Summer term 2025 May 14, 2025

Topology IV

Sheet 3

Exercise 1. Consider the fibre sequence

$$S^{d-1} \to \mathrm{BSO}(d-1) \to \mathrm{BSO}(d).$$

Show that the first map classifies the tangent bundle TS^d of S^{d-1}, i.e. the subbundle of $S^{d-1} \times \mathbb{R}^d$ on those pairs (x, y) such that $\langle x, y\rangle = 0$. Furthermore, show that the boundary map associated to this fibration factors as

$$\partial\colon \pi_d(\mathrm{BSO}(d)) \to \pi_d(\mathrm{BSG}(d)) \to \mathbb{Z} \cong \pi_{d-1}(S^{d-1})$$

where the latter map sends an oriented spherical fibration $E \to S^d$ to its Euler class.

Exercise 2. Suppose that there is a fibre sequence $S^d \to S^m \xrightarrow{\pi} S^n$. Then $d = n-1$, $m = 2n-1$, and $e(\pi) = \pm 1$. Compute the Euler classes of the tautological complex and quaternionic bundles over S^2 and S^4.

Exercise 3. Prove the following relations in $H^*(\mathrm{BG}; \mathbb{F}_2)$:

(1) $\mathrm{Sq}^1(w_{2n}) = w_1 w_{2n} + w_{2n+1}$.
(2) $\mathrm{Sq}^1(w_{2n+1}) = w_1 w_{2n+1}$.

Exercise 4. Let $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ be (oriented) spherical fibrations of rank $d-1$ and $d'-1$, respectively. Show that $e(\pi \star \pi') = e(\pi) \times e(\pi') \in H^{d+d'}(B \times B')$ and consequently that, if $B = B'$, $e(\pi \oplus \pi') = e(\pi) \cdot e(\pi')$.

This sheet will be discussed on May 23.