

TOPOLOGIE IV – EXERCISE SHEET 7

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The formula for the tensor product on $\text{Pr}_{\mathcal{V}}^{\text{L}}$. Let \mathcal{V} a presentably symmetric monoidal category, or in other words a commutative algebra object in Pr^{L} . For \mathcal{C}, \mathcal{D} and \mathcal{E} three \mathcal{V} -modules, the chain of identifications

$$\begin{aligned} \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{C}, \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{D}, \mathcal{E})) &\simeq \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{C}, \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{E}^{\text{op}}, \mathcal{D}^{\text{op}})) \\ &\simeq \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{E}^{\text{op}}, \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{C}, \mathcal{D}^{\text{op}})) \\ &\simeq \text{Fun}_{\mathcal{V}}^{\text{L}}(\text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{C}, \mathcal{D}^{\text{op}})^{\text{op}}, \mathcal{E}) \\ &\simeq \text{Fun}_{\mathcal{V}}^{\text{L}}(\text{Fun}_{\mathcal{V}}^{\text{R}}(\mathcal{C}^{\text{op}}, \mathcal{D}), \mathcal{E}) \end{aligned}$$

is natural in all variables. More explicitly, this identification sends $F \in \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{C}, \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{D}, \mathcal{E}))$ to the left adjoint of the functor $\mathcal{E} \rightarrow \text{Fun}_{\mathcal{V}}^{\text{R}}(\mathcal{C}^{\text{op}}, \mathcal{D})$ given by the formula

$$z \mapsto (x \mapsto F(x)^{\text{R}}(z))$$

where $(-)^{\text{R}}$ denotes the action of passing to right adjoints. In particular, we have

$$\mathcal{C} \otimes_{\mathcal{V}} \mathcal{D} \simeq \text{Fun}_{\mathcal{V}}^{\text{R}}(\mathcal{C}^{\text{op}}, \mathcal{D})$$

since both objects have the same universal property. As an exercise, describe the functoriality of the right-hand side in variables \mathcal{C} and \mathcal{D} by tracing back through the above identifications.

An important special case occurs when $\mathcal{C} \equiv \text{Psh}(\mathcal{I}) \otimes \mathcal{V}$ for some small category \mathcal{I} , where the above isomorphism can be rewritten

$$\text{Fun}(\mathcal{I}, \text{Fun}_{\mathcal{V}}^{\text{L}}(\mathcal{D}, \mathcal{E})) \simeq \text{Fun}_{\mathcal{V}}^{\text{L}}(\text{Fun}(\mathcal{I}^{\text{op}}, \mathcal{D}), \mathcal{E})$$

Exercise 1. Let R be a commutative ring spectrum.

- (1) For \mathcal{C} any small category, we get by the above an isomorphism

$$\begin{aligned} \text{Fun}(\mathcal{C}^{\text{op}}, \text{Mod}(R)) &\simeq \text{Fun}(\mathcal{C}^{\text{op}}, \text{Fun}_R^{\text{L}}(\text{Mod}(R), \text{Mod}(R))) \\ &\simeq \text{Fun}_R^{\text{L}}(\text{Fun}(\mathcal{C}, \text{Mod}(R)), \text{Mod}(R)) \end{aligned}$$

sending $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \text{Mod}(R)$ to the left adjoint of the functor

$$\text{Hom}_R(\mathcal{F}, -): \text{Mod}(R) \rightarrow \text{Fun}(\mathcal{C}, \text{Mod}(R))$$

given by the formula

$$M \mapsto (x \mapsto \text{Hom}_R(\mathcal{F}(x), M))$$

But, for M an R -module and $\mathcal{X}: \mathcal{C} \rightarrow \text{Mod}(R)$:

$$\begin{aligned} \text{Hom}(\mathcal{X}, \text{Hom}_R(\mathcal{F}, M)) &\simeq \int_{x \in \mathcal{C}} \text{Hom}_R(\mathcal{X}(x), \text{Hom}_R(\mathcal{F}(x), M)) \\ &\simeq \int_{x \in \mathcal{C}} \text{Hom}_R(\mathcal{X}(x) \otimes \mathcal{F}(x), M) \\ &\simeq \text{Hom}_R\left(\int^{\mathcal{C}} \mathcal{X} \otimes \mathcal{F}, M\right) \end{aligned}$$

Finally, the above isomorphism is explicitly given by the formula

$$\mathcal{F} \rightarrow \int^{\mathcal{C}} (-) \otimes \mathcal{F}$$

This restricts to the correct formula when $\mathcal{C} \equiv X$ is a groupoid since, since in this case $\mathrm{Tw}(X) \simeq X$.

