Summer term 2025 June 13, 2025

Topology IV

Sheet 6

Exercise 1. Let X be an anima and $R \in \mathrm{CAlg}(\mathrm{Sp})$. Show that

$$\mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}(X, \mathrm{Mod}(R)), \mathrm{Mod}(R)), \quad \mathcal{F} \mapsto r_!(- \otimes \mathcal{F})$$

is an equivalence; Show that this implies an equivalence

$$\mathrm{Fun}(X \times Y, \mathrm{Mod}(R)) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}(X, \mathrm{Mod}(R)), \mathrm{Fun}(Y, \mathrm{Mod}(R)))$$

and make this equivalence explicit.

Exercise 2. In this exercise you may use the following theorem of Wall: For a compact anima X there is an obstruction $o(X) \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$ which vanishes if and only if X is a finite anima.

Now show that every compact spectrum is finite.

Exercise 3. Let α be any of the following maps $S^4 \to S^2 \vee S^3$: The trivial map, the composite $S^4 \xrightarrow{\eta^2} S^2 \to S^2 \vee S^3$, or the composite $S^4 \xrightarrow{\eta} S^3 \to S^2 \vee S^3$. Moreover, let $[i_2, i_3]\colon S^4 \to S^2 \vee S^3$ be the attaching map for the 5-cell of $S^2 \times S^3$. Let $X(\alpha)$ be the cofibre of the map $[i_2, i_3] + \alpha \in \pi_4(S^2 \vee S^3)$. Show that there exists a class $[X(\alpha)] \in H_5(X(\alpha); \mathbb{Z})$ such that

$$- \cap [X]\colon H^k(X(\alpha); \mathbb{Z}) \to H_{5-k}(X(\alpha); \mathbb{Z})$$

is an isomorphism, i.e. that $X(\alpha)$ satisfies Poincaré duality and compute the action of the Steenrod algebra on $H^*(X(\alpha); \mathbb{F}_2)$.

This sheet will be discussed on June 6.