LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2025 June 4, 2025

Algebraic K-theory

Sheet 5

Exercise 1. Let $f\colon X \to Y$ be a map of anima. Show that the following conditions are equivalent.

(1) For every local coefficient system $\mathcal{L}$ on Y, the map $f_*\colon H_*(X; f^*\mathcal{L}) \to H_*(Y;\mathcal{L})$ is an isomorphism,

(2) the map $\bar{X} = X \times_Y \widetilde{Y} \to \widetilde{Y}$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$; here $\widetilde{Y} \to Y$ denotes the universal cover, and

(3) for every point $y \in Y$, the map $\mathrm{fib}_y(f) \to *$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$.

Solution. We recall that $H_*(Y;\mathcal{L})$ is the homology of the chain complex $C_*(\widetilde{Y}) \otimes_{\mathbb{Z}[\pi_1(Y)]} \mathcal{L}$. Moreover, we have

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} f^*(\mathcal{L}) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{Z}\pi_1(Y) \otimes_{\mathbb{Z}\pi_1(Y)} \mathcal{L} = C_*(\bar{X}) \otimes_{\mathbb{Z}\pi_1(Y)} \mathcal{L}.$$

Consequently, since $C_*(\bar{X}) \to C_*(\widetilde{Y})$ is $\mathbb{Z}\pi_1(Y)$-equivariant (and degreewise free), it it is a quasi-isomorphism, it is a $\mathbb{Z}\pi_1(Y)$-chain homotopy equivalence; in particular the map remains a quasi-isomorphism upon tensoring with arbitrary $\mathbb{Z}\pi_1(Y)$-modules, showing that (2) implies (1). Conversely, (1) implies (2) since we may choose $\mathcal{L} = \mathbb{Z}\pi_1(Y)$. To see that (2) is equivalent to (3), not that there is a fibre sequence

$$\mathrm{fib}(f) \to \bar{X} \to \widetilde{Y}.$$

Since $\widetilde{Y}$ is simply connected, the Serre spectral sequence is

$$H_p(\widetilde{Y}; H_q(\mathrm{fib}(f)) \Rightarrow H_{p+q}(\bar{X};\mathbb{Z}).$$

If $\mathrm{fib}(f) \to *$ is a homology isomorphism, the spectral sequence collapses and immediately gives that $\bar{X} \to \widetilde{Y}$ is a homology isomorphism. Conversely, if $\bar{X} \to \widetilde{Y}$ is a homology isomorphism, then there cannot be any differentials eminating out of the x-axes and the $E^\infty_{p,q}$-term must be trivial for $q > 0$. Inductively, this implies that $E^2_{0,q}$ must be trivial for $q > 0$: Suppose n is the smallest positive number such that $H_n(\mathrm{fib}(f)) = E^2_{0,n} \neq 0$. Then, as $E^\infty_{0,n} = 0$, there must be a non-trivial differential hitting $H_n(\mathrm{fib}(f))$. By the minimality of n, such a differential must come from the x-axes; but we have just argued that all differentials out of the x-axes are trivial. □

Exercise 2. Without using the construction, show that if the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ admits a left adjoint L, then the unit map $X \to LX$ induces an isomorphism on $H_*(-;\mathbb{Z})$. Hint: Show that $X \to LX$ induces an isomorphism on $H^*(-;\mathbb{Z})$ and show that a map which induces an isomorphism on $H^*(-;\mathbb{Z})$ in fact also induces an isomorphism on $H_*(-;\mathbb{Z})$.

Solution. $X \to LX$ induces an equivalence $\mathrm{Map}(LX, K(\mathbb{Z}, n)) \to \mathrm{Map}(X, K(\mathbb{Z}, n))$ since $K(\mathbb{Z}, n)$ is in particular hypoabelian; it follows that $H^*(LX; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ is an isomorphism. Now, we show that the functor $\mathrm{map}_{\mathbb{Z}}(-, \mathbb{Z})\colon \mathcal{D}(\mathbb{Z})^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ is conservative; then we apply this to $C_*(X; \mathbb{Z}) \to C_*(LX; \mathbb{Z})$, where applying $\mathrm{map}_{\mathbb{Z}}(-, \mathbb{Z})$ gives $C^*(LX; \mathbb{Z}) \to C^*(X; \mathbb{Z})$. To see the conservativity, pick $M \in \mathcal{D}(\mathbb{Z})$ with $\mathrm{map}_{\mathbb{Z}}(M, \mathbb{Z}) = 0$. Then

$$0 = \mathrm{map}_{\mathbb{Z}}(M, \mathbb{Z})/p = \mathrm{map}_{\mathbb{Z}}(M, \mathbb{F}_p) = \mathrm{map}_{\mathbb{F}_p}(M \otimes_{\mathbb{Z}} \mathbb{F}_p, \mathbb{F}_p)$$

and since $\mathbb{F}_p$ is a field, we find $M \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$. This holds for every prime p, showing that M is rational. If M is non-trivial, it contains some (shift of) $\mathbb{Q}$ as a direct summand, but $\mathrm{map}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is non-trivial (its π_{-1} is $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ which is non-trivial). □

Exercise 3. Show that the canonical map $(X \times Y)^+ \to X^+ \times Y^+$ is an equivalence for all anima X and Y.

