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# Algebraic $K$-theory

Sheet 3

**Exercise 1.** Show that the following condition on a Segal anima $X$ are equivalent.

(1) $X$ is complete,

(2) $X^\times$ is a constant simplicial anima,

(3) the diagram

$$\begin{tikzcd} X_0 \arrow[r, "s"] \arrow[d, "\Delta"] & X_3 \arrow[d, "f\times g"] \\ X_0 \times X_0 \arrow[r, "s\times s"] & X_1 \times X_1 \end{tikzcd}$$

is a pullback, where $f$ and $g$ are induced by the two maps $[1] \to [3]$ in $\Delta$ given by $0 \mapsto 0$, $1 \mapsto 2$ and $0 \mapsto 1$, $1 \mapsto 3$, respectively.

(4) The map $X_0 \to \mathrm{Map}_{\mathrm{sAn}}(J, X)$ induced by $J \to *$ is an equivalence. Here, $J$ denotes the nerve of the contractible groupoid with two elements.

*Solution.* (1)⇔(2): Since $\Delta^{\mathrm{op}}$ is contractible, $X^\times$ being constant is equivalent to all simplicial structure maps being equivalences. Since $X^\times$ is Segal, this is implied by the degeneracy $X_0^\times \to X_1^\times$ being an equivalence which is the definition of $X$ being complete.

In general, we claim that there is a pullback diagram

$$\begin{tikzcd} X_1^\times \arrow[r] \arrow[d] & X_3^\times \arrow[r] & X_3 \arrow[d] \\ X_0 \times X_0 \arrow[rr] & & X_1 \times X_1 \end{tikzcd}$$

where the left upper horizontal map is induced by $[3] \to [1]$ given by the morphism $0 \le 0 \le 1 \le 1$ and the left vertical map is the source and target map. Now note that the diagram appearing in condition (3) is obtained from the just described pullback upon precomposing with the degeneracy $X_0 \to X_1^\times$. Hence (1)⇔(3). Finally, for (1)⇔(4), we need to argue that the canonical $\mathrm{Map}_{\mathrm{sAn}}(J, X) \to X_1$ induces an equivalence $\mathrm{Map}_{\mathrm{sAn}}(J, X) \to X_1^\times$. The argument is a bit more involved than I initially thought (and due to Charles Rezk), I'll add the details later. □

**Exercise 2.** Let $\mathcal{C}$ be an $\infty$-category. Show that its Rezk nerve $\mathrm{N}(\mathcal{C})$ is a complete Segal anima.

*Solution.* Unravelling the definitions, $\mathrm{N}(\mathcal{C})$ is Segal if and only if for all $n \ge 0$, the spine inclusion $I^n \to \Delta^n$ induces an equivalence $\iota\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \iota\mathrm{Fun}(I^n, \mathcal{C})$. This is true even before passing to groupoid cores as $I^n \to \Delta^n$ is inner anodyne and hence a categorical (aka Joyal) equivalence. To see that $\mathrm{N}(\mathcal{C})$ is complete, we note that $\mathrm{N}(\mathcal{C})_1^{\times}$ is really the groupoid of equivalences in $\mathcal{C}$, that is the full subcategory of $\iota\mathrm{Fun}(\Delta^1, \mathcal{C})$ on those functors $\Delta^1 \to \mathcal{C}$ representing an equivalence in $\mathcal{C}$. Hence, this is simply $\iota\mathrm{Fun}(\Delta^1, \iota\mathcal{C}) \simeq \iota\mathcal{C}$ since $\Delta^1$ is contractible, so $\mathrm{N}(\mathcal{C})$ is indeed complete. $\square$

**Exercise 3.** Let $M \in \mathrm{Mon}(\mathrm{An})$. Characterize when its underlying simplicial anima $\mathrm{Bar}(M)$ is complete.

*Solution.* Unravelling the definitions, we have $\mathrm{Bar}(M)_1^{\times}$ is the maximal subgroup in $M$, that is we a pullback diagram

$$\begin{tikzcd}
\mathrm{Bar}(M)_1^{\times} \arrow[r] \arrow[d] & M \arrow[d] \\
\pi_0(M)^{\times} \arrow[r] & \pi_0(M)
\end{tikzcd}$$

where $\pi_0(M)^{\times}$ denotes the invertible elements in the ordinary monoid $\pi_0(M)$. Since $\mathrm{Bar}(M)_0 = *$, we see that $M$ is complete if and only if the above pullback is contractible. This is the case if and only if $\pi_0(M)^{\times} = \{1\}$ and the component of the unit element in $M$ is contractible. In particular, if $M \in \mathrm{Grp}(\mathrm{An})$, then $\mathrm{Bar}(M)$ is complete if and only if $M$ is trivial. $\square$

**Exercise 4.** Let $\mathcal{C}$ be an $\infty$-category with small limits. Show that $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{Mon}(\mathcal{C})$ again have small limits and that the forgetful functors

$$\mathrm{CMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$$

preserve small limits.

*Solution.* We show that the subcategories

$$\mathrm{CMon}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}_p, \mathcal{C}) \quad \text{and} \quad \mathrm{Mon}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

are closed under limits. This implies both claims, since the forgetful functors are the composite of the above functors with an evaluation functor which also preserves limits.

To see this, suppose $I$ is a small category and consider a diagram $F\colon I \to \mathrm{CMon}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}_p, \mathcal{C})$. Then $\lim_{i\in I} F_i(\emptyset) = \lim_{i \in I} * = *$ since a limit of terminal objects is terminal. Likewise, we need to show that for all finite sets $S$, the Segal map $\lim_{i\in I} F_i(S) \to \prod_{s\in S} \lim_{i\in I} F_i(s)$ is an equivalence. Since limits commute, this map is equivalent to the map

$$\lim_{i\in I} F_i(S) \to \lim_{i\in I} \prod_{s\in S} F_i(s)$$

which is a limit of equivalences (as all $F_i$ are Segal objects), and hence an equivalence.

The same argument applies to $\mathrm{Mon}(\mathcal{C})$ rather than $\mathrm{CMon}(\mathcal{C})$ upon replacing $\mathrm{Fin}_p$ with $\Delta^{\mathrm{op}}$. $\square$

This sheet will be discussed on May 29 2025.