LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Algebraic K-theory

Sheet 4

Exercise 1. In this exercise, you may use the following fact about colimits in anima. Let I be a small ∞-category and and let $\bar{\tau}\colon \bar{X} \to \bar{Y}$ a natural transformation of functors $I^{\triangleright} \to \mathrm{An}$. Suppose $\bar{Y}$ is a colimit cone and that $\tau = \bar{\tau}_{|I}\colon X \to Y$ is a cartesian transformation of functors $I \to \mathrm{An}$, where $X = \bar{X}_{|I}$ and $Y = \bar{Y}_{|I}$. That is, for all morphisms $i \to j$ in I, the square

$$\begin{array}{ccc} X(i) & \longrightarrow & Y(i) \\ \downarrow & & \downarrow \\ X(j) & \longrightarrow & Y(j) \end{array}$$

is cartesian. Then $\bar{X}$ is a colimit cone if and only if $\bar{\tau}\colon \bar{X} \to \bar{Y}$ is a cartesian transformation.

Now, show the following result (called Rezk's equifibrancy criterion). Given a pullback diagram of functors $I \to \mathrm{An}$

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle \tau} \\ Y' & \longrightarrow & Y \end{array}$$

where τ is a cartesian transformation. Then the square of colimits

$$\begin{array}{ccc} \mathrm{colim}_I X' & \longrightarrow & \mathrm{colim}_I X \\ \downarrow & & \downarrow \\ \mathrm{colim}_I Y' & \longrightarrow & \mathrm{colim}_I Y \end{array}$$

is again a pullback diagram.

Solution. Let $\bar{Y'}, \bar{Y}$, and $\bar{X}$ be colimit cones over Y', Y, and X respectively. Consider the pullback diagram of functors $I^{\triangleright} \to \mathrm{An}$

$$\begin{array}{ccc} \bar{Y'} \times_{\bar{Y}} \bar{X} & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ \bar{Y'} & \longrightarrow & \bar{Y} \end{array}$$

whose restriction to I is given by the diagram of the statement of the exercise (since that diagram is a pullback). By the general result on colimits explained above, it suffices to show that the left vertical map is a cartesian transformation. Note that τ', as a pullback of a cartesian transformation, is again a cartesian transformation. Since the restriction to I is τ' it remains to show that for each $i \in I$, the left square in the diagram

$$\begin{array}{ccccc} X'(i) & \longrightarrow & \mathrm{colim}_I Y' \times_{\mathrm{colim}_I Y} \mathrm{colim}_I X & \longrightarrow & \mathrm{colim}_I X \\ \downarrow & & \downarrow & & \downarrow \\ Y'(i) & \longrightarrow & \mathrm{colim}_I Y' & \longrightarrow & \mathrm{colim}_i Y \end{array}$$

is a pullback. Since the right square is one, it suffices to show that the big square is. This square can also be factored as

$$\begin{array}{ccccc} X'(i) & \longrightarrow & X(i) & \longrightarrow & \operatorname{colim}_I X \\ \downarrow & & \downarrow & & \downarrow \\ Y'(i) & \longrightarrow & Y(i) & \longrightarrow & \operatorname{colim}_I Y \end{array}$$

whose left square is a pullback by assumption and the right square is a pullback since $\bar{X} \to \bar{Y}$ is a cartesian transformation since both $\bar{X}$ and $\bar{Y}$ are colimit cones. $\square$

Exercise 2. Let X be a Segal anima. Show that its décalage $\operatorname{dec}(X)$ is again a Segal anima and participates in a pullback diagram

$$\begin{array}{ccc} \operatorname{const}(X_1) & \longrightarrow & \operatorname{dec}(X) \\ \downarrow & & \downarrow \\ \operatorname{const}(X_0) & \longrightarrow & X \end{array}$$

as in the lecture.

