LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2025 July 10, 2025

Algebraic K-theory

Sheet 8

Exercise 1. Let $p\colon \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration such that for all morphisms $f\colon d \to d'$ in $\mathcal{D}$, the fibre transport functor $\mathcal{E}_d \to \mathcal{E}_{d'}$ is a weak equivalence, i.e. induces an equivalence $|\mathcal{E}_d| \xrightarrow{\simeq} |\mathcal{E}_{d'}|$. Show that p is a *realization fibration*, i.e. that for any functor $\mathcal{C} \to \mathcal{D}$, the square

$$\begin{array}{ccc} |\mathcal{E} \times_{\mathcal{D}} \mathcal{C}| & \longrightarrow & |\mathcal{E}| \\ \downarrow & & \downarrow \\ |\mathcal{C}| & \longrightarrow & |\mathcal{D}| \end{array}$$

of anima is a pullback. Deduce that bicartesian (that is cartesian and cocartesian) fibrations are realization fibrations.

Exercise 2. Let $\mathcal{A}$ be an abelian category, $S \in \mathcal{A}$ a simple object and $\mathcal{A}_{\{S\}}$ the full subcategory consisting of sums of S. Show that $\mathcal{A}_{\{S\}} \simeq \mathrm{Mod}(\mathrm{End}_{\mathcal{A}}(S))$.

Exercise 3. Show that the Serre quotient $\mathcal{A}/\mathcal{B}$ as indicated in the lecture is indeed an abelian category with the claimed universal property. Show that for R a coherent ring and $S \subseteq R$ a multiplicatively closed subset, $\mathrm{Mod}^{\mathrm{fp}}(R) \to \mathrm{Mod}^{\mathrm{fp}}(R[\frac{1}{S}])$ is indeed a Serre quotient map (necessarily by its kernel).

Exercise 4. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be a full exact subcategory of an exact ∞-category. Assume that $\mathcal{E}_0$ is *dense* in $\mathcal{E}$, i.e. that for all $x \in \mathcal{E}$ there is an $x' \in \mathcal{E}$ such that $x \oplus x' \in \mathcal{E}_0$. Show that $K_0(\mathcal{E}_0) \to K_0(\mathcal{E})$ is injective.

This sheet will be discussed on 17 July 2025.