

LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

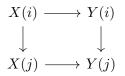


Summer term 2025

Algebraic *K*-theory

Sheet 4

Exercise 1. In this exercise, you may use the following fact about colimits in anima. Let I be a small ∞ -category and and let $\bar{\tau} \colon \bar{X} \to \bar{Y}$ a natural transformation of functors $I^{\triangleright} \to An$. Suppose \bar{Y} is a colimit cone and that $\tau = \bar{\tau}_{|I} \colon X \to Y$ is a cartesian transformation of functors $I \to An$, where $X = \bar{X}_{|I}$ and $Y = \bar{Y}_{|I}$. That is, for all morphisms $i \to j$ in I, the square



is cartesian. Then \bar{X} is a colimit cone if and only if $\bar{\tau} \colon \bar{X} \to \bar{Y}$ is a cartesian transformation.

Now, show the following result (called Rezk's equifibrancy criterion). Given a pullback diagram of functors $I \to \mathrm{An}$

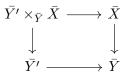


where τ is a cartesian transformation. Then the square of colimits

$$\begin{array}{ccc} \operatorname{colim}_{I} X' & \longrightarrow & \operatorname{colim}_{I} X \\ & & & \downarrow \\ & & & \downarrow \\ \operatorname{colim}_{I} Y' & \longrightarrow & \operatorname{colim}_{I} Y \end{array}$$

is again a pullback diagram.

Solution. Let \bar{Y}', \bar{Y} , and \bar{X} be colimit cones over Y', Y, and X respectively. Consider the pullback diagram of functors $I^{\triangleright} \to An$

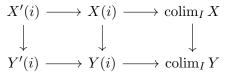


whose restriction to I is given by the diagram of the statement of the exercise (since that diagram is a pullback). By the general result on colimits explained above, it suffices to show that the left vertical map is a cartesian transformation. Note that τ' , as a pullback of a cartesian transformation, is again a cartesian transformation. Since the restriction to I is τ' it remains to show that for each $i \in I$, the left square in the diagram

$$\begin{array}{cccc} X'(i) & \longrightarrow \operatorname{colim}_{I} Y' \times_{\operatorname{colim}_{I} Y} \operatorname{colim}_{I} X & \longrightarrow \operatorname{colim}_{I} X \\ & & \downarrow & & \downarrow \\ Y'(i) & \longrightarrow \operatorname{colim}_{I} Y' & \longrightarrow \operatorname{colim}_{i} Y \end{array}$$

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is a pullback. Since the right square is one, it suffices to show that the big square is. This square can also be factored as



whose left square is a pullback by assumption and the right square is a pullback since $\bar{X} \to \bar{Y}$ is a cartesian transformation since both \bar{X} and \bar{Y} are colimit cones.

Exercise 2. Let X be a Segal anima. Show that its décalage dec(X) is again a Segal anima and participates in a pullback diagram

$$const(X_1) \longrightarrow dec(X)$$

$$\downarrow \qquad \qquad \downarrow$$

$$const(X_0) \longrightarrow X$$

as in the lecture.

Solution. That there is a commutative diagram of the claimed shape is immediate. To see that it is a pullback we may argue levelwise, so consider $n \ge 0$. Then there is a commutative diagram

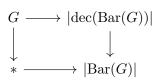
$$\begin{array}{cccc} X_1 & \longrightarrow & X_{1+n} & \xrightarrow{\simeq} & X_1 \times_{X_0} X_n & \longrightarrow & X_1 \\ & \downarrow^{d_0} & \downarrow^{d_0} & \downarrow & & \downarrow^{d_0} \\ X_0 & \longrightarrow & X_n & \xrightarrow{\qquad} & X_n & \longrightarrow & X_0 \end{array}$$

where the left and right squares are induced by the commutative squares in Δ (together with the Segal property of X):

$$\begin{array}{cccc} [n] & \longrightarrow & [0] & & & [0] & \stackrel{0}{\longrightarrow} & [n] \\ \downarrow d_0 & \downarrow 1 & & 1 \downarrow & \downarrow d_0 \\ [1+n] & \longrightarrow & [1] & & & [1] \xrightarrow[(0<1)]{} & [1+n] \end{array}$$

where the map $[1+n] \rightarrow [1]$ sends 0 to 0 and all other elements to 1. Now in the upper large diagram, right most square is a pullback square and the big square is a pullback because both horizontal composites are the identity. Hence, the left most square is a pullback as well.

