TOPOLOGIE IV – EXERCISE SHEET 3

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Quotients of ∞-groups. Given a morphism $H \to G$ of group objects in Gpd, the quotient G/H is defined as the fiber

$$G/H :\simeq \mathrm{fib}(\mathrm{B}H \to \mathrm{B}G)$$

Pasting cartesian squares

$$\begin{array}{ccccc} G & \longrightarrow & G/H & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{B}H & \longrightarrow & \mathrm{B}G \end{array}$$

shows that the structural map $G/H \to \mathrm{B}H$ classifies an H-principal bundle $G \to G/H$.

When G is a topological group and H a subgroup such that the projection $G \to G/H$ to the classical quotient is a principal H-bundle, then shifting and applying B to the fiber sequence

$$H \longrightarrow G \longrightarrow G/H$$

yields

$$G/H \longrightarrow \mathrm{B}H \longrightarrow \mathrm{B}G$$

so that the notation is justified.

Exercise 1. The map

$$S^d \simeq \mathrm{SO}(d+1)/\mathrm{SO}(d) \to \mathrm{BSO}(d)$$

classifies the principal $\mathrm{SO}(d)$-bundle

$$\mathrm{ev}_1 \colon \mathrm{SO}(d+1) \to S^d$$

given by evaluating on the first element of the canonical basis. Observe that the fiber of ev_1 above a unit vector $x \in S^d$ are direct orthonormal bases $(y_1, \ldots, y_d)$ of $x^\perp$, or in other words direct orthonormal frames of the tangent bundle $\mathrm{T}S^d \to S^d$, with its canonical metric and orientation, at x. Finally

$$S^d \to \mathrm{BSO}(d)$$

classifies the tangent bundle of S^d.

The horizontal lines in the following diagram of pointed groupoids

$$\begin{array}{ccccc} S^d & \longrightarrow & \mathrm{BSO}(d) & \longrightarrow & \mathrm{BSO}(d+1) \\ \| & & \downarrow & & \downarrow \\ S^d & \longrightarrow & \mathrm{BSF}(d) & \longrightarrow & \mathrm{BSG}(d+1) \\ \downarrow{\scriptstyle [S^d]} & & \downarrow & & \downarrow{\scriptstyle e} \\ \mathrm{K}(\mathbb{Z}, d) & \longrightarrow & * & \longrightarrow & \mathrm{K}(\mathbb{Z}, d+1) \end{array}$$

being fiber sequences, we obtain by naturality of the long exact sequence induced on homotopy groups

$$\begin{array}{ccccc} \pi_{d+1}\big(\mathrm{BSO}(d+1)\big) & \longrightarrow & \pi_{d+1}\big(\mathrm{BSG}(d+1)\big) & \xrightarrow{\quad e_* \quad} & \mathbb{Z} \\ \big\downarrow{\scriptstyle\partial} & & \big\downarrow{\scriptstyle\partial} & & \big\| \\ \pi_d\big(S^d\big) & = & \pi_d\big(S^d\big) & = & \mathbb{Z} \end{array}$$

This is the desired factorization.

Exercise 2. Consider a fiber sequence

$$S^d \longrightarrow S^m \xrightarrow{\ p\ } S^n$$

with typical fiber S^d where the integers d, m and n are non-negative. If $n = 0$, then $m = 0$ as well and p is the identity. But this is absurd because $d \geq 0$.

Only four terms are non-zero in the *reduced* Gysin sequence (with $\mathbb{F}_2$-coefficients)

$$\cdots \longrightarrow \mathrm{H}^{k-d-1}(S^n) \xrightarrow{e(p)} \overline{\mathrm{H}}^k(S^n) \longrightarrow \overline{\mathrm{H}}^k(S^m) \longrightarrow \cdots$$

Observe that:

(1) the group $\overline{\mathrm{H}}^{n+d+1}(S^n)$ is trivial and therefore the boundary map

$$\partial\colon \overline{\mathrm{H}}^{n+d}(S^m) \to \mathrm{H}^n(S^n) \simeq \mathbb{F}_2$$

is surjective. This implies $m = n + d$.

(2) observation (1) implies in particular $\overline{\mathrm{H}}^d(S^m) \simeq 0$, and the map

$$e(p)\colon \mathrm{H}^0(S^n) \to \overline{\mathrm{H}}^{d+1}(S^n)$$

is injective. In particular $n = d + 1$ and $e(p)$ is invertible.

If $n \geq 2$, then p is oriented and the same argument as above using coefficients $\mathbb{Z}$ instead of $\mathbb{F}_2$ shows that the oriented Euler class is invertible.

