TOPOLOGIE IV – EXERCISE SHEET 2

MARCUS NICOLAS

Universality of colimits in Gpd**.** Given a map $f\colon X \to Y$ between groupoids, the pullback functor sits inside a commutative diagram

$$\begin{array}{ccc} \mathrm{Gpd}_{/Y} & \xrightarrow{\quad f^* \quad} & \mathrm{Gpd}_{/X} \\ \Vert & & \Vert \\ \mathrm{Fun}(Y, \mathrm{Gpd}) & \xrightarrow{\quad f^* \quad} & \mathrm{Fun}(X, \mathrm{Gpd}) \end{array}$$

Since colimits in those functor categories are formed pointwise, they are preserved by precomposition. In particular, the base change along f functor

$$f^*\colon \mathrm{Gpd}_{/Y} \to \mathrm{Gpd}_{/X}$$

preserves colimits. We say that colimits are *universal* in Gpd. As an exercice, show that colimits in Cat are not universal.

Truncated maps. Let $\mathcal{C}$ be a category with finite limits. For $k \geq -2$, a map $f\colon x \to y$ of $\mathcal{C}$ is $(k+1)$-truncated if and only if the diagonal $\Delta_f\colon x \to x \times_y x$ is k-truncated.

Using this characterisation, one can show that any left exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories with finite limits preserves truncatedness of objects and morphisms. If F is conservative, then it furthermore reflects truncatedness. For instance, since the forgetful functor $\mathrm{Gpd}_* \to \mathrm{Gpd}$ is conservative and preserves limits, it preserves and reflects truncatedness.

Lifting problems and finding sections. Let $\mathcal{C}$ be a category. For any cospan in $\mathcal{C}$

$$x \xrightarrow{f} b \xleftarrow{p} e$$

whose limit exists, consider the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}_{/x}}(x, f^*p) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x, f^*p) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x, e) \\ \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f_*} \\ * & \xrightarrow{\quad \mathrm{id}_x \quad} & \mathrm{Hom}_{\mathcal{C}}(x, x) & \xrightarrow{u_*} & \mathrm{Hom}_{\mathcal{C}}(x, b) \end{array}$$

In particular, the two following lifting problems are equivalent

$$\begin{array}{ccc} & & e \\ & \nearrow & \big\downarrow{\scriptstyle p} \\ x & \xrightarrow[f]{} & b \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & & x \times_b e \\ & \nearrow & \big\downarrow{\scriptstyle f^*p} \\ x & = & x \end{array}$$

As a slogan, every lifting problem is equivalent to the problem of constructing a section.

A criterion for connectivity. For X a pointed groupoid and $n \geq 0$, the following are equivalent:

(i) X is n-connected, or in other words $\tau_{\leq n} X \simeq *$

(ii) $\mathrm{Hom}_*(X, Y) \simeq *$ for every pointed and n-truncated groupoid Y

(iii) for any m-truncated morphism $f\colon Y \to Z$ between pointed groupoids, the map
$$f_*\colon \operatorname{Hom}_*(X,Y) \to \operatorname{Hom}_*(X,Z)$$
is $(m-n-1)$-truncated.

Proof. Clearly (iii) implies (ii).

We now show that (ii) implies (i). For any groupoid Y, the evaluation map $\mathrm{ev}_*\colon \operatorname{Hom}(X,Y) \to Y$ is an equivalence, since all of its fibers are contractible by assumption. Yoneda lemma then implies that $\tau_{\leq n} X \simeq *$.

