Summer term 2025 July 4, 2025

Topology IV

Sheet 8

Exercise 1. You may use that all of the following spaces are closed manifolds and hence Poincaré duality complexes. Recall or compute the (co)homology rings of each of the following spaces, the action of the Steenrod algebra on their mod 2 cohomology, and compute all Wu and Stiefel–Whitney classes.

(1) $\mathbb{RP}^n$ for $n \geq 1$,
(2) $\mathbb{CP}^n$ for $n \geq 1$,
(3) $\mathbb{HP}^n$ for $n \geq 1$,
(4) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ for $n \geq 1$,
(5) $\mathrm{Sp}(n)$ for $n \geq 1$,
(6) $\mathrm{SU}(3)/\mathrm{SO}(3)$, and
(7) $T^n = (S^1)^{\times n}$ for $n \geq 1$.
(8) $X(\alpha)$ as in Exercise 3 Sheet 7.

Exercise 2. Show that there is a pointed S^3-bundle over S^2 with non-trivial w_2. Show that this bundle is equivalent to $X(\eta)$, in particular showing that $X(\eta)$ is homotopy equivalent to a closed manifold.

This sheet will be discussed on June 6.