(2) For \mathcal{C} and \mathcal{D} two small categories, the identification:

$$\begin{aligned} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{Mod}(R)) &\simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}(R))) \\ &\simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}_R^{\mathrm{L}}(\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)), \mathrm{Mod}(R))) \\ &\simeq \mathrm{Fun}_R^{\mathrm{L}}(\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)), \mathrm{Fun}(\mathcal{D}, \mathrm{Mod}(R))) \end{aligned}$$

sends $\mathcal{F}: \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \rightarrow \mathrm{Mod}(R)$ to the functor

$$\mathcal{X} \mapsto \left(y \mapsto \int^{\mathcal{C}} (-) \otimes \mathcal{F}(-, y) \right)$$

Exercise 2. Let X be a compact spectrum. Since Sp is generated under filtered colimits by finite spectra, X is a retract of some finite spectrum. Without loss of generality, we can thus assume that X is connective and that $X \otimes \mathbb{Z}$ is n -coconnective for some $n \geq 0$. We prove by induction on n that X is finite.

- (1) If $n = 0$, then $X \otimes \mathbb{Z} \simeq H_*(X \otimes \mathbb{Z}) \simeq M$ for some classical \mathbb{Z} -module M . The compactness of X implies that M is projective and finitely generated, which in this case means free of finite rank r . Choose an identification $\mathbb{Z}^r \simeq M$, which by stable Hurewicz lifts to a map $\varphi: \mathbb{S}^r \rightarrow X$. In particular

$$\begin{aligned} (\mathrm{cofib} \varphi) \otimes \mathbb{Z} &\simeq \mathrm{cofib} \varphi \otimes \mathbb{Z} \\ &\simeq 0 \end{aligned}$$

But $\mathrm{cofib} \varphi$ is connective since X is, and stable Hurewicz implies $\mathrm{cofib} \varphi \simeq 0$. The morphism φ is therefore an isomorphism and the \mathbb{S} -module X is free of finite rank.

- (2) Assume that the result holds for some $n \geq 0$ and that $X \otimes \mathbb{Z}$ is $(n+1)$ -coconnective. Since the chain complex $X \otimes \mathbb{Z}$ is perfect, the group $H_0(X \otimes \mathbb{Z})$ is finitely generated and there exists a presentation $\mathbb{Z}^r \twoheadrightarrow H_0(X \otimes \mathbb{Z})$. By the stable Hurewicz theorem, this morphism lifts to $\varphi: \mathbb{S}^r \rightarrow X$, and from the long exact sequence induced in homology

$$\cdots \longrightarrow H_k(\mathbb{Z}^r) \xrightarrow{\varphi_*} H_k(X \otimes \mathbb{Z}) \longrightarrow H_{k-1}((\mathrm{fib} \varphi) \otimes \mathbb{Z}) \longrightarrow \cdots$$

one sees that $(\mathrm{fib} \varphi) \otimes \mathbb{Z}$ is connective and n -coconnective¹. Because X was supposed to be connective, the spectrum $\mathrm{fib} \varphi \simeq \Omega(\mathrm{cofib} \varphi)$ is (-1) -connective a priori, and therefore also connective by the stable Hurewicz theorem. The induction assumption implies that $\mathrm{fib} \varphi$ is finite, and this concludes since $X \simeq \mathrm{cofib}(\mathrm{fib} \varphi \rightarrow \mathbb{S}^r)$.

Exercise 3. Denote i_2 and i_3 the inclusions of S^2 and S^3 respectively in the wedge sum $S^2 \vee S^3$, and

$$[i_2, i_3]: S^4 \rightarrow S^2 \vee S^3$$

the induced Whitehead bracket. It sits inside a cocartesian square

$$\begin{array}{ccc} S^4 & \xrightarrow{[i_2, i_3]} & S^2 \vee S^3 \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & S^2 \times S^3 \end{array}$$

so that $[i_2, i_3]$ is homotopic to the attaching map for the 5-cell of $S^2 \times S^3$.