Exercise 3. In this exercise you may use that $|\operatorname{dec}(X)| \simeq X_0$ for any simplicial anima X. Let $G \in \operatorname{Grp}(\operatorname{An})$ be a group in anima. Show that the square



is a pullback and deduce that $G \simeq \Omega |\text{Bar}(G)|$, where $|-| = \text{colim}_{\Delta^{\text{op}}}$. Why does the proof not apply in case M is a monoid rather than a group?

Solution. By Exercise 2, the diagram

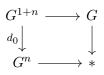
$$const(G) \longrightarrow dec(Bar(G))$$

$$\downarrow \qquad \qquad \downarrow$$

$$const(*) \longrightarrow Bar(G)$$

is a pullback diagram. Now, since Δ^{op} is contractible (it has an initial object), we have $\operatorname{colim}_{\Delta^{\text{op}}} \operatorname{const}(G) \simeq G$ and $\operatorname{colim}_{\Delta^{\text{op}}} * \simeq *$. Hence the statement follows from Exercise 1 as soon as we show that the map $\operatorname{dec}(\operatorname{Bar}(G)) \to \operatorname{Bar}(G)$ is a cartesian transformation. So let $[m] \to [n]$ be a map in Δ . Then we need to show that the left square in the diagram

is a pullback; here the right map is induced by a map $[0] \to [m]$. By pasting pullbacks, it therefore suffices to treat the case of the maps $i: [0] \to [n]$ for $0 \le i \le n$. In that case, the diagram becomes



where the top horizontal map is induced by $[1] \rightarrow [1+n]$ sending 0 to 0 and 1 to 1+i. For i = 0, this is true, as appeared in Exercise 2. However, for i > 0, this is not a priori the case, and it is exactly at this point where the proof uses that G is a group rather than a monoid. Indeed, for i > 0, the map $G^{1+n} \rightarrow G$ is given by multiplying together the first *i* product factors. But since G is a group, there is a self-equivalence $G^{1+n} \rightarrow G^{1+n}$ which commutes with the projection onto the last *n*-factors and which translates the multiplication of the first *i* factors map to the projection onto the first factor (for i = 1 this is precisely the shear map witnessing that G is a group, rather than a mere monoid). It follows that, in case G is a group, the diagram under investigation is equivalent to a pullback diagram, and hence a pullback diagram as desired.

Exercise 4. Let $M \in \text{CMon}(An)$ be a commutative monoid. Show that Bar(M) is indeed left Kan extended from both its restrictions to $\Delta_{<1}^{\text{op}}$ and $\Delta_{<1,\text{inj}}^{\text{op}}$.

