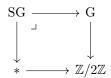
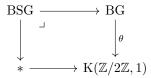
TOPOLOGIE IV - EXERCISE SHEET 6

MARCUS NICOLAS

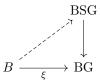
The obstruction to orientability. Applying B(-) to the cartesian square of ∞ -groups



yields a cartesian square of pointed groupoids



By definition, the pullback $\xi^*\theta$ of θ along a stable spherical fibration $\xi\colon B\to \mathrm{BG}$ is exactly the obstruction to the existence of a lift



or in other words, to the obstruction to orientability. Since there exists non-orientable spherical fibrations, the class $\theta \in H^1(BG, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ cannot vanish. Therefore $\theta = w_1$.

Stable Hurewicz. If X is a spectrum, there is a natural comparison morphism

$$\pi_0(X \otimes \mathbb{S}) \to \pi_0(X \otimes \mathbb{Z})$$

induced by the ring map $\mathbb{S} \to \mathbb{Z}$.

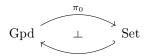
• If $E \equiv \mathbb{S}$, then this map is the identification

$$\pi_0^{\rm s}(S^0) \simeq {\rm H}_0(S^0; \mathbb{Z})$$

obtained by taking the colimit of the Hurewicz isomorphisms

$$\pi_n(S^n) \simeq \mathrm{H}_n(S^n; \mathbb{Z}) \simeq \mathrm{H}_0(S^0; \mathbb{Z})$$

• Since π_0 : Gpd \rightarrow Set preserves finite products, the adjunction



induces

$$\operatorname{CGrp}(\operatorname{Gpd}) \simeq \operatorname{Sp}_{\geq 0} \xrightarrow{\hspace{1cm} \bot \hspace{1cm}} \operatorname{Ab} \simeq \operatorname{CGrp}(\operatorname{Set})$$

In particular both composites $\pi_0((-)\otimes\mathbb{S})$ and $\pi_0((-)\otimes\mathbb{Z})$ preserve colimits when seen as functors $\mathrm{Sp}_{\geq 0}\to\mathrm{Ab}$.

Since the smallest full subcategory of $\mathrm{Sp}_{\geq 0}$ closed under colimits and containing $\mathbb S$ is $\mathrm{Sp}_{\geq 0}$ itself, it follows that the canonical comparison is an isomorphism

$$\pi_0(X) \simeq \pi_0(X \otimes \mathbb{Z})$$

for any connected spectrum X. Equivalently, there is a canonical isomorphism

$$\pi_n(X) \simeq \pi_n(X \otimes \mathbb{Z})$$

for any (n-1)-connected spectrum X.

Exercise 1. Since Stiefel-Whitney classes are stable and natural, it suffices to prove the desired formula

$$\operatorname{Sq}^{i}(w_{j}(p)) = \sum_{k=0}^{i} {j+k-i-1 \choose k} w_{j+k}(p) w_{i-k}(p)$$

for any given real vector bundle p. The proof proceeds by induction on the rank n of p.

- The case n=0 is clear, since in this case w(p)=1.
- Assuming that Wu's second formula holds for rank n vector bundles, we establish that it must also hold for p of rank n + 1. Since we are working with \mathbb{F}_2 -coefficients, the splitting principle applies and we may assume that p splits as

$$p \simeq q \oplus \lambda$$

where λ is a line bundle. Observe that the formula trivially holds when i > j as well as in the edge case i = j. Assuming i < j, compute

$$\operatorname{Sq}^{i}(w_{j}(p)) = \operatorname{Sq}^{i}(w_{j}(q) + w_{1}(\lambda)w_{j-1}(q))$$

=
$$\operatorname{Sq}^{i}(w_{j}(q)) + w_{1}(\lambda)\operatorname{Sq}^{i}(w_{j-1}(q)) + w_{1}(\lambda)^{2}\operatorname{Sq}^{i-1}(w_{j-1}(q))$$

By the induction hypothesis:

