Summer term 2025 22. Mai 2025

Algebraic K-theory

Sheet 2

Exercise 1. Let R be a ring. Show that the elementary matrices $E_{i,j}(r) \in \mathrm{GL}(R)$ satisfy the following relations:

(1) $E_{i,j}(r)E_{i,j}(r') = E_{i,j}(r + r')$,

(2) $[E_{i,j}(r), E_{j,k}(r')] = E_{i,k}(rr')$, if $i \neq k$ and

(3) $[E_{i,j}(r), E_{k,l}(r')] = 1$ if $i \neq l$ and $j \neq k$.

Solution. Recall that for matrices $A, B \in M_n(R)$ and $1 \leq k, l \leq n$, we have

$$(A \cdot B)_{k,l} = \sum_{s=1}^{n} A_{k,s} \cdot B_{s,l}.$$

Hence we obtain

$$(A \cdot E_{i,j}(r))_{k,l} = \sum_{s=1}^{n} A_{k,s} \cdot E_{i,j}(r)_{s,l} = A_{k,l} + \delta_{j,l} r A_{k,i}.$$

Hence, we find $A \cdot E_{i,j}(r)$ is the matrix obtained from A by adding r times the i-th column to the jth column, similarly for $E_{i,j}(r) \cdot A$. From here, the exercise is to multiply matrices. Perhaps I'll add the actual computations at some point. □

Exercise 2. Let R be a ring. Show that the center $C(E(R))$ of the group of elementary matrices is trivial. Hint: Show that if $A \in \mathrm{GL}_n(R)$ commutes with all elements of $E_n(R)$, then A is a homothetie, i.e. a diagonal matrix $(r, \dots, r)$ with $r \in C(R)$. Deduce that no element of $E_{n-1}(R)$ lies in the center of $E_n(R)$ and let n go to infinity.

Solution. Note that $E_{i,j}(r) = 1 + V_{i,j}(r)$ where $V_{i,j}(r)$ is the matrix with 0s everywhere except at spot (i, j) where the entry is r. Hence, if A commutes with $E_{i,j}(1)$ it also commutes with $V_{i,j}(1)$. But $AV_{i,j}(1)$ is the matrix obtained from A by putting the jth column in the $i$the column and setting all other columns to zero. Likewise, $V_{i,j}(1)A$ is obtained from A by putting the ith row in the jth row and setting all other rows to zero. For these two matrices to be equal (for all $i \neq j$) there can only be terms on the diagonal and all these terms have to be equal. We conclude that if $A \in \mathrm{GL}_n(R)$ commutes with $E_n(R)$, then A is a diagonal matrix with common terms on the diagonal. Now consider $A \in E_{n-1}(R)$ and view $A \oplus 1 \in E_n(R)$. If this matrix commutes with all elements of $E_n(R)$, then A is diagonal with common term on the diagonal. But the lowest diagonal term of A is a 1, so this implies that A is the identity matrix. Finally, every element in $E(R)$ is contained in $E_{n-1}(R)$ for some n, and if it commutes with all elements of $E(R)$ it in particular commutes with all elements of $E_n(R)$. The result follows. □

Exercise 3. Let R be a ring and $C \in \mathrm{GL}_n(R)$. Show that the following matrices are elementary:

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -C^{-1} & 1 \end{pmatrix}$$

Solution. For $1 \leq i, j \leq n$, consider the matrices $V_{i,j}(r)$ which have r at spot (i, j) and 0 in every other spot. Consider then the matrices $S_{i,j}(r)$

$$\begin{pmatrix} 1 & V_{i,j}(r) \\ 0 & 1 \end{pmatrix}$$

and note that they pairwise commute and are contained in $E_{2n}(R)$. Taking the product over all $1 \leq i, j \leq n$ of the matrices $S_{i,j}(C_{i,j})$ gives the matrix

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$$

The argument for the second matrix is analogous. □

Exercise 4. Show that the map $R^{n-1} \to \mathrm{St}(R)$ given by

$$(r_1, \dots, r_{n-1}) \mapsto e_{1,n}(r_1) \cdot e_{2,n}(r_2) \cdots e_{n-1,n}(r_{n-1})$$

is an injective group homomorphism.

Solution. We claim that the lements $\{e_{i,n}(r_i)\}_{i=1,\dots,n-1}$ pairwise commute. Indeed, we want to apply the defining relation (3), and so need that $i \neq n$ and $j \neq n$ for $e_{i,n}(r)$ and $e_{j,n}(r')$ to commute. Since $i, j < n$, this is true. It then remains to note from defining relation (1) that $e_{i,n}(r+s) = e_{i,n}(r) \cdot e_{i,n}(s)$. We therefore obtain a canonical group homomorphism as stated. To see that it is injective, let us compose with the tautological map $\mathrm{St}(R) \to \mathrm{GL}(R)$. Then the product becomes the following matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & r_1 \\ 0 & 1 & \dots & 0 & r_2 \\ & & \dots & & \\ & & \dots & & \\ 0 & 0 & \dots & 1 & r_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

This shows that the composite

$$R^{n-1} \to \mathrm{St}(R) \to \mathrm{GL}(R)$$

is injective, so that in particular the first map is injective as claimed. □

Exercise 5. Use Matsumoto's theorem to show that $K_2(\mathbb{F}_q) = 0$ where $\mathbb{F}_q$ is a finite field.