Finally, we turn to the implication (i) implies (iii). Since the functor $\operatorname{Hom}_*(X,-)$ preserves limits, the map
$$f_*\colon \operatorname{Hom}_*(X,Y) \to \operatorname{Hom}_*(X,Z)$$
induced by composition with some $f\colon Y \to Z$ is $(k+1)$-truncated if and only if the map
$$(\Delta_f)_*\colon \operatorname{Hom}_*(X,Y) \to \operatorname{Hom}_*(X,Z)$$
induced by the diagonal $\Delta_f\colon Y \to Y \times_Z Y$ is k-truncated. By induction, it thus suffices to show that f_* is an equivalence when f is $(n-1)$-truncated. In this case, the fiber above $u\colon X \to Z$ of f_* is the groupoid of pointed sections of $u^*f\colon W \to X$

$$\begin{array}{ccc}
\operatorname{Hom}_{*/X}(X, u^*f) & \longrightarrow & \operatorname{Hom}_*(X,Y) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle f_*} \\
* & \xrightarrow{\;u\;} & \operatorname{Hom}_*(X,Z)
\end{array}$$

By assumption X is connected, and thus u^*f has typical fiber F for some $(n-1)$-truncated groupoid F. Since $\operatorname{Aut}(F)$ is a reunion of connected components of the $(n-1)$-truncated groupoid $\operatorname{End}(F) :\simeq \operatorname{Hom}(F,F)$, it is itself $(n-1)$-truncated and $\operatorname{BAut}(F)$ is n-truncated. Since X is n-connected, the classifying map $X \to \operatorname{BAut}(F)$ is constant, and pasting cartesian squares

$$\begin{array}{ccccc}
X \times F & \longrightarrow & F & \longrightarrow & \operatorname{BAut}_*(F) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & * & \longrightarrow & \operatorname{BAut}(F)
\end{array}$$

yields an identification $W \simeq X \times F$ over X. Using the base point of W to turn F into a pointed groupoid:
$$\begin{aligned}
\operatorname{Hom}_{*/X}(X, u^*f) &\simeq \operatorname{Hom}_{*/X}(X, X \times F) \\
&\simeq \operatorname{Hom}_*(X, F) \\
&\simeq *
\end{aligned}$$
where the last step uses again that X is n-connected. Finally the map
$$f_*\colon \operatorname{Hom}_*(X,Y) \to \operatorname{Hom}_*(X,Z)$$
has contractible fibers, and therefore is an equivalence. This concludes the proof. □

Exercise 1. Observe that $* \wedge (-)$ and $S^0 \wedge (-)$ are left adjoint to the functors $*\colon \mathrm{Gpd}_* \to \mathrm{Gpd}_*$ and $\mathrm{id}_{\mathrm{Gpd}_*}$ respectively, and thus
$$* \wedge (-) \simeq * \quad \text{et} \quad S^0 \wedge (-) \simeq \mathrm{id}_{\mathrm{Gpd}_*}$$
Since $(-) \wedge (-)$ preserves colimits in each variable, we have a pushout square

$$\begin{array}{ccc}
\mathrm{id}_{\mathrm{Gpd}_*} & \longrightarrow & * \\
\downarrow & \ulcorner & \downarrow \\
* & \longrightarrow & S^1 \wedge (-)
\end{array}$$

and a canonical natural isomorphism $S^1 \wedge (-) \simeq \Sigma$. In particular

$$\begin{aligned}\Sigma(- \wedge -) &\simeq \Sigma(-) \wedge (-) \\ &\simeq (-) \wedge \Sigma(-)\end{aligned}$$

since the smash product is commutative.

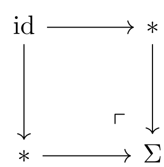
Exercise 2. Let X and Y two pointed groupoids being respectively m- and n-connected with m and n non-negative. For Z pointed and $(m+n+1)$-truncated, the criterion above shows that $\mathrm{Hom}_*(Y, Z)$ is m-truncated and thus

$$\begin{aligned}\mathrm{Hom}_*(X \wedge Y, Z) &\simeq \mathrm{Hom}_*(X, \mathrm{Hom}_*(Y, Z)) \\ &\simeq *\end{aligned}$$

Since this holds uniformly in Z, the smash product $X \wedge Y$ must be $(m+n+1)$-connected.