(a)
$$\operatorname{Sq}^{i}(w_{j}(q)) = \sum_{k=0}^{i} {j+k-i-1 \choose k} w_{j+k}(q) w_{i-k}(q)$$

(b)
$$w_1(\lambda) \operatorname{Sq}^i(w_{j-1}(q))$$

$$= w_1(\lambda) \sum_{k=0}^i {j+k-i-2 \choose k} w_{j+k-1}(q) w_{i-k}(q)$$

$$= w_1(\lambda) \sum_{k=0}^i {j+k-i-1 \choose k} - {j+k-i-2 \choose k-1} w_{j+k-1}(q) w_{i-k}(q)$$

$$= w_1(\lambda) \left(\sum_{k=0}^i {j+k-i-1 \choose k} w_{j+k-1}(q) w_{i-k}(q) + \sum_{k=-1}^{i-1} {j+k-i-1 \choose k} w_{j+k}(q) w_{i-k-1}(q) \right)$$

$$= w_1(\lambda) \left(\sum_{k=0}^i {j+k-i-1 \choose k} w_{j+k-1}(q) w_{i-k}(q) + \sum_{k=0}^i {j+k-i-1 \choose k} w_{j+k}(q) w_{i-k-1}(q) \right)$$

$$= w_1(\lambda) \sum_{k=0}^i {j+k-i-1 \choose k} (w_{j+k-1}(q) w_{i-k}(q) + w_{j+k}(q) w_{i-k-1}(q))$$

Here, the first step uses that the identity $\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}$ holds for $0 \le n \le m$ but also for m = -1 and n = 0.

(c)
$$w_1(\lambda)^2 \operatorname{Sq}^{i-1}(w_{j-1}(q)) = w_1(\lambda)^2 \sum_{k=0}^{i-1} {j+k-i-1 \choose k} w_{j+k-1}(q) w_{i-k-1}(q)$$

$$= w_1(\lambda)^2 \sum_{k=0}^{i} {j+k-i-1 \choose k} w_{j+k-1}(q) w_{i-k-1}(q)$$

Finally:

$$\operatorname{Sq}^{i}(w_{j}(p)) = \sum_{k=0}^{i} {j+k-i-1 \choose k} (w_{j+k}(q) + w_{1}(\lambda)w_{j+k-1}(q)) (w_{i-k}(q) + w_{1}(\lambda)w_{i-k-1}(q))$$

$$= \sum_{k=0}^{i} {j+k-i-1 \choose k} w_{j+k}(p)w_{i-k}(p)$$

Exercise 2. Let $\xi \colon B \to \mathrm{BG}$ be an oriented stable spherical fibration. We proved during last exercise session that ξ admits a Thom isomorphism in \mathbb{Z} -cohomology. More explicitly, the composite

$$\operatorname{Hom}_{\mathbb{Z}}(B\otimes\mathbb{Z};\mathbb{Z}) \xrightarrow{u(\xi)^*} \operatorname{Hom}_{\mathbb{Z}}(B\otimes\operatorname{M}(\xi);\mathbb{Z}) \xrightarrow{\Delta_{\xi}^*} \operatorname{Hom}_{\mathbb{Z}}(\operatorname{M}(\xi);\mathbb{Z})$$

$$\mathbb{C}^{-*}(B;\mathbb{Z}) \xrightarrow{(-)\cdot u(\xi)} \operatorname{C}^{-*}(\operatorname{M}(\xi);\mathbb{Z})$$

is an equivalence of connective spectra. In particular, the composite

$$M(Th(\xi)) \xrightarrow{\Delta_{\xi}} B \otimes M(\xi) \xrightarrow{u(\xi)} B \otimes \mathbb{Z}$$

induces an isomorphism on \mathbb{Z} -cohomology. Since this is also the case of the unit $M(\xi) \to M(\xi) \otimes \mathbb{Z}$, the induced \mathbb{Z} -linear map

$$M(\xi) \longrightarrow M(\xi) \otimes \mathbb{Z} \longrightarrow B \otimes \mathbb{Z}$$

induces an isomorphism on Z-cohomology as well.

