TOPOLOGIE IV – EXERCISE SHEET 8

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An explicit splitting principle for $\mathrm{BO}(n)$**.** The sum map $\oplus\colon \mathrm{O}(1)^n \to \mathrm{O}(n)$ induces a morphism

$$\theta\colon \mathrm{BO}(1)^n \to \mathrm{BO}(n)$$

which by definition classifies the assignment $(\lambda_1, \dots, \lambda_n) \to \bigoplus_k \lambda_k$. In particular, denoting

- λ_k the universal line bundle over $\mathrm{BO}(\{k\})$ for $1 \leq k \leq n$
- p the universal rank n vector bundle $\mathrm{BO}(n-1) \to \mathrm{BO}(n)$

we obtain a cartesian square

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle \bigoplus_k \lambda_k}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathrm{BO}(1)^n & \xrightarrow{\theta} & \mathrm{BO}(n) \end{array}$$

Given $0 \leq i \leq n$, naturality of the Stiefel–Whitney classes together with the Cartan formula yields

$$\begin{aligned} \theta^*(w_i) &= \theta^*\big(w_i(p)\big) \\ &= w_i\Big(\bigoplus_i \lambda_i\Big) \\ &= \sigma_i\big(c_1(\lambda_1), \dots, c_1(\lambda_n)\big) \end{aligned}$$

in $\mathrm{H}^i(\mathrm{BO}(1);\mathbb{F}_2)^{\otimes n}$, where $\sigma_i \in \mathbb{Z}[x_1, \dots, x_n]$ denotes the i-th elementary symmetric polynomial.

In particular, the morphism

$$\theta^*\colon \mathrm{H}^i(\mathrm{BO}(n);\mathbb{F}_2) \to \mathrm{H}^i(\mathrm{BO}(1);\mathbb{F}_2)^{\otimes n} \simeq \mathbb{F}_2[x_1, \dots, x_n]$$

is injective. The same proof works with integer coefficients for BU and BSp.

A combinatorial identity. For all $n \geq 0$ and any integer $k \leq n$, we have:

$$\binom{n+1}{k} \equiv \sum_{k \leq 2i \leq n} \binom{n-i}{i}\binom{i}{k-i} \pmod 2$$

Here, we use the convention that $\binom{a}{b} = 0$ if $b < 0$.

Proof. The proof proceeds by induction on n, and uses repeatedly that $\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$ for $a \geq b$.

- The base cases $n = 0$ and $n = 1$ are trivial.
- Assuming the result for both n and $n-1$, then

$$\begin{aligned} &\sum_{k \leq 2i \leq n+1} \tbinom{n+1-i}{i}\tbinom{i}{k-i} \\ &\equiv \sum_{k \leq 2i \leq n+1} \Big(\tbinom{n-i}{i} + \tbinom{n-i}{i-1}\Big)\tbinom{i}{k-i} \\ &\equiv \sum_{k \leq 2i \leq n+1} \tbinom{n-i}{i}\tbinom{i}{k-i} + \sum_{k \leq 2i \leq n+1} \tbinom{n-i}{i-1}\Big(\tbinom{i-1}{k-i} + \tbinom{i-1}{k-i-1}\Big) \\ &\equiv \sum_{k \leq 2i \leq n} \tbinom{n-i}{i}\tbinom{i}{k-i} + \sum_{k-1 \leq 2i \leq n-1} \tbinom{n-1-i}{i}\tbinom{i}{k-1-i} + \sum_{k-2 \leq 2i \leq n-1} \tbinom{n-1-i}{i}\tbinom{i}{k-2-i} \end{aligned}$$

$$\equiv \tbinom{n+1}{k} + \tbinom{n}{k-1} + \tbinom{n}{k-2}$$
$$\equiv \tbinom{n+1}{k} + \tbinom{n+1}{k-1}$$
$$\equiv \tbinom{n+2}{k}$$

for all $k \leq n$. The case $k = n+1$ is clear, since then the sum has at most one term.

□

Exercise 1.