Solution. That there is a commutative diagram of the claimed shape is immediate. To see that it is a pullback we may argue levelwise, so consider $n \geq 0$. Then there is a commutative diagram

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_{1+n} & \xrightarrow{\simeq} & X_1 \times_{X_0} X_n & \longrightarrow & X_1 \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow & & \downarrow{\scriptstyle d_0} \\ X_0 & \longrightarrow & X_n & = & X_n & \longrightarrow & X_0 \end{array}$$

where the left and right squares are induced by the commutative squares in Δ (together with the Segal property of X):

$$\begin{array}{ccc} [n] & \longrightarrow & [0] \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle 1} \\ [1+n] & \longrightarrow & [1] \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{0} & [n] \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle d_0} \\ [1] & \xrightarrow[(0\leq 1)]{} & [1+n] \end{array}$$

where the map $[1+n] \to [1]$ sends 0 to 0 and all other elements to 1. Now in the upper large diagram, right most square is a pullback square and the big square is a pullback because both horizontal composites are the identity. Hence, the left most square is a pullback as well. $\square$

Exercise 3. In this exercise you may use that $|\operatorname{dec}(X)| \simeq X_0$ for any simplicial anima X. Let $G \in \operatorname{Grp}(\mathrm{An})$ be a group in anima. Show that the square

$$\begin{array}{ccc} G & \longrightarrow & |\operatorname{dec}(\operatorname{Bar}(G))| \\ \downarrow & & \downarrow \\ * & \longrightarrow & |\operatorname{Bar}(G)| \end{array}$$

is a pullback and deduce that $G \simeq \Omega|\mathrm{Bar}(G)|$, where $|-| = \mathrm{colim}_{\Delta^{\mathrm{op}}}$. Why does the proof not apply in case M is a monoid rather than a group?

Solution. By Exercise 2, the diagram

$$\begin{array}{ccc} \mathrm{const}(G) & \longrightarrow & \mathrm{dec}(\mathrm{Bar}(G)) \\ \downarrow & & \downarrow \\ \mathrm{const}(*) & \longrightarrow & \mathrm{Bar}(G) \end{array}$$

is a pullback diagram. Now, since Δ^{op} is contractible (it has an initial object), we have $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{const}(G) \simeq G$ and $\mathrm{colim}_{\Delta^{\mathrm{op}}} * \simeq *$. Hence the statement follows from Exercise 1 as soon as we show that the map $\mathrm{dec}(\mathrm{Bar}(G)) \to \mathrm{Bar}(G)$ is a cartesian transformation. So let $[m] \to [n]$ be a map in Δ. Then we need to show that the left square in the diagram

$$\begin{array}{ccccc} \mathrm{Bar}(G)_{1+n} & \longrightarrow & \mathrm{Bar}(G)_{1+m} & \longrightarrow & \mathrm{Bar}(G)_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bar}(G)_n & \longrightarrow & \mathrm{Bar}(G)_m & \longrightarrow & \mathrm{Bar}(G)_0 \end{array}$$

is a pullback; here the right map is induced by a map $[0] \to [m]$. By pasting pullbacks, it therefore suffices to treat the case of the maps $i\colon [0] \to [n]$ for $0 \le i \le n$. In that case, the diagram becomes

$$\begin{array}{ccc} G^{1+n} & \longrightarrow & G \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ G^n & \longrightarrow & * \end{array}$$

where the top horizontal map is induced by $[1] \to [1+n]$ sending 0 to 0 and 1 to $1+i$. For $i = 0$, this is true, as appeared in Exercise 2. However, for $i > 0$, this is not a priori the case, and it is exactly at this point where the proof uses that G is a group rather than a monoid. Indeed, for $i > 0$, the map $G^{1+n} \to G$ is given by multiplying together the first i product factors. But since G is a group, there is a self-equivalence $G^{1+n} \to G^{1+n}$ which commutes with the projection onto the last n-factors and which translates the multiplication of the first i factors map to the projection onto the first factor (for $i = 1$ this is precisely the shear map witnessing that G is a group, rather than a mere monoid). It follows that, in case G is a group, the diagram under investigation is equivalent to a pullback diagram, and hence a pullback diagram as desired. □

Exercise 4. Let $M \in \mathrm{CMon}(\mathrm{An})$ be a commutative monoid. Show that $\mathrm{Bar}(M)$ is indeed left Kan extended from both its restrictions to $\Delta^{\mathrm{op}}_{\le 1}$ and $\Delta^{\mathrm{op}}_{\le 1,\mathrm{inj}}$.