- (1) For any $\alpha: S^4 \rightarrow S^2 \vee S^3$, the diagram

¹This uses $n+1 \geq 1$, so that one cannot just merge the base case with this induction step.

$$\begin{array}{ccccc}
 S^4 & \xrightarrow{\alpha} & S^2 \vee S^3 & \longrightarrow & * \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 * & \longrightarrow & \text{cofib } \alpha & \longrightarrow & S^5
 \end{array}$$

shows that the reduced (co)homology of $\text{cofib } \alpha$ is concentrated in degrees 2, 3 and 5. Denote $c(\alpha) \in H^5(\text{cofib } \alpha)$ the unique lift of $[S^5]$.

(2) Pasting cocartesian squares, we obtain

$$\begin{array}{ccccc}
 S^4 & \xrightarrow{\Sigma\eta} & S^3 & \xrightarrow{i_3} & S^2 \vee S^3 \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 * & \longrightarrow & \Sigma\mathbb{P}^2(\mathbb{C}) & \longrightarrow & S^2 \vee \Sigma\mathbb{P}^2(\mathbb{C}) \simeq \text{cofib } i_3\eta
 \end{array}$$

The map $S^2 \simeq \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ induces an isomorphism on $H^2(-)$, so there exists $w \in H^2(\mathbb{P}^2(\mathbb{C}))$ lifting $[S^2]$ and then

$$H^*(\mathbb{P}^2(\mathbb{C})) \simeq \mathbb{Z}[w]/(w^3)$$

We have in particular:

$$\begin{aligned}
 \text{Sq}^2(\Sigma w) &= \Sigma \text{Sq}^2(w) \\
 &= \Sigma w^2 \\
 &= c(i_3\eta)
 \end{aligned}$$

in $H^5(\text{cofib } i_3\eta; \mathbb{F}_2)$.

(3) If $\alpha: S^4 \rightarrow S^2 \vee S^3$ factors through i_2 , writing $\alpha \simeq i_2\beta$ for some β we have an equivalence

$$\text{cofib } \alpha \simeq \text{cofib } \beta \vee S^3$$

and the inclusion of the second factor $S^3 \rightarrow \text{cofib } \alpha$ induces an isomorphism on $H^3(-)$. In particular in this case the Steenrod operation

$$\text{Sq}^2: H^3(\text{cofib } \alpha; \mathbb{F}_2) \longrightarrow H^5(\text{cofib } \alpha; \mathbb{F}_2)$$

is trivial.

(4) Let $\alpha: S^4 \rightarrow S^2 \vee S^3$ factoring through either i_2 or i_3 . Observe that the sum $[i_2, i_3] + \alpha$ factors as

$$S^4 \longrightarrow (S^4)^{\vee 2} \xrightarrow{[i_2, i_3] \vee \alpha} (S^2 \vee S^3)^{\vee 2} \xrightarrow{\nabla} S^2 \vee S^3$$

by definition. In particular, consider the following commutative diagram

$$\begin{array}{ccccccc}
 S^4 & \longrightarrow & (S^4)^{\vee 2} & \xrightarrow{[i_2, i_3] \vee \alpha} & (S^2 \vee S^3)^{\vee 2} & \xrightarrow{\nabla} & S^2 \vee S^3 \longrightarrow * \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & X(\alpha) \longrightarrow S^5 \\
 & & \downarrow & & \downarrow & & \downarrow f \\
 & & * & \longrightarrow & (S^2 \times S^3) \vee \text{cofib } \alpha & \xrightarrow{(g, h)} & Y(\alpha) \longrightarrow (S^5)^{\vee 2}
 \end{array}$$

and make the following remarks.

- The cofiber sequence

$$S^2 \vee S^3 \longrightarrow X(\alpha) \longrightarrow S^5$$

implies that the reduced (co)homology of $X(\alpha)$ is concentrated in degrees 2, 3 and 5. Denote $[X(\alpha)]$ the unique class in $H_5(X(\alpha))$ lifting the *homological* fundamental class $[S^5]$ via u . Using the notation $[S^5]$ to denote the *cohomological* fundamental class as well, the computation

$$\begin{aligned} u_*([X(\alpha)] \cap u^*[S^5]) &= u_*[X(\alpha)] \cap [S^5] \\ &= [S^5] \cap [S^5] \\ &= 1 \end{aligned}$$

implies

$$[X(\alpha)] \cap u^*[S^5] = 1$$

in $H_0(X(\alpha))$. In particular $[X(\alpha)] \cap (-)$ induces isomorphisms

$$H^5(X(\alpha)) \simeq H_0(X(\alpha)) \quad \text{and} \quad H^0(X(\alpha)) \simeq H_5(X(\alpha))$$

In the following, we will also use the notation $[X(\alpha)] := u_*[S^5]$, so that $[X(\alpha)] \cap [X(\alpha)] = 1$.