1) Recall that
$$\mathrm{H}^*\big(\mathbb{P}^\infty(\mathbb{R}); \mathbb{F}_2\big) \simeq \mathbb{F}_2[w_1]$$
and use the Cartan formula to deduce that
$$\mathrm{Sq}^i\big(w_1^k\big) = \binom{k}{i} w_1^{n+i}$$
for $i \leq k$. Since the map induced by the inclusion $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^\infty(\mathbb{R})$ for $n \geq 1$ on $\mathbb{F}_2$-cohomology is the quotient
$$\mathbb{F}_2[w_1] \to \mathbb{F}_2[w_1]/\big(w_1^{n+1}\big)$$
one computes
$$\begin{aligned} v_i\big(\mathbb{P}^n(\mathbb{R})\big) \cdot w_1^{n-i} &= \mathrm{Sq}^i\big(w_i^{n-i}\big) \\ &= \tbinom{n-i}{i} w_1^n \end{aligned}$$
for $0 \leq i \leq n$. Here the binomial coefficient is to be understood as 0 when $2i > n$. Therefore
$$v_i\big(\mathbb{P}^n(\mathbb{R})\big) = \begin{cases} \tbinom{n-i}{i} w_1^i & \text{if } 2i \leq n \\ 0 & \text{otherwise} \end{cases}$$
for $i \geq 0$. The total Stiefel–Whitney class is then computed using Wu's third formula:
$$\begin{aligned} w\big(\mathbb{P}^n(\mathbb{R})\big) &= \mathrm{Sq}\left(v\big(\mathbb{P}^n(\mathbb{R})\big)\right) \\ &= \sum_{2i \leq n} \sum_{j \leq i} \tbinom{n-i}{i} \mathrm{Sq}^j\big(w_1^i\big) \\ &= \sum_{2i \leq n} \sum_{j \leq i} \tbinom{n-i}{i} \tbinom{i}{j} w_1^{i+j} \\ &= \sum_{k=0}^{n} \sum_{k \leq 2i \leq n} \tbinom{n-i}{i} \tbinom{i}{k-i} w_1^k \\ &= \sum_{k=0}^{n} \tbinom{n+1}{k} w_1^k \end{aligned}$$

and finally
$$w\big(\mathbb{P}^n(\mathbb{R})\big) = (1 + w_1)^{n+1}$$
since $w_1^{n+1} = 0$.

2) For $n \geq 1$, all odd Wu classes of $\mathbb{P}^n(\mathbb{C})$ are obviously trivial and
$$v_{2i}\big(\mathbb{P}^n(\mathbb{C})\big) = \begin{cases} \tbinom{n-i}{i} w_2^i & \text{if } 2i \leq n \\ 0 & \text{otherwise} \end{cases}$$
for $i \geq 0$. Also
$$w\big(\mathbb{P}^n(\mathbb{C})\big) = (1 + w_2)^{n+1}$$
The proofs are similar to 1).

3) The same computation as in 1) yields

$$v_{4i}\big(\mathbb{P}^n(\mathbb{H})\big) = \begin{cases} \binom{n-i}{i} w_4^i & \text{if } 2i \leq n \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 1$ and $i \geq 0$, and

$$w\big(\mathbb{P}^n(\mathbb{H})\big) = (1+w_4)^{n+1}$$

6) Let $Q :\simeq \mathrm{SU}(3)/\mathrm{SO}(3)$. Since both SU(3) and SO(3) are connected compact Lie groups of dimension 8 and 3 respectively, Q is a connected 5-dimensional compact manifold sitting in a fiber sequence

$$\mathrm{SO}(3) \longrightarrow \mathrm{SU}(3) \longrightarrow Q$$

Make the following observations.

- Applying $\mathrm{B}(-)$ to the following cartesian square

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(3) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{U}(2) & \longrightarrow & \mathrm{U}(3) \end{array}$$

and taking horizontal fibers gives an identification $\mathrm{SU}(3)/\mathrm{SU}(2) \simeq \mathrm{U}(3)/\mathrm{U}(2) \simeq S^5$. In particular, we get a fiber sequence

$$S^3 \simeq \mathrm{SU}(2) \longrightarrow \mathrm{SU}(3) \longrightarrow S^5$$

and $\pi_i\big(\mathrm{SU}(3)\big) \simeq 0$ for $i \leq 2$.