Solution. Denote by $\bar{M}$ the restriction of M to $\Delta^{\mathrm{op}}_{\le 1,\mathrm{inj}}$ and by $\hat{M}$ the left Kan extension of $\bar{M}$ along the inclusion $\Delta^{\mathrm{op}}_{\le 1,\mathrm{inj}} \subseteq \Delta^{\mathrm{op}}$. By the pointwise formula for left Kan extensions, for $[n] \in \Delta^{\mathrm{op}}$, we have to study the category $(\Delta^{\mathrm{op}}_{\le 1,\mathrm{inj}})_{/[n]} \simeq [(\Delta_{\le 1,\mathrm{inj}})_{[n]/}]^{\mathrm{op}}$. Now $(\Delta_{\le 1,\mathrm{inj}})_{[n]/}$ is given by the category whose objects are all simplicial maps $[n] \to [1]$ and the unique map $[n] \to [0]$. Now, there are two constant maps $[n] \rightrightarrows [1]$, given by the composite $[n] \to [0] \rightrightarrows [1]$. The non-constant maps $[n] \to [1]$ are precisely what we called Dedekind cuts in the lecture, and are in bijection to $\langle n\rangle = \{1,\dots,n\}$ where $j \in \langle n\rangle$ corresponds to the map $[n]\to[1]$ sending $j-1$ to 0 and j to 1. We find that $(\Delta_{\leq 1,\mathrm{inj}})_{[n]/}$ is isomorphic to the disjoint union of the category classifying pushouts with the discrete category $\{1,\dots,n\}$. Its opposite is then given by the disjoint union of the category classifying pullbacks and the discrete category $\{1,\dots,n\}$. In particular, we find that

$$\hat{M}_n = \operatorname*{colim}_{(\Delta^{\mathrm{op}}_{\leq 1,\mathrm{inj}})_{/[n]}} \bar{M} = M_0 \amalg \coprod_{j=1}^{n} M_1.$$

Now in CMon(An), the coproduct is simply the product, and $M_0 = *$, so that we find

$$\hat{M}_n = * \times M_1^{\times n} = M_1^{\times n}.$$

We now argue that $\hat{M}$ is a Segal object. To that end consider for $1\leq i\leq n$ the Segal map $\rho_i\colon [1]\to[n]$ classifying the morphism $(i-1\leq i)$ in $[n]$. This induces the following functor $(\Delta_{\leq 1,\mathrm{inj}})_{[n]/}\to(\Delta_{\leq 1,\mathrm{inj}})_{[1]/}$: It is an isomorphism on the pushout-classifying component of both categories. To see what happens to the objects of the discrete category $\{1,\dots,n\}$, pick $1\leq j\leq n$. Then we need to study the composite

$$[1] \xrightarrow{\rho_i} [n] \xrightarrow{j} [1]$$

where the latter map, as indicated above, sends $j-1$ to 0 and j to 1. If $i\leq j-1$, then this composite is constant at 0 and if $i-1\geq j$ then it is constant at 1, and if $i=j$ it is the identity. This shows that the functor under consideration sends $j\in\{1,\dots,n\}$ to one of the objects of the pushout-classifying category if $j\neq i$, and sends i to the unique object of $\{1\}$ not in the pushout-classifying category. On colimits, we deduce that the Segal map ρ_i induces the map $M^n\to M$ which is the projection to the ith product factor. In particular, $\hat{M}$ is a Segal object. Now the counit of the restriction-Kan extension adjunction provides a map $\hat{M}\to M$. This map induces on 0- and 1-simplices the maps

$$*\to* \quad\text{ and }\quad *\times M_1\to M_1$$

where the first map is the identity, and the second map is the degeneracy on $*$ and the identity on M_1. Using that $*\times M_1\simeq M_1$ via the projection, we find that $\hat{M}\to M$ is an equivalence on 0- and 1-simplices. Since both $\hat{M}$ and M are Segal objects, it then follows that the map $\hat{M}\to M$ is an equivalence, showing that M is left Kan extended from its restriction to $\Delta^{\mathrm{op}}_{\leq 1,\mathrm{inj}}$. To see that it is also left Kan extended from its restriction to $\Delta^{\mathrm{op}}_{\leq 1}$, it remains to argue that the left Kan extension of $\bar{M}$ to $\Delta^{\mathrm{op}}_{\leq 1}$ is equivalent to the restriction of M to $\Delta^{\mathrm{op}}_{\leq 1}$. But this is a formal consequence of the fact that $\Delta^{\mathrm{op}}_{\leq 1}\subseteq\Delta^{\mathrm{op}}$ is a full subcategory: Indeed, this says that left Kan extending from $\Delta^{\mathrm{op}}_{\leq 1}$ to Δ^{op}, and then restricting back again is the identity. Since composites of left Kan extensions are the is the left Kan extension of the composite, the claim follows. □

This sheet will be discussed on 5 June 2025.