Solution. Denote by \overline{M} the restriction of M to $\Delta_{\leq 1, \text{inj}}^{\text{op}}$ and by \widehat{M} the left Kan extension of \overline{M} along the inclusion $\Delta_{\leq 1, \text{inj}}^{\text{op}} \subseteq \Delta^{\text{op}}$. By the pointwise formula for left Kan extensions, for $[n] \in \Delta^{\text{op}}$, we have to study the category $(\Delta_{\leq 1, \text{inj}}^{\text{op}})/[n] \simeq [(\Delta_{\leq 1, \text{inj}})_{[n]/}]^{\text{op}}$. Now $(\Delta_{\leq 1, \text{inj}})_{[n]/}$ is given by the category whose objects are all simplicial maps $[n] \to [1]$ and the unique map $[n] \to [0]$. Now, there are two constant maps $[n] \Rightarrow [1]$, given by the composite $[n] \to [0] \Rightarrow [1]$. The non-constant maps $[n] \to [1]$ are precisely what we called Dedekind cuts in the lecture, and are in bijection to $\langle n \rangle = \{1, \ldots, n\}$ where $j \in \langle n \rangle$ corresponds to the map $[n] \rightarrow [1]$ sending j-1 to 0 and j to 1. We find that $(\Delta_{\leq 1, inj})_{[n]/}$ is isomorphic to the disjoint union of the category classifying pushouts with the discrete category $\{1, \ldots, n\}$. Its opposite is then given by the disjoint union of the category classifying pullbacks and the discrete category $\{1, \ldots, n\}$. In particular, we find that

$$\hat{M}_n = \operatorname{colim}_{(\Delta_{\leq 1, \operatorname{inj}}^{\operatorname{op}})/[n]} \bar{M} = M_0 \amalg \coprod_{j=1}^n M_1.$$

Now in CMon(An), the coproduct is simply the product, and $M_0 = *$, so that we find

$$\hat{M}_n = * \times M_1^{\times n} = M_1^{\times n}.$$

We now argue that \hat{M} is a Segal object. To that end consider for $1 \leq i \leq n$ the Segal map $\rho_i: [1] \rightarrow [n]$ classifying the morphism $(i - 1 \leq i)$ in [n]. This induces the following functor $(\Delta_{\leq 1, inj})_{[n]/} \rightarrow (\Delta_{\leq 1, inj})_{[1]/}$: It is an isomorphism on the pushout-classifying component of both categories. To see what happens to the objects of the discrete category $\{1, \ldots, n\}$, pick $1 \leq j \leq n$. Then we need to study the composite

$$[1] \xrightarrow{\rho_i} [n] \xrightarrow{\jmath} [1]$$

where the latter map, as indicated above, sends j-1 to 0 and j to 1. If $i \leq j-1$, then this composite is constant at 0 and if $i-1 \geq j$ then it is constant at 1, and if i = j it is the identity. This shows that the functor under consideration sends $j \in \{1, \ldots, n\}$ to one of the objects of the pushout-classifying category if $j \neq i$, and sends i to the unique object of $\{1\}$ not in the pushout-classifying category. On colimits, we deduce that the Segal map ρ_i induces the map $M^n \to M$ which is the projection to the *i*th product factor. In particular, \hat{M} is a Segal object. Now the counit of the restriction-Kan extension adjunction provides a map $\hat{M} \to M$. This map induces on 0- and 1-simplices the maps

$$* \to *$$
 and $* \times M_1 \to M_1$

where the first map is the identity, and the second map is the degeneracy on * and the identity on M_1 . Using that $* \times M_1 \simeq M_1$ via the projection, we find that $\hat{M} \to M$ is an equivalence on 0- and 1-simplices. Since both \hat{M} and M are Segal objects, it then follows that the map $\hat{M} \to M$ is an equivalence, showing that M is left Kan extended from its restriction to $\Delta_{\leq 1, \text{inj}}^{\text{op}}$. To see that it is also left Kan extended from its restriction to $\Delta_{\leq 1}^{\text{op}}$ is equivalent to the restriction to $\Delta_{\leq 1}^{\text{op}}$. But this is a formal consequence of the fact that $\Delta_{\leq 1}^{\text{op}} \subseteq \Delta^{\text{op}}$ is a full subcategory: Indeed, this says that left Kan extending from $\Delta_{\leq 1}^{\text{op}}$ to $\Delta_{\leq 1}^{\text{op}}$, and then restricting back again is the identity. Since composites of left Kan extensions are the is the left Kan extension of the composite, the claim follows.

This sheet will be discussed on 5 June 2025.