \simeq 0$ for $i \leq 2$.

• The canonical map $S^3 \simeq SU(2) \to SO(3)$ is a double cover, and in particular $\pi_1(SO(3)) \simeq \mathbb{Z}/2\mathbb{Z}$. The long exact sequence in homotopy groups thus shows that

$$\pi_1(Q) \simeq 0$$
 and $\pi_2(Q) \simeq \mathbb{Z}/2\mathbb{Z}$

In particular, Hurewicz and Poincaré duality together imply that

$$\mathbf{H}^{k}(Q; \mathbb{F}_{2}) \simeq \begin{cases} \mathbb{F}_{2} & \text{if } k \text{ is } 0, 2, 3 \text{ or} \\ 0 & \text{otherwise} \end{cases}$$

More precisely, relative Hurewicz applied to the pair $f: Q \to BSO(3)$ shows that the induced map $f^*: H^2(BSO(3)) \to H^2(Q)$

in integral cohomology is an isomorphism. In particular f^*w_2 is the generator of $\mathrm{H}^2(Q;\mathbb{F}_2)$.

• Since Sq¹ is the Bockstein morphism associated to the short exact sequence of abelian groups $0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{F}_2 \longrightarrow 0$

we have an exact sequence

But

$$0 \longrightarrow \mathrm{H}^{2}(Q; \mathbb{F}_{2}) \longrightarrow \mathrm{H}^{2}(Q; \mathbb{Z}/4\mathbb{Z}) \longrightarrow \mathrm{H}^{2}(Q; \mathbb{F}_{2}) \xrightarrow{\mathrm{Sq}^{1}} \mathrm{H}^{3}(Q; \mathbb{F}_{2})$$

the universal coefficient theorem implies

$$H^{2}(Q; \mathbb{Z}/4\mathbb{Z}) \simeq \operatorname{Hom}_{\mathbb{Z}}(H_{2}(Q), \mathbb{Z}/4\mathbb{Z})$$

$$\simeq \mathbb{F}_{2}$$

and the blue map is thus an isomorphism. This shows that Sq^1 induces an isomorphism

$$\operatorname{Sq}^1$$
: $\operatorname{H}^2(Q; \mathbb{F}_2) \simeq \operatorname{H}^3(Q; \mathbb{F}_2)$

In particular f^* Sq¹(w_2) is the generator of H³($Q; \mathbb{F}_2$).

• Using either the sum map $BO(1)^3 \rightarrow BO(3)$ or Wu's second formula, observe that

$$Sq^{1}(w_{2}) = w_{3} + w_{1}w_{2}$$
 $Sq^{2}(w_{3}) = w_{2}w_{3}$

in $H^*(BO(3); \mathbb{F}_2)$. In particular, the relations

$$Sq^{1}(w_{2}) = w_{3}$$
 $Sq^{2}(w_{3}) = w_{2}w_{3}$

hold in $H^*(BSO(3); \mathbb{F}_2)$.

Combining the two previous observations, the action of the Steenrod algebra on the \mathbb{F}_2 -cohomology of Q is completely described by the fact that f^*w_3 is the generator of $\mathrm{H}^3(Q;\mathbb{F}_2)$, and by the relations

$$\operatorname{Sq}^{1}(f^{*}w_{2}) = f^{*}w_{3} \qquad \operatorname{Sq}^{2}(f^{*}w_{3}) = f^{*}(w_{2}w_{3})$$

Remark that $f^*(w_2w_3)$ is the generator of $\mathrm{H}^5(Q;\mathbb{F}_2)$ by Poincaré duality. Finally:

$$v(Q) = 1 + f^* w_2$$
 and $w(Q) = 1 + f^* w_2 + f^* w_3$

7) Using Künneth, we have

$$\mathrm{H}^*(T^n; \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, \dots, x_n] / (x_1^2, \dots, x_n^2)$$

where elements x_i have degree 1. In particular, Cartan's formula implies that all Steenrod squares must vanish and thus

$$v(T^n) = 1$$
 and $w(T^n) = 1$

8) Reusing the notations introduced in the correction to the last exercise sheet, we have

$$v(X(\alpha)) = w(X(\alpha)) = \begin{cases} 1 + f^*\lambda & \text{if } \alpha \simeq i_3\eta\\ 1 & \text{otherwise} \end{cases}$$

Exercise 2. Since the suspension morphism $BSO(3) \rightarrow BSO(4)$ induces an isomorphism

$$\mathbb{Z}/2\mathbb{Z} \simeq \pi_2(\mathrm{BSO}(3)) \simeq \pi_2(\mathrm{BSO}(4))$$

the generator determines a canonical oriented spherical fibration

$$S^3 \xrightarrow{j} E \xrightarrow{p} S^2$$

obtained by suspending fibrewise a spherical fibration of rank 3 over S^2 .

- There exists a section s to p.
- The groupoid E is a simply connected compact manifold of dimension 5, and thus satisfies Poincaré duality. In particular, the morphism

$$(s,j)\colon S^2 \vee S^3 \to E$$

induces an isomorphism on $H^2(-)$ and $H^3(-)$.

• The Hurewicz isomorphism

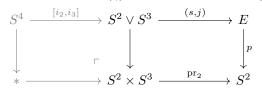
$$\pi_2(\mathrm{BSO}(4)) \simeq \mathrm{H}_2(\mathrm{BSO}(4)) \simeq \mathbb{F}_2 w_2^{\vee}$$

sends the classifying map $\xi\colon S^2\to \mathrm{BSO}(4)$ to $w_2^\vee.$ In particular

$$\xi_*\left(\left[S^2\right] \cap w_2(p)\right) = \xi_*\left[S^2\right] \cap w_2$$
$$= w_2^{\vee} \cap w_2$$
$$= 1$$

and therefore $w_2(p) = 1$ in $\mathrm{H}^2(S^2; \mathbb{F}_2)$.

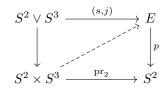
We will now construct an identification $E \simeq X(\eta)$. The commutative diagram



shows that $(s, j) \circ [i_2, i_3]: S^4 \to E$ is in the kernel of the second map in the following short exact sequence

$$\mathbb{Z}/2\mathbb{Z} \simeq \pi_4(S^3) \xrightarrow{j_*} \pi_4(E) \xrightarrow{p_*} \pi_4(S^2)$$

In particular, it is either null homotopic or homotopic to η . If it were null homotopic, there would exist a dashed lift in the following diagram



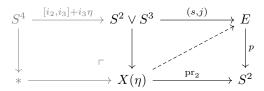
inducing an isomorphism on $H^2(-)$ and $H^3(-)$. By Poincaré duality and Whitehead, it has to be an isomorphism. But this is absurd, since $w_2(p) = 1$ and $w_2(pr_2) = 0$.

Therefore, in $\pi_4(E)$, we have

$$(s,j) \circ ([i_2,i_3] + i_3\eta) = j_*(\eta + \eta)$$

= 0

and there exists a dashed lift



It induces an isomorphism on $H^2(-)$ and $H^3(-)$, and is thus an equivalence by Poincaré duality and White-head.