LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2025 May 30, 2025

Topology IV

Sheet 5

Exercise 1. Let $v(\pi)$ be the total Wu class of a spherical fibration π. Show that $v(\pi\oplus\pi') = v(\pi)\cdot v(\pi')$, i.e. that the total Wu class satisfies the Cartan formula.

Exercise 2. Let PSp be the ∞-category of pre-spectra, that is, the ∞-category of diagrams $\mathbb{N}\times\mathbb{N} \to \mathrm{An}_*$ with the property that all off diagonal entries the zero object $* \in \mathrm{An}_*$; write X_n for $X(n,n)$ when X is a pre-spectrum. Show that there is an adjunction between An_* and PSp where the right adjoint is $\mathrm{PSp} \to \mathrm{An}_*$ given by evaluation at 0. The ∞-category Sp of spectra is the full subcategory of PSp on those diagrams $X \in \mathrm{PSp}$ such that the canonically induced maps $X_n \to \Omega X_{n+1}$ are equivalences for all n. For $X \in \mathrm{PSp}$, define $\pi_*(X) = \mathrm{colim}\pi_{*+n}(X_n)$. Show that the inclusion $\mathrm{Sp} \subseteq \mathrm{PSp}$ admits a left adjoint L called spectrification and that the unit map $X \to LX$ induces an isomorphism on π_*. In fact, Sp turns out to be the localization of PSp at the maps inducing isomorphisms on homotopy groups.

Show that $\Omega\colon \mathrm{Sp} \to \mathrm{Sp}$ is simply given by shifting and that it is an equivalence (its inverse is then necessarily given by the internal suspension functor of Sp).

Exercise 3. Let $\xi\colon B \to \mathrm{BG}$ be a stable spherical fibration. Denote by $\xi_n\colon B \times_{\mathrm{BG}} \mathrm{BG}(n)$ the induced family of finite rank spherical fibrations. Show that $\{\mathrm{Th}(\xi_n)\}_{n\geq 0}$ naturally forms a prespectrum $\mathrm{Th}(\xi)$. Denote by $M(\xi) = L(\mathrm{Th}(\xi))$ its associated spectrum. Show that $M(\xi) = \mathrm{colim}_B(B \to \mathrm{BG} \to \mathrm{Sp})$. Here, you may use (or show) that $\mathrm{BG} \subseteq \mathrm{Sp}$ is the full subgroupoid of Sp on the sphere spectrum, that is of the spectrification of the suspension-prespectrum of S^0.

Moreover, show that there is a tautological Thom diagonal map $M(\xi) \to B \otimes M(\xi)$, where the tensor product refers to the colimit of the B-indexed constant diagram on $M(\xi)$.

Exercise 4. For an abelian group, let us denote also by A the spectrum $\{K(A,n)\}_{n\geq 0}$. Define for a spectrum X its cohomology with coefficients in A by $H^k(X;A) = \pi_0\mathrm{Map}_{\mathrm{Sp}}(X, \Sigma^k A)$. Show that an oriented stable spherical fibration ξ over B has a Thom class $u(\xi) \in H^0(M(\xi);\mathbb{Z})$, that is a class whose restriction (induced by a basepoint in B) to $H^0(\mathbb{S};\mathbb{Z})$ is a generator, and that the multiplication with the Thom class map induces an isomorphism

$$H^*(B;\mathbb{Z}) \xrightarrow{\cong} H^*(M(\xi);\mathbb{Z})$$

This sheet will be discussed on June 6.