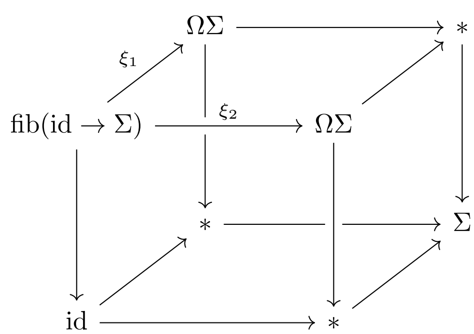
The result is false when m and n are allowed to be negative. For instance, smashing with the (-1)-connected space $S^0 \vee S^0$ does not preserve connectedness.

Exercise 3 ([DH21, lemma 2.17])**.** Mather's second cube lemma follows immediately from the universality of pushouts in Gpd. Let now $\mathcal{C}$ be a category with universal pushouts. Recall that the endofunctor $\Sigma \colon \mathcal{C}_* \to \mathcal{C}_*$ is defined by the following cocartesian square



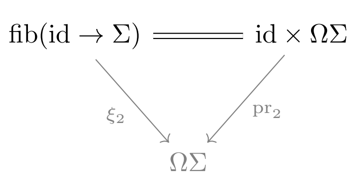
in the category of endomorphisms of pointed objects $\mathcal{C}_*$.

Consider now the following cube



where the top face is obtained by pulling back the bottom face along the base point $* \to \Sigma$. Since the bottom face is a pushout, the one must be as well by assumption.

By pasting cartesian squares all four other squares appearing in the cube are cartesian. Looking at the front face yields an identification



above $\Omega\Sigma$. Finally, we obtain a pushout square

$$\begin{array}{ccc} \mathrm{id} \times \Omega\Sigma & \xrightarrow{\mathrm{pr}_2} & \Omega\Sigma \\ {\scriptstyle a}\downarrow & \ulcorner & \downarrow \\ \Omega\Sigma & \longrightarrow & * \end{array}$$

where a is the composite

$$\mathrm{id} \times \Omega\Sigma = \mathrm{fib}(\mathrm{id} \to \Sigma) \xrightarrow{\xi_1} \Omega\Sigma$$

Observe that a is in general only conjugated to pr_2 by an automorphism of $\mathrm{id} \times \Omega\Sigma$ but is not homotopic to it. Indeed, base changing the defining pushout square of Σ along $\mathrm{pr}_1 \colon \Sigma \times \Omega\Sigma \to \Sigma$ yields

$$\begin{array}{ccc} \mathrm{id} \times \Omega\Sigma & \xrightarrow{\mathrm{pr}_2} & \Omega\Sigma \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \ulcorner & \downarrow \\ \Omega\Sigma & \longrightarrow & \Sigma \times \Omega\Sigma \end{array}$$

but $\Sigma \times \Omega\Sigma$ is not terminal in general.

Exercise 4 ([DH21, theorem 1.4])**.** Let $\mathcal{C}$ be a category with finite products and pushouts. Remember that for two pointed objects x and y, the natural identification $x \star y \simeq \Sigma(x \wedge y)$ is obtained by computing both sides as the colimit of the following diagram

$$\begin{array}{ccccc} * & \longleftarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ x & \longleftarrow & x \vee y & \longrightarrow & y \\ \downarrow & & \downarrow & & \downarrow \\ x & \longleftarrow & x \times y & \longrightarrow & y \end{array}$$

In particular, the structure maps $x \to x \star y$ and $y \to x \star y$ both naturally factor through the point, which is not obvious from the definition.

The following diagram

$$\begin{array}{ccccc} x \times y & & \xrightarrow{\mathrm{pr}_2} & & y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & & \ulcorner & \downarrow \\ x & \longrightarrow & * & \longrightarrow & x \star y \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma x & \longrightarrow & \Sigma x \vee (x \star y) \end{array}$$

gives canonical identifications

$$\begin{aligned} \mathrm{cofib}(\mathrm{pr}_2 \colon x \times y \to y) &\simeq \Sigma x \vee (x \star y) \\ &\simeq \Sigma x \vee \Sigma(x \wedge y) \end{aligned}$$

