TOPOLOGIE IV – EXERCISE SHEET 7

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The formula for the tensor product on Pr^{L}. For $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ three presentable categories, the chain of identifications

$$\begin{aligned}\mathrm{Fun}^{\mathrm{L}}\big(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})\big) &\simeq \mathrm{Fun}^{\mathrm{L}}\big(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{E}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})\big) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}\big(\mathcal{E}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})\big) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}\big(\,\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}\big) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}\big(\,\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}), \mathcal{E}\big)\end{aligned}$$

is natural in all variables. More explicitly, this identification sends $F \in \mathrm{Fun}^{\mathrm{L}}\big(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})\big)$ to the left adjoint of the functor $\mathcal{E} \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ given by the formula

$$z \mapsto \big(x \mapsto F(x)^{\mathrm{R}}(z)\big)$$

where $(-)^{\mathrm{R}}$ denotes the action of passing to right adjoints. In particular, we have

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

since both objects have the same universal property. As an exercise, describe the functoriality of the right-hand side in variables $\mathcal{C}$ and $\mathcal{D}$ by tracing back through the above identifications.

An important special case occurs when $\mathcal{C} \equiv \mathrm{Psh}(\mathcal{I})$ for some small category $\mathcal{I}$, where the above isomorphism can be rewritten

$$\mathrm{Fun}\big(\mathcal{I}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})\big) \simeq \mathrm{Fun}^{\mathrm{L}}\big(\,\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{D}), \mathcal{E}\big)$$

Exercise 1. Let R be a commutative ring spectrum.

(1) For $\mathcal{C}$ any small category, we get by the above an isomorphism

$$\begin{aligned}\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}(R)) &\simeq \mathrm{Fun}\big(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}(R), \mathrm{Mod}(R))\big) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}\big(\,\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)), \mathrm{Mod}(R)\big)\end{aligned}$$

sending $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}(R)$ to the left adjoint of the functor

$$\mathrm{Hom}_R(\mathcal{F}, -)\colon\ \mathrm{Mod}(R) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R))$$

given by the formula

$$M \mapsto \big(x \mapsto \mathrm{Hom}_R(\mathcal{F}(x), M)\big)$$

But, for M an R-module and $\mathcal{X}\colon \mathcal{C} \to \mathrm{Mod}(R)$:

$$\begin{aligned}\mathrm{Hom}(\mathcal{X}, \mathrm{Hom}_R(\mathcal{F}, M)) &\simeq \mathrm{Hom}(\mathcal{X} \otimes \mathcal{F}, \underline{M}) \\ &\simeq \mathrm{Hom}_R\left(r_!(\mathcal{X} \otimes \mathcal{F}), M\right)\end{aligned}$$

where $r_!\colon \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)) \to \mathrm{Mod}(R)$ is the colimit functor. Finally, the above isomorphism is explicitly given by the formula

$$\mathcal{F} \to r_!(- \otimes \mathcal{F})$$

since $r_!(- \otimes \mathcal{F})$ is left adjoint to $\mathrm{Hom}_R(\mathcal{F}, -)$.

(2) For $\mathcal{C}$ and $\mathcal{D}$ two small categories, the identification:

$$\begin{aligned}\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{Mod}(R)) &\simeq \mathrm{Fun}\left(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}(R))\right) \\ &\simeq \mathrm{Fun}\left(\mathcal{D}, \mathrm{Fun}^{\mathrm{L}}\big(\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)), \mathrm{Mod}(R)\big)\right) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}\left(\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}(R)), \mathrm{Fun}(\mathcal{D}, \mathrm{Mod}(R))\right)\end{aligned}$$

sends $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Mod}(R)$ to the functor

$$\mathcal{X} \mapsto \big(y \mapsto r_!(- \otimes \mathcal{F}(-, y))\big)$$

Exercise 2. Let X be a compact spectrum. Since Sp is both generated under filtered colimits by finite spectra, X is a retract of some finite spectrum. Without loss of generality, we can thus assume that X is connective and that $X \otimes \mathbb{Z}$ is n-coconnective for some $n \geq 0$. We prove by induction on n that X is finite.