• If B is finite, then

$$\mathrm{M}(\xi) \simeq \operatornamewithlimits{colim}_B \xi \simeq \operatornamewithlimits{colim}_B \mathbb{S}$$

is as well, and both $\mathrm{M}(\xi)\otimes\mathbb{Z}$ and $B\otimes\mathbb{Z}$ are dualizable. Therefore, the map

is an equivalence.

• In general, B is a filtered colimit of finite groupoids

$$B \simeq \operatorname{colim}_{i \in I} B_i$$

and

$$\begin{split} \mathbf{M}(\xi) &\simeq \operatornamewithlimits{colim}_{B} \xi \\ &\simeq \operatornamewithlimits{colim}_{i \in I} \operatornamewithlimits{colim}_{B_{i}} \xi_{i} \\ &\simeq \operatornamewithlimits{colim}_{i \in I} \mathbf{M}(\xi_{i}) \end{split}$$

We conclude by taking a filtered colimit on the maps

$$M(\xi_i) \otimes \mathbb{Z} \simeq B_i \otimes \mathbb{Z}$$

for $i \in I$.

Conversely if $M(\xi) \otimes \mathbb{Z} \simeq B \otimes \mathbb{Z}$, then $M(\xi) \to B \otimes \mathbb{Z}$ induces an isomorphism on \mathbb{Z} -cohomology and Thom isomorphism holds.

Exercise 3. Fix E a commutative ring spectrum.

(1) For $\xi \colon B \to \operatorname{Pic}(\mathbb{S})$ a stable spherical fibration, compute

$$C^{-*}(M(\xi); E) \simeq \operatorname{Hom}_{\mathbb{S}}(M(\xi), E)$$

$$\simeq \operatorname{Hom}_{E}(M(\xi) \otimes E, E)$$

$$\simeq \operatorname{Hom}_{E}(\operatorname{colim} \xi \otimes E, E)$$

$$\simeq \operatorname{Hom}_{\operatorname{Fun}(B, \operatorname{Mod}(E))}(\xi \otimes E, \underline{E})$$

and, similarly

$$C^{-*}(B; E) \simeq \operatorname{Hom}_{E}(B \otimes E, E)$$

 $\simeq \operatorname{Hom}_{E}(\operatorname{colim} \underline{E}, E)$
 $\simeq \operatorname{Hom}_{\operatorname{Fun}(B, \operatorname{Mod}(E))}(\underline{E}, \underline{E})$

Observe now that the following data are equivalent:

- (i) a Thom class, in other words an arrow $u: M(\xi) \to E$ whose restriction $x^*u: \mathbb{S} \to E$ along any point $x: * \to B$ is a unit of the commutative ring $\pi_0(E)$
- (ii) a natural transformation $u: \xi \otimes E \to \underline{E}$ between functors $B \to \text{Pic}(E)$
- (iii) a trivialization of the composite $\xi \otimes E \colon B \to \operatorname{Pic}(\mathbb{S}) \to \operatorname{Pic}(E)$

In this case, we obtain a Thom isomorphism in E-cohomology

$$C^{-*}(B; E) \xrightarrow{(-) \cdot u} C^{-*}(M(\xi); E)$$

$$\parallel \qquad \qquad \parallel$$

$$\operatorname{Hom}(\underline{E}, \underline{E}) \xrightarrow{\underline{u}^*} \operatorname{Hom}(\xi \otimes E, \underline{E})$$

(2) The connected component of Pic(E) containing E is exactly $BAut_E(E)$. Since $Aut_E(E)$ is the subgroupoid of $End_E(E) \simeq E$ on invertible connected components, it follows that for $n \geq 0$:

$$\pi_{n+1}(\operatorname{Pic}(E)) := \begin{cases} \pi_0(E)^{\times} & \text{if } n = 0\\ \pi_n(E) & \text{otherwise} \end{cases}$$

For $\xi \colon B \to \operatorname{Pic}(\mathbb{S})$ a rank 0 spherical fibration, the composite

$$B \xrightarrow{\xi} \operatorname{Pic}(\mathbb{S}) \xrightarrow{(-)\otimes E} \operatorname{Pic}(E)$$

factors through $BAut_E(E)$.

