Summer term 2025 June 12, 2025

Algebraic K-theory

Sheet 5

We are in the situation of Milnor patching as in the lecture.

Exercise 1. Assume that $S \in M_{n,m}(B')$ is the image of an invertible matrix $T \in M_{n,m}(B)$. Then $M(A'^n, B^m, S)$ is finite free and the A-linear maps $A'^n \leftarrow M(A'^n, B^m, S) \to B^m$ induce isomorphisms

$$M(A'^n, B^m, S) \otimes_A A' \cong A'^n \quad \text{and} \quad M(A'^n, B^m, S) \otimes_A B \cong B^m.$$

The resulting isomorphism

$$B'^n \cong A'^n \otimes_{A'} B' \cong M(A'^n, B^m, S) \otimes_A B' \cong B^m \otimes_B B' \cong B'^m$$

is S.

Solution. Our conventions are so that elements of $M_{n,m}(R)$ determine R-linear maps $R^n \to R^m$. Consider the following commutative diagram of spans

$$\begin{array}{ccccc} A'^n & \xrightarrow{f'} & B'^n & \xleftarrow{p} & B^n \\ \Vert & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} \\ A'^n & \xrightarrow{S\circ f'} & B'^m & \xleftarrow{p} & B^m \end{array}$$

all whose vertical maps are isomorphisms. Therefore, on pullbacks we obtain an isomorphism

$$A^n \xrightarrow{\cong} M(A'^n, B^m, S)$$

showing that the latter is finite free over A. The second claim is now immediate, since the analogous result is true for $S = \mathrm{id}$ and the final claim follows similarly. □

Exercise 2. Let $S \in M_{n,m}(B')$ be an invertible matrix and assume that $B \to B'$ is surjective. Then the invertible matrix

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \in M_{m+n,m+n}(B')$$

is the image of an invertible matrix in $M_{m+n,m+n}(B)$.

Solution. We have

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} = \begin{pmatrix} 1_n & S \\ 0 & 1_m \end{pmatrix} \cdot \begin{pmatrix} 1_n & 0 \\ -S^{-1} & 1_m \end{pmatrix} \cdot \begin{pmatrix} 1_n & S \\ 0 & 1_m \end{pmatrix} \cdot \begin{pmatrix} 0 & -1_n \\ 1_m & 0 \end{pmatrix}$$

and all of the matrices appearing on the right hand side can be lifted to matrices of the same shape since $B \to B'$ is surjective. Since any matrix of these shapes is invertible, the claim follows. □

Exercise 3. Assume P is a finite free A'-module, Q is a finite free B-module, and $\alpha\colon P \otimes_{A'} B' \cong Q \otimes_B B'$ is an isomorphism of B'-modules. Then $M(P,Q,\alpha)$ is finite projective and the tautological maps $M(P,Q,\alpha) \otimes_A A' \to P$ and $M(P,Q,\alpha) \otimes_A B \to Q$ are isomorphisms and the resulting composite isomorphism

$$P \otimes_A' B' \cong M(P,Q,\alpha) \otimes_A B' \cong Q \otimes_B B'$$

is α.

Proof. Fix isomorphisms $P \cong A'^n$ and $Q \cong B^m$ so that α is equivalently described by an invertible matrix $S \in M_{n,m}(B')$. Then we compute

$$M(A'^n, B^m, S) \oplus M(A'^m, B^n, S^{-1}) = M(A'^{n+m}, B^{n+m}, \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}) \cong A^{n+m}$$

where the latter isomorphism follows from Exercise 2 and Exercise 1. It follows that $M(P,Q,\alpha) \cong M(A'^n, B^m, S)$ is finite projective. Moreover, the sum of the maps

$$M(A'^n, B^m, S) \to A'^n \quad \text{and } M(A'^m, B^n, S^{-1}) \to A'^m$$

identifies with the corresponding map

$$M(A'^{n+m}, B^{n+m}, S \oplus S^{-1}) \to A'^{n+m}$$

which is an isomorphism by Exercise 1. Hence, both maps appearing in the upper display are also isomorphisms and therefore also $M(P,Q,\alpha) \otimes_A A' \to P$ and $M(P,Q,\alpha) \otimes_A B \to Q$. Similarly, the sum of the composite isomorphisms

$$B'^n \to B'^m \quad \text{and } B'^m \oplus B'^n$$

identifies with the composite isomorphism

$$B'^{n+m} \to B'^{n+m}$$

which Exercise 1 shows to be $S \oplus S'$. Consequently, we obtain the the two composite isomorphisms in the upper display are given by S and S^{-1} respectively, showing all claims. ☐

Exercise 4. Let now (P,Q,α) be a general object of $\mathrm{Proj}(A') \times_{\mathrm{Proj}(B')} \mathrm{Proj}(B)$. Show that there exists (P',Q',α') such that $P \oplus P'$ and $Q \oplus Q'$ are free and finish the proof of Milnor's patching theorem.

