

LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT



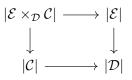
Summer term 2025

July 10, 2025

Algebraic *K*-theory

Sheet 8

Exercise 1. Let $p: \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration such that for all morphisms $f: d \to d'$ in \mathcal{D} , the fibre transport functor $\mathcal{E}_d \to \mathcal{E}_{d'}$ is a weak equivalence, i.e. induces an equivalence $|\mathcal{E}_d| \xrightarrow{\simeq} |\mathcal{E}_{d'}|$. Show that p is a realization fibration, i.e. that for any functor $\mathcal{C} \to \mathcal{D}$, the square



of anima is a pullback. Deduce that bicartesian (that is cartesian and cocartesian) fibrations are realization fibrations.

Exercise 2. Let \mathcal{A} be an abelian category, $S \in \mathcal{A}$ a simple object and $\mathcal{A}_{\{S\}}$ the full subcategory consisting of sums of S. Show that $\mathcal{A}_{\{S\}} \simeq \operatorname{Mod}(\operatorname{End}_{\mathcal{A}}(S)).$

Exercise 3. Show that the Serre quotient \mathcal{A}/\mathcal{B} as indicated in the lecture is indeed an abelian category with the claimed universal property. Show that for R a coherent ring and $S \subseteq R$ a multiplicatively closed subset, $\operatorname{Mod}^{\operatorname{fp}}(R) \to \operatorname{Mod}^{\operatorname{fp}}(R[\frac{1}{S}])$ is indeed a Serre quotient map (necessarily by its kernel).

Exercise 4. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be a full exact subcategory of an exact ∞ -category. Assume that \mathcal{E}_0 is *dense* in \mathcal{E} , i.e. that for all $x \in \mathcal{E}$ there is an $x' \in \mathcal{E}$ such that $x \oplus x' \in \mathcal{E}_0$. Show that $K_0(\mathcal{E}_0) \to K_0(\mathcal{E})$ is injective.

This sheet will be discussed on 17 July 2025.