• If $E \equiv \mathbb{F}_2$, then all homotopy groups $\pi_{n+1}(\operatorname{Pic}(\mathbb{F}_2))$ vanish and there exists an \mathbb{F}_2 -oriented Thom class

$$u(\xi) \colon \mathbf{M}(\xi) \to \mathbb{F}_2$$

and therefore a Thom isomorphism with \mathbb{F}_2 -coefficients.

• If $E \equiv \mathbb{Z}$ and assuming B connected and pointed, the obstruction to the existence of a Thom class is exactly the induced map

$$\pi_1(B) \to \pi_1(\operatorname{Pic}(\mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z}$$

vanishing if and only if ξ is orientable, if and only if the first Stiefel–Whitney class $w_1(\xi)$ with \mathbb{F}_2 -coefficients vanishes.

Exercise 4. In both cases, we show that invertible modules are shifts of the unit.

(1) Let X be an invertible \mathbb{Z} -module, with inverse Y. In particular both X and Y are perfect, and thus are represented by bounded complexes of projective (which here are free since we work over \mathbb{Z}) modules. Fix two such representatives X and Y in $\operatorname{Ch}_{\geq -m}(\mathbb{Z})$ for some $m \geq 0$.

The classical tensor product $X \otimes Y$ is already derived, since both X and Y are cofibrant for the projective model structure on $\operatorname{Ch}_{\geq -m}(\mathbb{Z})$. Since X and Y are degreewise projective, Künneth formula yields split short exact sequences

$$0 \longrightarrow \bigoplus_{k=-\infty}^{\infty} \mathrm{H}_k(X) \otimes \mathrm{H}_{n-k}(Y) \longrightarrow \mathrm{H}_n(\mathbb{Z}) \longrightarrow \bigoplus_{k=-\infty}^{\infty} \mathrm{Tor}_1\big(\mathrm{H}_k(X), \mathrm{H}_{n-k-1}(Y)\big) \longrightarrow 0$$

for all n, and because $\operatorname{Tor}_1(A,B)$ is always torsion for \mathbb{Z} -modules of finite type A and B, the left term at n=0 cannot vanish. Therefore, there exists r such that $\operatorname{H}_r(X) \otimes \operatorname{H}_{-r}(Y) \simeq \mathbb{Z}$, and this implies in turn

$$H_r(X) \simeq H_{-r}(Y) \simeq \mathbb{Z}$$

As an immediate consequence, we obtain $H_{n-r}(Y) \simeq 0$ and $H_{n+r}(X) \simeq 0$ for $n \neq 0$, so that $X \simeq \mathbb{Z}[r]$ and $Y \simeq \mathbb{Z}[-r]$.

(2) Let X be an inversible S-module, and assume without loss of generality that $X \otimes \mathbb{Z} \simeq \mathbb{Z}$. Since X is dualizable, it is (-m)-connective for some $m \geq 0$, and stable Hurewicz then implies that X is connective and yields an identification

$$\pi_0(X) \simeq \pi_0(X \otimes \mathbb{Z}) \simeq \mathbb{Z}$$

Since the first map is obtained by the composition

$$\pi_0(X) \simeq \pi_0 \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, X)$$

$$\to \pi_0 \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, X \otimes \mathbb{Z})$$

$$\simeq \pi_0 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, X \otimes \mathbb{Z})$$

$$\simeq \pi_0(X \otimes \mathbb{Z})$$

there exists a morphism $\alpha \colon \mathbb{S} \to X$ inducing an equivalence after applying $(-) \otimes \mathbb{Z}$. In particular

$$(\operatorname{cofib} \alpha) \otimes \mathbb{Z} \simeq \operatorname{cofib} \alpha \otimes \mathbb{Z}$$

Both $\mathbb S$ and X are connective so cofib α is connective as well, and stable Hurewicz then implies cofib $\alpha \simeq 0$. Finally, α is an equivalence $\mathbb S \simeq X$.