Solution. Recall that Matsumoto's theorem says that

$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / \langle a \otimes 1 - a \mid a \in F^\times \setminus \{1\}\rangle.$$

Since the right hand side involves a tensor product over $\mathbb{Z}$, it is more convenient to write the group $F^\times$ additively. However, since $F^\times \subseteq F$ this is confusing, so we follow Milnor's suggestion to give a name to the isomorphism $F^\times \to K_1(F)$, say ℓ, and then write $K_2(F)$ as the quotient of $K_1(F) \otimes_{\mathbb{Z}} K_1(F)$ by the subgroup generated by $\ell(a) \otimes \ell(1-a)$ for $a \in F^\times \setminus \{1\}$. With this notation, we have $\ell(ab) = \ell(a) + \ell(b)$, as we write $K_1(F)$ additively. First, some general relations that follow in $K_2(F)$:

(1) $\ell(a) \otimes \ell(-a) = 0$,

(2) $\ell(a) \otimes \ell(a) = \ell(a) \otimes \ell(-1)$, and

(3) $\ell(a) \otimes \ell(b) = -\ell(b) \otimes \ell(a)$.

Indeed, to see (1), note the equality in $F^\times$ given by $-a = \frac{1-a}{1-a^{-1}}$. Hence we have

$$\ell(a) \otimes \ell(-a) = \ell(a) \otimes [\ell(1-a) - \ell(1-a^{-1})] = \ell(a) \otimes \ell(1-a) + \ell(a^{-1}) \otimes \ell(1-a^{-1}) = 0 + 0 = 0$$

giving (1). Then, since $a = (-1)(-a)$, we get

$$\ell(a) \otimes \ell(a) = \ell(a) \otimes [\ell(-1) + \ell(-a)] = \ell(a) \otimes \ell(-1) + \ell(a) \otimes \ell(-a) = \ell(a) \otimes \ell(-1)$$

showing (2). Moreover, using (1) three times, we obtain (3):

$$0 = \ell(ab) \otimes \ell(-ab) = \ell(a) \otimes [\ell(-a) + \ell(b)] + \ell(b) \otimes [\ell(a) + \ell(-b)] = \ell(a) \otimes \ell(b) + \ell(b) \otimes \ell(a)$$

Now for F a finite field, we have $F^\times$ is a cyclic group, say of order $q-1$ (i.e. $F = \mathbb{F}_q$ is a finite field with $q = p^n$ elements). Pick a generator ξ. Then any two units in F are given by ξ^n and ξ^m for some n, m. Then

$$\ell(\xi^n) \otimes \ell(\xi^m) = nm \cdot [\ell(\xi) \otimes \ell(\xi)].$$

But using (3) above, we find that $\ell(\xi) \otimes \ell(\xi)$ is of order 2. Thus $K_2(F)$ is generated by an element of order 2 and is hence either cyclic of order two or trivial. In fact, we also find $\ell(1) = \ell(\xi^{q-1}) = (q-1)\ell(\xi)$, showing that

$$(q-1)^2 \cdot \ell(\xi) \otimes \ell(\xi) = \ell(1) \otimes \ell(1)$$

which is the trivial element in $K_2(F)$ as we recall that $\ell(1) = 0 \in K_1(F)$. However, if q is even, then $(q-1)^2 \equiv 1 \mod 2$, so the fact that $\ell(\xi) \otimes \ell(\xi)$ has order two implies that it vanishes. It remains to argue the case where q is odd, in which, possibly $\ell(\xi) \otimes \ell(\xi)$ is a non-trivial element of order 2 in $K_2(F)$.

Now to treat the case where q is odd, we first claim that there are elements $u, v \in nS = F^\times \setminus (F^\times)^2$ such that $1 = u + v$. Indeed, consider the set nS and the set $1 - nS$. Both these sets have $(q-1)/2$ many elements and are subsets of $F \setminus \{0, 1\}$ which has $q-2$ many elements. Hence the intersection is non-trivial, showing that there is a non-square u for which $1 - u = v$ is also a non-square. It follows that $\ell(u) \otimes \ell(v) = \ell(u) \otimes \ell(1-u) = 0$. Now since $\xi \in F^\times$ is a generator, it is not a square. Moreover, the multiplication by squares acts transitively on the non-squares. This implies that there exists $a, b \in F^\times$

such that $a^2u = \xi = b^2v$. Furthermore, as we have already argued that $K_2(F)$ is 2-torsion, we know that for any unit a, we have $0 = 2 \cdot \ell(a) \otimes \ell(v) = \ell(a^2) \otimes \ell(v)$, and similarly, $0 = \ell(u) \otimes \ell(b^2)$. Consequently, we obtain

$$0 = \ell(u) \otimes \ell(v) = \ell(a^2u) \otimes \ell(v) = \ell(a^2u) \otimes \ell(b^2v) = \ell(\xi) \otimes \ell(\xi)$$

showing that the generator of $K_2(F)$ is trivial, and hence finally that $K_2(F) = 0$ as wanted. $\square$

This sheet will be discussed on 22 May 2023.