In particular, the Euler classes of the Hopf fibrations

$$S^0 \longrightarrow S^1 \xrightarrow{\gamma_1^{\mathbb{R}}} \mathbb{P}^1(\mathbb{R}) \simeq S^1$$

$$S^1 \longrightarrow S^3 \xrightarrow{\gamma_1^{\mathbb{C}}} \mathbb{P}^1(\mathbb{C}) \simeq S^2$$

$$S^3 \longrightarrow S^7 \xrightarrow{\gamma_1^{\mathbb{H}}} \mathbb{P}^1(\mathbb{H}) \simeq S^4$$

are invertible. Since the sign of the oriented Euler class depends on the identification of the fiber, we can assume that the Euler classes of the last two sequences are both 1.

Exercise 3. Observe that the exercise makes sense because Stiefel–Whitney classes are stable, and that it suffices to prove these relations for a given spherical fibration $p\colon E \to B$ of rank $d-1$ (and in particular for the universal one $\mathrm{BF}(d-1) \to \mathrm{BG}(d)$).

If $u \in \mathrm{H}^d(\mathrm{Th}(p); \mathbb{F}_2)$ is the Thom class and $n \geq 0$, we compute in $\mathrm{H}^{n+1+d}(\mathrm{Th}(p); \mathbb{F}_2)$

$$\begin{aligned} \mathrm{Sq}^1 \mathrm{Sq}^n(u) &= \mathrm{Sq}^1(w_n \cdot u) \\ &= \mathrm{Sq}^1(u) \cdot w_n + \mathrm{Sq}^1(w_n) \cdot u \\ &= \big(w_1 w_n + \mathrm{Sq}^1(w_n)\big) \cdot u \end{aligned}$$

where the second step uses the Cartan formula. But remember that the Adem relations imply

$$\mathrm{Sq}^1 \mathrm{Sq}^n = \begin{cases} \mathrm{Sq}^{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Using Thom isomorphism, we obtain finally

$$\mathrm{Sq}^1(w_n) = \begin{cases} w_1 w_n + w_{n+1} & \text{if } n \text{ is even} \\ w_1 w_n & \text{if } n \text{ is odd} \end{cases}$$

in $\mathrm{H}^{n+1}(B;\mathbb{F}_2)$.

Exercise 4. For $p\colon E \to B$ and $p'\colon E' \to B'$ two spherical fibrations of rank $d-1$ and $d'-1$ respectively with B and B' connected.

Since the fundamental class $\left[S^{k+1}\right]$ for some $k \geq 0$ is by definition the image of $\left[S^k\right]$ under the suspension isomorphism

$$\left[S^1\right] \otimes (-)\colon \overline{\mathrm{H}}^k\left(S^k;\mathbb{Z}\right) \simeq \overline{\mathrm{H}}^{k+1}\left(S^{k+1};\mathbb{Z}\right)$$

where we implicitly use Künneth to compute $\overline{\mathrm{H}}^*$ of a smash product, it follows that

$$\begin{aligned} \left[S^d\right] \otimes \left[S^{d'}\right] &= \left[S^1\right]^{\otimes(d+d')} \\ &= \left[S^{d+d'}\right] \end{aligned}$$

in $\overline{\mathrm{H}}^{d+d'}\left(S^{d+d'};\mathbb{Z}\right)$.

Choosing base points for B and B', the commutative diagram

$$\begin{array}{ccc} S^d \wedge S^{d'} & = & S^{d+d'} \\ \| & & \| \\ \mathrm{Th}\left(S^{d-1}\right) \wedge \mathrm{Th}\left(S^{d'-1}\right) & = & \mathrm{Th}\left(S^{d+d'-1}\right) \\ \downarrow & & \downarrow \\ \mathrm{Th}(p) \wedge \mathrm{Th}(p') & = & \mathrm{Th}(p \star p') \end{array}$$

then yields the equality of unoriented Thom classes

$$u(p \star p') = u(p) \otimes u(p')$$

Pulling back along the following commutative square

$$\begin{array}{ccc} B \wedge B' & \longleftarrow & B \times B' \\ \downarrow & & \downarrow \\ \mathrm{Th}(p) \wedge \mathrm{Th}(p') & = & \mathrm{Th}(p \star p') \end{array}$$

finally shows the desired relation on Euler classes

$$e(p \star p') = e(p) \times e(p')$$

If $B \equiv B'$, denote $\Delta_B\colon B \to B \times B$ the diagonal and compute

$$\begin{aligned} e(p \oplus p') &= e(\Delta_B^*(p \star p')) \\ &= \Delta_B^*\, e(p \star p') \\ &= \Delta_B^*(e(p) \times e(p')) \\ &= e(p) \cdot e(p') \end{aligned}$$

The proof in the oriented case is the same.