- The canonical map $S^3 \simeq \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a double cover, and in particular $\pi_1\big(\mathrm{SO}(3)\big) \simeq \mathbb{Z}/2\mathbb{Z}$.

The long exact sequence in homotopy groups thus shows that

$$\pi_1(Q) \simeq 0 \quad \text{and} \quad \pi_2(Q) \simeq \mathbb{Z}/2\mathbb{Z}$$

In particular, Hurewicz and Poincaré duality together imply that

$$\mathrm{H}^k(Q;\mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2 & \text{if } k \text{ is } 0,\, 2,\, 3 \text{ or } 5 \\ 0 & \text{otherwise} \end{cases}$$

More precisely, relative Hurewicz applied to the pair $f\colon Q \to \mathrm{BSO}(3)$ shows that the induced map

$$f^*\colon \mathrm{H}^2\big(\mathrm{BSO}(3)\big) \to \mathrm{H}^2\big(Q\big)$$

in integral cohomology is an isomorphism. In particular f^*w_2 is the generator of $\mathrm{H}^2(Q;\mathbb{F}_2)$.

- Since Sq^1 is the Bockstein morphism associated to the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

we have an exact sequence

$$0 \longrightarrow \mathrm{H}^2(Q;\mathbb{F}_2) \longrightarrow \mathrm{H}^2(Q;\mathbb{Z}/4\mathbb{Z}) \longrightarrow \mathrm{H}^2(Q;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^1} \mathrm{H}^3(Q;\mathbb{F}_2)$$

But the universal coefficient theorem implies

$$\begin{aligned} \mathrm{H}^2(Q;\mathbb{Z}/4\mathbb{Z}) &\simeq \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{H}_2(Q), \mathbb{Z}/4\mathbb{Z}\big) \\ &\simeq \mathbb{F}_2 \end{aligned}$$

and the blue map is thus an isomorphism. This shows that Sq^1 induces an isomorphism

$$\mathrm{Sq}^1\colon \mathrm{H}^2(Q;\mathbb{F}_2) \simeq \mathrm{H}^3(Q;\mathbb{F}_2)$$

In particular $f^*\,\mathrm{Sq}^1(w_2)$ is the generator of $\mathrm{H}^3(Q;\mathbb{F}_2)$.

- Using either the sum map $\mathrm{BO}(1)^3 \to \mathrm{BO}(3)$ or Wu's second formula, observe that

$$\mathrm{Sq}^1(w_2) = w_3 + w_1w_2 \qquad \mathrm{Sq}^2(w_3) = w_2w_3$$

in $\mathrm{H}^*(\mathrm{BO}(3);\mathbb{F}_2)$. In particular, the relations

$$\mathrm{Sq}^1(w_2) = w_3 \qquad \mathrm{Sq}^2(w_3) = w_2w_3$$

hold in $\mathrm{H}^*(\mathrm{BSO}(3);\mathbb{F}_2)$.

Combining the two previous observations, the action of the Steenrod algebra on the $\mathbb{F}_2$-cohomology of Q is completely described by the fact that f^*w_3 is the generator of $\mathrm{H}^3(Q;\mathbb{F}_2)$, and by the relations

$$\mathrm{Sq}^1\big(f^*w_2\big) = f^*w_3 \qquad \mathrm{Sq}^2\big(f^*w_3\big) = f^*(w_2w_3)$$

Remark that $f^*(w_2w_3)$ is the generator of $\mathrm{H}^5(Q;\mathbb{F}_2)$ by Poincaré duality. Finally:

$$v(Q) = 1 + f^*w_2 \quad \text{and} \quad w(Q) = 1 + f^*w_2 + f^*w_3$$

7) Using Künneth, we have

$$\mathrm{H}^*\big(T^n;\mathbb{F}_2\big) \simeq \mathbb{F}_2[x_1,\dots,x_n]/\big(x_1^2,\dots,x_n^2\big)$$

where elements x_i have degree 1. In particular, Cartan's formula implies that all Steenrod squares must vanish and thus

$$v\big(T^n\big) = 1 \quad \text{and} \quad w\big(T^n\big) = 1$$

8) Reusing the notations introduced in the correction to the last exercise sheet, we have

$$v\big(X(\alpha)\big) = w\big(X(\alpha)\big) = \begin{cases} 1 + f^*\lambda & \text{if } \alpha \simeq i_3\eta \\ 1 & \text{otherwise} \end{cases}$$