(1) If $n = 0$, then $X \otimes \mathbb{Z} \simeq \mathrm{H}_*(X \otimes \mathbb{Z}) \simeq M$ for some classical $\mathbb{Z}$-module M. The compacity of X implies that M is projective and finitely generated, which in this case means free of finite rank r. Choose an identification $\mathbb{Z}^r \simeq M$, which by stable Hurewicz lifts to a map $\varphi\colon \mathbb{S}^r \to X$. In particular

$$\begin{aligned}(\mathrm{cofib}\,\varphi) \otimes \mathbb{Z} &\simeq \mathrm{cofib}\,\varphi \otimes \mathbb{Z} \\ &\simeq 0\end{aligned}$$

But $\mathrm{cofib}\,\varphi$ is connective since X is, and stable Hurewicz implies $\mathrm{cofib}\,\varphi \simeq 0$. The morphism φ is therefore an isomorphism and the $\mathbb{S}$-module X is free of finite rank.

(2) Assume that the result holds for some $n \geq 0$ and that $X \otimes \mathbb{Z}$ is $(n+1)$-coconnective. Since the chain complex $X \otimes \mathbb{Z}$ is perfect, the group $\mathrm{H}_0(X \otimes \mathbb{Z})$ is finitely generated and there exists a presentation $\mathbb{Z}^r \twoheadrightarrow \mathrm{H}_0(X \otimes \mathbb{Z})$. By the stable Hurewicz theorem, this morphism lifts to $\varphi\colon \mathbb{S}^r \to X$, and from the long exact sequence induced in homology

$$\cdots \longrightarrow \mathrm{H}_k\left(\mathbb{Z}^r\right) \xrightarrow{\varphi_*} \mathrm{H}_k(X \otimes \mathbb{Z}) \longrightarrow \mathrm{H}_{k-1}\big((\mathrm{fib}\,\varphi) \otimes \mathbb{Z}\big) \longrightarrow \cdots$$

one sees that $(\mathrm{fib}\,\varphi)\otimes\mathbb{Z}$ is connective and n-coconnective[1]. Because X was supposed to be connective, the spectrum $\mathrm{fib}\,\varphi \simeq \Omega(\mathrm{cofib}\,\varphi)$ is (-1)-connective a priori, and therefore also connective by the stable Hurewicz theorem. The induction assumption implies that $\mathrm{fib}\,\varphi$ is finite, and this concludes since $X \simeq \mathrm{cofib}\left(\mathrm{fib}\,\varphi \to \mathbb{S}^r\right)$.

Exercise 3. Denote i_2 and i_3 the inclusions of S^2 and S^3 respectively in the wedge sum $S^2 \vee S^3$, and

$$[i_2, i_3]\colon S^4 \to S^2 \vee S^3$$

the induced Whitehead bracket. It sits inside a cocartesian square

$$\begin{array}{ccc} S^4 & \xrightarrow{[i_2,i_3]} & S^2 \vee S^3 \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & S^2 \times S^3 \end{array}$$

so that $[i_2, i_3]$ is homotopic to the attaching map for the 5-cell of $S^2 \times S^3$.

(1) For any $\alpha\colon S^4 \to S^2 \vee S^3$, the diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{\alpha} & S^2 \vee S^3 & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \mathrm{cofib}\,\alpha & \longrightarrow & S^5 \end{array}$$

[1] This uses $n + 1 \geq 1$, so that one cannot just merge the base case with this induction step.

shows that the reduced (co)homology of $\operatorname{cofib}\alpha$ is concentrated in degrees 2, 3 and 5. Denote $c(\alpha) \in \mathrm{H}^5(\operatorname{cofib}\alpha)$ the unique lift of $\left[S^5\right]$.

(2) Pasting cocartesian squares, we obtain

$$\begin{array}{ccccc} S^4 & \xrightarrow{\Sigma\eta} & S^3 & \xrightarrow{i_3} & S^2 \vee S^3 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma\mathbb{P}^2(\mathbb{C}) & \longrightarrow & S^2 \vee \Sigma\mathbb{P}^2(\mathbb{C}) \simeq \operatorname{cofib} i_3\eta \end{array}$$

The map $S^2 \simeq \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ induces a isomorphism on $\mathrm{H}^2(-)$, so there exists $w \in \mathrm{H}^2\big(\mathbb{P}^2(\mathbb{C})\big)$ lifting $\left[S^2\right]$ and then

$$\mathrm{H}^*\big(\mathbb{P}^2(\mathbb{C})\big) \simeq \mathbb{Z}[w]/(w^3)$$

We have in particular:

$$\begin{aligned} \mathrm{Sq}^2(\Sigma w) &= \Sigma\,\mathrm{Sq}^2(w) \\ &= \Sigma w^2 \\ &= c(i_3\eta) \end{aligned}$$

in $\mathrm{H}^5(\operatorname{cofib} i_3\eta; \mathbb{F}_2)$.