- Similarly, the cofiber sequence

$$S^2 \vee S^3 \longrightarrow Y(\alpha) \longrightarrow S^5 \vee S^5$$

shows that the reduced (co)homology of $Y(\alpha)$ is also concentrated in degrees 2, 3 and 5. Fix bases

$$H^2(Y(\alpha)) \simeq \mathbb{Z}\lambda, \quad H^3(Y(\alpha)) \simeq \mathbb{Z}\mu \quad \text{and} \quad H^5(Y(\alpha)) \simeq \mathbb{Z}\sigma \oplus \mathbb{Z}\tau$$

such that

- λ and μ lift $[S^2]$ and $[S^3]$ respectively;
- (σ, τ) lifts the canonical basis of $H^5(S^5 \vee S^5)$, so that

$$g^*\sigma = c([i_2, i_3]) = [S^2] \otimes [S^3] \quad \text{and} \quad h^*\tau = c(\alpha)$$

- As a consequence, the map

$$f: X(\alpha) \rightarrow Y(\alpha)$$

induces isomorphisms on $H^2(-)$ and $H^3(-)$, and

$$f^*\sigma = f^*\tau = [X(\alpha)]$$

Under our assumption, α factors either through i_2 or i_3 . This implies that $\text{cofib}(\alpha)$ is of the form $Z \vee S^3$ or $S^2 \vee Z$ for some pointed groupoid Z , and thus

$$h^*\lambda \cdot h^*\mu = 0$$

Since (g, h) induces an isomorphism on $H^5(-)$, this shows

$$\lambda \cdot \mu = \sigma$$

and therefore

$$f^*\lambda \cdot f^*\mu = [X(\alpha)]$$

Since

$$\begin{aligned} ([X(\alpha)] \cap f^*\lambda) \cap f^*\mu &= [X(\alpha)] \cap [X(\alpha)] \\ &= 1 \end{aligned}$$

the class $[X(\alpha)] \cap f^*\lambda$ must be a generator of $H_3(X(\alpha)) \simeq \mathbb{Z}$. The class $[X(\alpha)] \cap f^*\mu$ is a generator of $H_2(X(\alpha)) \simeq \mathbb{Z}$ for the same reason. Finally, the morphism

$$[X(\alpha)] \cap (-): H^k(X(\alpha)) \rightarrow H_{5-k}(X(\alpha))$$

is an isomorphism for all k and $X(\alpha)$ satisfies Poincaré duality.

- (5) Let $\alpha: S^4 \rightarrow S^2 \vee S^3$ be such that one of the following conditions is satisfied

- (a) α factors through i_2
- (b) α is homotopic to $i_3\eta$

In particular, the discussion from (4) applies and we reuse notation thereof. To describe the action of the Steenrod algebra on $H^*(Y(\alpha))$, it suffices to compute $Sq^1(\lambda)$ and $Sq^2(\mu)$ since the reduced cohomology of $Y(\alpha)$ is concentrated in degree 2, 3 and 5.

- Since the map $S^2 \vee S^3 \rightarrow Y(\alpha)$ is an isomorphism on $H^2(-; \mathbb{F}_2)$ and $H^3(-; \mathbb{F}_2)$, it follows that

$$Sq^1: H^2(Y(\alpha); \mathbb{F}_2) \longrightarrow H^3(Y(\alpha); \mathbb{F}_2)$$

is null, or in other words $Sq^1(\lambda) = 0$.

- The map $(g, h): (S^2 \times S^3) \vee \text{cofib } \alpha \rightarrow Y(\alpha)$ induces an isomorphism on $H^5(-; \mathbb{F}_2)$, and
 - $Sq^2(g^*\mu) = Sq^2[S^3] = 0$ by the Cartan formula;
 - if (a) is true, then $\text{cofib } \alpha \simeq Z \vee S^3$ for some pointed groupoid Z , and $Sq^2(h^*\mu) = 0$;
 - in case (b), the discussion (2) shows that $Sq^2(h^*\mu) = c(\alpha)$.

Finally

$$Sq^1(f^*\lambda) = 0 \quad Sq^2(f^*\mu) = \begin{cases} 0 & \text{if (a)} \\ [X(\alpha)] & \text{if (b)} \end{cases}$$

and this completely describes the Steenrod operations on the cohomology of $X(\alpha)$.