Solution. Pick $\bar{P}$ and $\bar{Q}$ such that $P \oplus \bar{P} \cong A'^n$ and $Q \oplus \bar{Q} \cong B^m$. Define $P' = \bar{P} \oplus A'^m$ and $Q' = \bar{Q} \oplus B^n$. Then we need to argue that there is an isomorphism α' between $P' \otimes_{A'} B'$ and $Q' \otimes_B B'$. To see this, we compute

$$\begin{aligned} P' \otimes_A &\cong \bar{P} \otimes_{A'} B' \oplus B'^m \\ &\cong \bar{P} \otimes_{A'} B' \oplus [Q \otimes_B B' \oplus \bar{Q} \otimes_B B'] \\ &\cong \bar{P} \otimes_{A'} B' \oplus P \otimes_{A'} B' \oplus \bar{Q} \otimes_B B' \\ &\cong B'^n \oplus \bar{Q} \otimes_B B' \\ &\cong Q' \otimes_B B' \end{aligned}$$

as needed. It then follows that

$$M(P,Q,\alpha) \oplus M(P',Q',\alpha') \cong M(P \oplus P', Q \oplus Q', \alpha \oplus \alpha')$$

and the latter is finite projective by Exercise 3, so it follows that $M(P,Q,\alpha)$ is also finite projective. Finally, just like in the argument in Exercise 3, the maps

$$M(P,Q,\alpha) \otimes_A A' \to P \quad \text{and} \quad M(P,Q,\alpha) \otimes_A B \to Q$$

are direct summand of the same maps for $M(P \oplus P', Q \oplus Q', \alpha \oplus \alpha')$ which are isomorphisms by Exercise 3, and are hence themselves isomorphisms; Similarly, the composite isomorphism

$$P \otimes_A A' \cong M(P,Q,\alpha) \otimes_A B' \cong Q \otimes_B B'$$

is α, again by reducing the the case of $M(P \oplus P', Q \oplus Q', \alpha \oplus \alpha')$ and using Exercise 3. □

Exercise 5. Show that the map $\partial \colon \mathrm{GL}(B') \to K_0(A)$ defined in the lecture is a monoid homomorphism.

Solution. Pick $S, T \in \mathrm{GL}(B')$ and find $n \geq 0$ so that $S, T \in \mathrm{GL}_n(B')$. Then we have

$$M(A'^n, B^n, ST) \oplus A^n \cong M(A'^{2n}, B^{2n}, ST \oplus 1_n).$$

Moreover, we have

$$\begin{pmatrix} ST & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \cdot \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$$

and the $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$ can be lifted to $U \in \mathrm{GL}_{2n}(B)$ by Exercise 2. Consider then the diagram

$$\begin{array}{ccccc} A'^{2n} & \xrightarrow{\phi} & B'^{2n} & \xleftarrow{p} & B^{2n} \\ \| & & \downarrow \theta & & \downarrow \eta \\ A'^{2n} & \xrightarrow{\psi} & B'^{2n} & \xleftarrow{p} & B^{2n} \end{array}$$

where ϕ is multiplication by $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, θ is multiplication by $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$, ψ is multiplication by $\begin{pmatrix} ST & 0 \\ 0 & 1_n \end{pmatrix}$ and η is multiplication by U. As all of the vertical maps in this diagram are isomorphisms, this shows that

$$M(A'^{2n}, B^{2n}, ST \oplus 1_n) \cong M(A'2n, B^{2n}, \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}) \cong M(A'^n, B^n, S) \oplus M(A'^n, B^n, T).$$

With this, we finally have

$$\begin{aligned} \partial[ST] &= [M(A'^n, B^n, ST)] - [A^n] \\ &= [M(A'^{2n}, B^{2n}, ST \oplus 1_n) - [A^{2n}] \\ &= [M(A'^n, B^n, S)] - [A^m] + [M(A'^n, B^n, T)] - [A^n] \\ &= \partial[S] + \partial[T] \end{aligned}$$

□

This sheet will be discussed on 3 July 2025.