(3) If $\alpha\colon S^4 \to S^2 \vee S^3$ factors through i_2, wrting $\alpha \simeq i_2\beta$ for some β we have an equivalence

$$\operatorname{cofib}\alpha \simeq \operatorname{cofib}\beta \vee S^3$$

and the inclusion of the second factor $S^3 \to \operatorname{cofib}\alpha$ induces an isomorphism on $\mathrm{H}^3(-)$. In particular in this case the Steenrod operation

$$\mathrm{Sq}^2\colon \mathrm{H}^3(\operatorname{cofib}\alpha; \mathbb{F}_2) \longrightarrow \mathrm{H}^5(\operatorname{cofib}\alpha; \mathbb{F}_2)$$

is trivial.

(4) Let $\alpha\colon S^4 \to S^2 \vee S^3$ factoring through either i_2 or i_3. Observe that the sum $[i_2, i_3] + \alpha$ factors as

$$S^4 \longrightarrow \big(S^4\big)^{\vee 2} \xrightarrow{[i_2,i_3]\vee\alpha} \big(S^2 \vee S^3\big)^{\vee 2} \xrightarrow{\nabla} S^2 \vee S^3$$

by definition. In particular, consider the following commutative diagram

$$\begin{array}{ccccccccc} S^4 & \longrightarrow & \big(S^4\big)^{\vee 2} & \xrightarrow{[i_2,i_3]\vee\alpha} & \big(S^2 \vee S^3\big)^{\vee 2} & \xrightarrow{\nabla} & S^2 \vee S^3 & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & X(\alpha) & \xrightarrow{u} & S^5 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ & & * & \longrightarrow & \big(S^2 \times S^3\big) \vee \operatorname{cofib}\alpha & \xrightarrow{(g,h)} & Y(\alpha) & \longrightarrow & \big(S^5\big)^{\vee 2} \end{array}$$

and make the following remarks.

- The cofiber sequence

$$S^2 \vee S^3 \longrightarrow X(\alpha) \longrightarrow S^5$$

implies that the reduced (co)homology of $X(\alpha)$ is concentrated in degrees 2, 3 and 5. Denote $\left[X(\alpha)\right]$ the unique class in $\mathrm{H}_5\big(X(\alpha)\big)$ lifting the *homological* fundamental class $\left[S^5\right]$ via u. Using the notation $\left[S^5\right]$ to denote the *cohomological* fundamental class as well, the computation

$$\begin{aligned} u_*\big(\left[X(\alpha)\right] \cap u^*\left[S^5\right]\big) &= u_*\left[X(\alpha)\right] \cap \left[S^5\right] \\ &= \left[S^5\right] \cap \left[S^5\right] \end{aligned}$$

$$= 1$$

implies

$$\big[X(\alpha)\big] \cap u^*\big[S^5\big] = 1$$

in $\mathrm{H}_0\big(X(\alpha)\big)$. In particular $\big[X(\alpha)\big] \cap (-)$ induces isomorphisms

$$\mathrm{H}^5\big(X(\alpha)\big) \simeq \mathrm{H}_0\big(X(\alpha)\big) \quad \text{and} \quad \mathrm{H}^0\big(X(\alpha)\big) \simeq \mathrm{H}_5\big(X(\alpha)\big)$$

In the following, we will also use the notation $\big[X(\alpha)\big] := u_*\big[S^5\big]$, so that $\big[X(\alpha)\big] \cap \big[X(\alpha)\big] = 1$.

- Similarly, the cofiber sequence

$$S^2 \vee S^3 \longrightarrow Y(\alpha) \longrightarrow S^5 \vee S^5$$

shows that the reduced (co)homology of $Y(\alpha)$ is also concentrated in degrees 2, 3 and 5. Fix bases

$$\mathrm{H}^2\big(Y(\alpha)\big) \simeq \mathbb{Z}\lambda, \quad \mathrm{H}^3\big(Y(\alpha)\big) \simeq \mathbb{Z}\mu \quad \text{and} \quad \mathrm{H}^5\big(Y(\alpha)\big) \simeq \mathbb{Z}\sigma \oplus \mathbb{Z}\tau$$

such that
  - λ and μ lift $\big[S^2\big]$ and $\big[S^3\big]$ respectively;
  - (σ, τ) lifts the canonical basis of $\mathrm{H}^5\big(S^5 \vee S^5\big)$, so that

$$g^*\sigma = c\big([i_2, i_3]\big) = \big[S^2\big] \otimes \big[S^3\big] \quad \text{and} \quad h^*\tau = c(\alpha)$$