Exercise 2. Since the suspension morphism $\mathrm{BSO}(3) \to \mathrm{BSO}(4)$ induces an isomorphism

$$\mathbb{Z}/2\mathbb{Z} \simeq \pi_2(\mathrm{BSO}(3)) \simeq \pi_2(\mathrm{BSO}(4))$$

the generator determines a canonical oriented spherical fibration

$$S^3 \xrightarrow{\ j\ } E \xrightarrow{\ p\ } S^2$$

obtained by suspending fibrewise a spherical fibration of rank 3 over S^2.

- There exists a section s to p.
- The groupoid E is a simply connected compact manifold of dimension 5, and thus satifies Poincaré duality. In particular, the morphism

$$(s,j)\colon S^2 \vee S^3 \to E$$

  induces an isomorphism on $\mathrm{H}^2(-)$ and $\mathrm{H}^3(-)$.
- The Hurewicz isomorphism

$$\pi_2\big(\mathrm{BSO}(4)\big) \simeq \mathrm{H}_2\big(\mathrm{BSO}(4)\big) \simeq \mathbb{F}_2 w_2^\vee$$

  sends the classifying map $\xi\colon S^2 \to \mathrm{BSO}(4)$ to $w_2^\vee$. In particular

$$\begin{aligned} \xi_*\Big(\big[S^2\big] \cap w_2(p)\Big) &= \xi_*\big[S^2\big] \cap w_2 \\ &= w_2^\vee \cap w_2 \\ &= 1 \end{aligned}$$

  and therefore $w_2(p) = 1$ in $\mathrm{H}^2\big(S^2;\mathbb{F}_2\big)$.

We will now construct an identification $E \simeq X(\eta)$. The commutative diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{[i_2,i_3]} & S^2 \vee S^3 & \xrightarrow{(s,j)} & E \\ \downarrow & \ulcorner & \downarrow & & \downarrow p \\ * & \longrightarrow & S^2 \times S^3 & \xrightarrow{\mathrm{pr}_2} & S^2 \end{array}$$

shows that $(s,j) \circ [i_2,i_3]\colon S^4 \to E$ is in the kernel of the second map in the following short exact sequence

$$\mathbb{Z}/2\mathbb{Z} \simeq \pi_4\big(S^3\big) \xrightarrow{j_*} \pi_4(E) \xrightarrow{p_*} \pi_4\big(S^2\big)$$

In particular, it is either null homotopic or homotopic to η. If it were null homotopic, there would exist a dashed lift in the following diagram

$$\begin{array}{ccc} S^2 \vee S^3 & \xrightarrow{(s,j)} & E \\ \downarrow & \nearrow & \downarrow p \\ S^2 \times S^3 & \xrightarrow{\mathrm{pr}_2} & S^2 \end{array}$$

inducing an isomorphism on $\mathrm{H}^2(-)$ and $\mathrm{H}^3(-)$. By Poincaré duality and Whitehead, it has to be an isomorphism. But this is absurd, since $w_2(p) = 1$ and $w_2(\mathrm{pr}_2) = 0$.

Therefore, in $\pi_4(E)$, we have

$$\begin{aligned} (s,j) \circ \big([i_2, i_3] + i_3\eta\big) &= j_*(\eta + \eta) \\ &= 0 \end{aligned}$$

and there exists a dashed lift

$$\begin{array}{ccccc} S^4 & \xrightarrow{[i_2,i_3]+i_3\eta} & S^2 \vee S^3 & \xrightarrow{(s,j)} & E \\ \downarrow & \ulcorner & \downarrow & \nearrow & \downarrow p \\ * & \longrightarrow & X(\eta) & \xrightarrow{\mathrm{pr}_2} & S^2 \end{array}$$

It induces an isomorphism on $\mathrm{H}^2(-)$ and $\mathrm{H}^3(-)$, and is thus an equivalence by Poincaré duality and Whitehead.