When $\mathcal{C}$ has universal pushouts, then combining this with the result from previous exercise we obtain natural isomorphisms

$$\begin{aligned} \Sigma\Omega\Sigma &\simeq \mathrm{cofib}(\mathrm{pr}_2 \colon \mathrm{id} \times \Omega\Sigma \to \Omega\Sigma) \\ &\simeq \Sigma \vee \Sigma(\mathrm{id} \wedge \Omega\Sigma) \\ &\simeq \Sigma \vee (\mathrm{id} \wedge \Sigma\Omega\Sigma) \end{aligned}$$

of endofunctors of $\mathcal{C}_*$. Plugging in the formula for $\Sigma\Omega\Sigma$ then yields

$$\begin{aligned}\Sigma\Omega\Sigma &\simeq \Sigma \vee \big(\mathrm{id} \wedge (\Sigma \vee \Sigma(\mathrm{id} \wedge \Omega\Sigma))\big)\\ &\simeq \Sigma \vee \Sigma\big(\mathrm{id}^{\wedge 2}\big) \vee \big(\mathrm{id}^{\wedge 2} \wedge \Sigma\Omega\Sigma\big)\end{aligned}$$

and by induction

$$\Sigma\Omega\Sigma \simeq \bigvee_{i=1}^{n} \Sigma\big(\mathrm{id}^{\wedge i}\big) \vee \big(\mathrm{id}^{\wedge n} \wedge \Sigma\Omega\Sigma\big)$$

for all $n \geq 1$. In particular there is a well defined comparison morphism

$$\bigvee_{i \geq 1} \Sigma\big(\mathrm{id}^{\wedge i}\big) \to \Sigma\Omega\Sigma$$

between endofunctors of $\mathcal{C}_*$.

Fix now X a pointed and connected groupoid. For $n \geq 1$, both the left map and the composite in the following diagram

$$\bigvee_{i=1}^{n} \Sigma\big(X^{\wedge i}\big) \longrightarrow \bigvee_{i \geq 1} \Sigma\big(X^{\wedge i}\big) \longrightarrow \Sigma\Omega\Sigma X$$

are the canonical inclusions, and therefore are both at least n-connected by the second exercise. By the cancellation property for connected morphisms, the right map is also n-connected. Since this holds for all n, we get the James splitting

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i \geq 1} \Sigma\big(X^{\wedge i}\big)$$

Exercise 5. Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and $d := [\mathbb{K} \colon \mathbb{R}]$. For $n \geq 1$, recall that $\mathbb{P}^{n+1}(\mathbb{K})$ is obtained from $\mathbb{P}^n(\mathbb{K})$ via the following cell attachment in Top

$$\begin{array}{ccc} S^{d(n+1)-1} & \xrightarrow{\gamma_n^{\mathbb{K}}} & \mathbb{P}^n(\mathbb{K}) \\ \downarrow & & \downarrow \\ D^{d(n+1)} & \longrightarrow & \mathbb{P}^{n+1}(\mathbb{K}) \end{array}$$

along the tautological spherical fibration $S^{d(n+1)-1} \to \mathbb{P}^n(\mathbb{K})$. But all objects at play are cofibrant and the left vertical map is a cofibration, so this pushout is furthermore an homotopy pushout. The cocartesian square in Gpd

$$\begin{array}{ccc} S^{d(n+1)-1} & \xrightarrow{\gamma_n^{\mathbb{K}}} & \mathbb{P}^n(\mathbb{K}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{P}^{n+1}(\mathbb{K}) \end{array}$$

thus yields an identification

$$\mathrm{Th}\left(\gamma_n^{\mathbb{K}}\right) \simeq \mathbb{P}^{n+1}(\mathbb{K})$$

This implies

$$\mathrm{Th}\left(\gamma_\infty^{\mathbb{K}}\right) \simeq \mathbb{P}^{\infty}(\mathbb{K})$$

which is evident from the description of $\gamma_\infty^{\mathbb{K}}$ as the universal principal $(\mathbb{K}^\times)$-bundle.

References

[DH21] Sanath Devalapurkar and Peter Haine, *On the James and Hilton–Milnor splittings, and the metastable EHP sequence*, Documenta Mathematica **26** (2021), 1423–1464.