- As a consequence, the map

$$f\colon X(\alpha) \to Y(\alpha)$$

induces isomorphisms on $\mathrm{H}^2(-)$ and $\mathrm{H}^3(-)$, and

$$f^*\sigma = f^*\tau = \big[X(\alpha)\big]$$

Under our assumptions, α factors either through i_2 or i_3. This implies that $\mathrm{cofib}(\alpha)$ is of the form $Z \vee S^3$ or $S^2 \vee Z$ for some pointed groupoid Z, and thus

$$h^*\lambda \cdot h^*\mu = 0$$

Since (g, h) induces an isomorphism on $\mathrm{H}^5(-)$, this shows

$$\lambda \cdot \mu = \sigma$$

and therefore

$$f^*\lambda \cdot f^*\mu = \big[X(\alpha)\big]$$

Since

$$\big(\big[X(\alpha)\big] \cap f^*\lambda\big) \cap f^*\mu = \big[X(\alpha)\big] \cap \big[X(\alpha)\big]$$
$$= 1$$

the class $\big[X(\alpha)\big] \cap f^*\lambda$ must be a generator of $\mathrm{H}_3\big(X(\alpha)\big) \simeq \mathbb{Z}$. The class $\big[X(\alpha)\big] \cap f^*\mu$ is a generator of $\mathrm{H}_2\big(X(\alpha)\big) \simeq \mathbb{Z}$ for the same reason. Finally, the morphism

$$\big[X(\alpha)\big] \cap (-)\colon \mathrm{H}^k\big(X(\alpha)\big) \to \mathrm{H}_{5-k}\big(X(\alpha)\big)$$

is an isomorphism for all k and $X(\alpha)$ satifies Poincaré duality.

(5) Let $\alpha\colon S^4 \to S^2 \vee S^3$ be such that one of the following conditions is satisfied
  (a) α factors through i_2
  (b) α is homotopic to $i_3\eta$

In particular, the discussion from (4) applies and we reuse notation introduced thereof. To describe the action of the Steenrod algebra on $\mathrm{H}^*\big(Y(\alpha)\big)$, it suffices to compute $\mathrm{Sq}^1(\lambda)$ and $\mathrm{Sq}^2(\mu)$ since the reduced cohomology of $Y(\alpha)$ is concentrated in degree 2, 3 and 5.

- Since the map $S^2 \vee S^3 \to Y(\alpha)$ is an isomorphism on $\mathrm{H}^2(-;\mathbb{F}_2)$ and $\mathrm{H}^3(-;\mathbb{F}_2)$, it follows that
$$\mathrm{Sq}^1\colon \mathrm{H}^2\big(Y(\alpha);\mathbb{F}_2\big) \longrightarrow \mathrm{H}^3\big(Y(\alpha);\mathbb{F}_2\big)$$
is null, or in other words $\mathrm{Sq}^1(\lambda) = 0$.
- The map $(g,h)\colon \big(S^2 \times S^3\big) \vee \operatorname{cofib}\alpha \to Y(\alpha)$ induces an isomorphism on $\mathrm{H}^5(-;\mathbb{F}_2)$, and
  - $\mathrm{Sq}^2(g^*\mu) = \mathrm{Sq}^2\left[S^3\right] = 0$ by the Cartan formula;
  - if (a) is true, then $\operatorname{cofib}\alpha \simeq Z \vee S^3$ for some pointed groupoid Z, and $\mathrm{Sq}^2(h^*\mu) = 0$;
  - in case (b), the discussion (2) shows that $\mathrm{Sq}^2(h^*\mu) = c(\alpha)$.

Finally
$$\mathrm{Sq}^1(f^*\lambda) = 0 \qquad \mathrm{Sq}^2(f^*\mu) = \begin{cases} 0 & \text{if (a)} \\ \big[X(\alpha)\big] & \text{if (b)} \end{cases}$$
and this completely describes the Steenrod operations on the cohomology of $X(\alpha)$.