Summer term 2025 June 5, 2025

Topology IV

Sheet 6

Exercise 1. Prove Wu's formula in $H^*(\mathrm{BO};\mathbb{F}_2)$ using the splitting principle.

Recall that $\mathrm{Pic}(\mathbb{S})$ is the groupoid of tensor invertible objects in Sp – this anima classifies stable spherical fibrations of possibly non-zero virtual rank. Its 0 component is given by BG, so this classifies stable spherical fibrations of virtual rank 0.

Exercise 2. Let $\xi\colon B \to \mathrm{BG}$ be an oriented stable spherical fibration of virtual rank 0 with Thom spectrum $M(\xi)$. Denote by $u\colon M(\xi) \to \mathbb{Z}$ the Thom class. Recall the Thom diagonal map $M(\xi) \to B \otimes M(\xi)$. Consider the composite

$$M(\xi) \to B \otimes M(\xi) \xrightarrow{B\otimes u} B \otimes \mathbb{Z}$$

and use the adjunction $\mathrm{Sp} \leftrightarrows \mathrm{Mod}(\mathbb{Z})$ induced from the ring map $\mathbb{S} \to \mathbb{Z}$ to get a $\mathbb{Z}$-linear map

$$M(\xi) \otimes \mathbb{Z} \to B \otimes \mathbb{Z}.$$

Show that this map is an equivalence of $\mathbb{Z}$-modules and explain why this recovers what we called Thom isomorphism in the lecture.

Exercise 3. For $E \in \mathrm{CAlg}(\mathrm{Sp})$, denote similarly to $\mathrm{Pic}(\mathbb{S})$ by $\mathrm{Pic}(E)$ the groupoid of $\otimes_E$-invertible objects in $\mathrm{Mod}(E)$. Let $\xi\colon B \to \mathrm{Pic}(\mathbb{S})$ be a stable spherical fibration. Show that ξ-admits a Thom isomorphism in E-cohomology if and only if $B \to \mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(E)$ is null-homotopic. Here, we use that $\mathrm{Sp} \to \mathrm{Mod}(E)$, $X \mapsto X \otimes E$ is symmetric monoidal and hence induces a map $\mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(E)$.

Show that for $n > 0$ we have

$$\pi_n(\mathrm{Pic}(E)) = \begin{cases} \pi_0(E)^\times & \text{if } n = 1 \\ \pi_{n-1}(E) & \text{if } n > 1 \end{cases}.$$

Use this to reprove that that any stable spherical fibration ξ admits a Thom isomorphism in $\mathbb{F}_2$-cohomology, and that there is a Thom isomorphism in $\mathbb{Z}$-cohomology if and only $w_1(\xi) = 0$.

Exercise 4. Show that $\pi_0(\mathrm{Pic}(\mathbb{S})) \to \pi_0(\mathrm{Pic}(\mathbb{Z}))$ and $\mathbb{Z} \to \pi_0(\mathrm{Pic}(\mathbb{Z}))$, $1 \mapsto \Sigma\mathbb{Z}$, are isomorphisms. You may use here that $\mathrm{Mod}(\mathbb{Z}) \simeq \mathcal{D}(\mathbb{Z})$, so that $\mathrm{Pic}(\mathbb{Z})$ identifies with the groupoid of $\otimes_{\mathbb{Z}}$-invertible objects of the derived category of $\mathbb{Z}$.

This sheet will be discussed on June 6.