LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2025 July 10, 2025

# Topology IV

Sheet 9

**Exercise 1.** Let $X$ be an anima and $f\colon X \to X$ an equivalence. Define the mapping torus $Tf$ of $f$ as the colimit in anima of the functor $B\mathbb{Z} \to \mathrm{An}$ determined by $f$.

(1) Show that there is a long exact sequence
$$\dots \to H^{*-1}(X) \xrightarrow{1-f^*} H^{*-1}(X) \to H^*(Tf) \to H^*(X) \xrightarrow{1-f^*} \to \dots$$
(2) Show that $Tf$ is a Poincaré duality complex if $X$ is a Poincaré duality complex.
(3) Assume that $X$ is an oriented Poincaré duality complex of dimension $4d-1$ that and $f$ preserves the orientation. Show that $Tf$ is then also oriented and that $\mathrm{sign}(Tf) = 0$. Hint: Construct a Lagrangian in $H^{2d}(Tf; \mathbb{R})$.

Recall here that for an oriented $4d$-dimensional PD complex $X$ we set $\mathrm{sign}(X) = \mathrm{sign}(\mu_X)$ where $\mu_X\colon H^{2d}(X;\mathbb{Z})/\mathrm{tors} \otimes_{\mathbb{Z}} H^{2d}(X;\mathbb{Z})/\mathrm{tors} \to \mathbb{Z}$ is the intersection pairing $\mu_X(a,b) = \langle ab, [X]\rangle$ induced by Poincaré duality. Moreover, recall that for a unimodular symmetric bilinear form $b$ on a finite free $\mathbb{Z}$-module, the signature $\mathrm{sign}(b)$ is defined as the signature of $b \otimes_{\mathbb{Z}} \mathbb{R}$, the associated unimodular symmetric bilinear form over $\mathbb{R}$ and that this in turn is computed as the number of positive minus the number of negative eigenvalues of a representing matrix for $b$.

Moreover, recall that given a unimodular symmetric bilinear form $b$ on a finite dimensional $\mathbb{R}$-vector space $V$, a Lagrangian for $b$ is a sub vector space $L \subseteq V$ such that $b(l, l') = 0$ for all $l, l' \in L$ (such $L$'s are called isotropic subspaces) such that the induced sequence
$$0 \to L \to V \overset{b}{\cong} V^\vee \to L^\vee \to 0$$
is exact. Show that an isotropic subspace $L \subseteq V$ is a Lagrangian if and only if $2\dim_{\mathbb{R}}(L) = \dim_{\mathbb{R}}(V)$ and show that if a unimodular symmetric bilinear form $b$ admits a Lagrangian, then $\mathrm{sign}(b) = 0$.

**Exercise 2.** Let $b$ be a unimodular symmetric bilinear form on a finite free $\mathbb{Z}$ module $V$. Then $b$ is called *even* if $b(x,x) \in 2\mathbb{Z}$ for all $x \in V$, if it is not even it is called *odd*. The goal of this exercise is to show that if $b$ is even, then $\mathrm{sign}(b) \in 8\mathbb{Z}$.

For the suggested outline, you may use the following fact from algebra: Two indefinite unimodular forms $b$ and $b'$ on finite $\mathbb{Z}$-modules $V$ and $V'$ are isomorphic if and only if they have the same type (i.e. are either both even or both odd), rank (i.e. $\mathrm{rk}(V) = \mathrm{rk}(V')$), and signature (i.e. $\mathrm{sign}(b) = \mathrm{sign}(b')$).

(1) Show that $b$ admits a *characteristic element* $c \in V$, i.e. an element such that $b(x,x) \equiv b(c,x) \mod 2$ for all $x \in V$.
(2) Show that the value $b(c,c) \in \mathbb{Z}/8\mathbb{Z}$ is independent of the choice of a characteristic element $c$.
(3) Show that if $b'$ is another unimodular symmetric bilinear form, and $c'$ is a characteristic element for $b'$, then $c \oplus c'$ is one for $b \oplus b'$.
(4) Show that if $b$ is even, then $\mathrm{sign}(b) \in 8\mathbb{Z}$.

This sheet will be discussed on July 18.