Solution. First option: Use Exercise 4, and the fact that the maximal perfect subgroup of $\pi_1(X) \times \pi_1(Y)$ is the product of the maximal perfect subgroups: This implies that the map in question is a π_1-isomorphism. Moreover, it is also acyclic since products of acyclic maps are acyclic.

Second option: For formal reasons, $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ is closed under arbitrary limits. In particular, if A is hypoabelian and T is any anima, then $\mathrm{Map}(T, A)$ is hypoabelian. Then by Yoneda we have

$$\mathrm{Map}((X \times Y)^+, A) \simeq \mathrm{Map}(X \times Y, A) \simeq \mathrm{Map}(X, \mathrm{Map}(Y, A)) \simeq \mathrm{Map}(X^+, \mathrm{Map}(Y^+, A)) \simeq \mathrm{Map}(X^+ \times Y^+, A)$$

as needed. □

Exercise 4. Let $f\colon X \to Y$ be acyclic. Show that for all $x \in X$, the map $\pi_1(X) \to \pi_1(Y)$ is surjective with perfect kernel. Moreover, show that if f induces an isomorphism $\pi_1(X, x) \to \pi_1(Y, f(x))$ for all $x \in X$, then f is an equivalence.

Solution. Let $F = \mathrm{fib}_{f(x)}(f)$. Since f is acyclic, we have that $\widetilde{H}_*(F) = 0$. In particular, F is connected and $\pi_1(F)$ has trivial abelianzation, i.e. is perfect. Then we have a long exact sequence of homotopy groups

$$\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1$$

Hence, the kernel of $\pi_1(X) \to \pi_1(Y)$ is a quotient of $\pi_1(F)$ and therefore also perfect. For the second claim, since f is acyclic, the induced map $\bar{X} \to \widetilde{Y}$ is a homology isomorphism. Since f is a π_1-isomorphism, $\bar{X} = \widetilde{X}$ is simply connected, so that the Hurewicz theorems imply that $\widetilde{X} \to \widetilde{Y}$ is an equivalence. In particular, f also induces an isomorphism on π_n for $n \geq 2$, and is hence an equivalence. □

Exercise 5. Let $F \to E \to B$ be a fibre sequence with B hypoabelian. Show that $F^+ \to E^+ \to B$ is again a fibre sequence.

Solution. Since B is hypoabelian, the map $E \to B$ factors as $E \to E^+ \to B$. Let $F' = \mathrm{fib}(E^+ \to B)$. Then we have a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \| \\ F' & \longrightarrow & E^+ & \longrightarrow & B \end{array}$$

The left square is a pullback square, since the map on horizontal fibres is the identity of ΩB. It follows that $F \to F'$ is acyclic (since its a fibre agrees with the fibre of the map $E \to E^+$ which is homologically trivial since $E \to E^+$ is acyclic) and in particular induces a bijection on π_0. Moreover, $\pi_1(F) \to \pi_1(F')$ is surjective with perfect kernel K; in particular, this kernel is contained in the maximal perfect subgroup P of $\pi_1(F)$. We now show that P is contained in K. To that end, consider the diagram of exact sequences

$$\begin{array}{ccccc} \pi_2(B) & \longrightarrow & \pi_1(F) & \xrightarrow{i} & \pi_1(E) \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ \pi_2(B) & \longrightarrow & \pi_1(F') & \xrightarrow{j} & \pi_1(E^+) \end{array}$$

Now $i(P) \subseteq \pi_1(E)$ is perfect, so $j(f(P))$ is trivial. It follows that $f(P)$ is contained in the image of $\pi_2(B) \to \pi_1(F')$ which is abelian. At the same time $f(P)$ is perfect, so it is trivial, showing that $P \subseteq K$ and hence $P = K$. We deduce that F' is hypoabelian and that the induced map $F^+ \to F'$ is acyclic and a π_1-isomorphism, so the result follows from Exercise 4. □

This sheet will be discussed on 12 June 2025.