Summer term 2025 3. Juli 2025

Algebraic K-theory

Sheet 7

Exercise 1. The goal of this exercise is to directly prove the following version of Quillen's Theorem A. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor and assume that for all $d \in \mathcal{D}$ the categories $\mathcal{C}_{d/} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ are contractible. Then show that

(1) $|f|\colon |\mathcal{C}| \to |\mathcal{D}|$ is an equivalence, and

(2) for all functors $F\colon \mathcal{D} \to \mathcal{E}$, the induced map $\operatorname{colim}_{\mathcal{C}} Ff \to \operatorname{colim}_{\mathcal{D}} F$ is an equivalence.

Exercise 2. Given a functor $F\colon \Delta^{\mathrm{op}} \to \mathcal{E}$ for some cocomplete category $\mathcal{E}$, show that the canonical map $\operatorname{colim}_{\Delta^{\mathrm{op}}_{\mathrm{inj}}} F' \to \operatorname{colim}_{\Delta^{\mathrm{op}}} F$ is an equivalence, where $F' = F_{|\Delta^{\mathrm{op}}_{\mathrm{inj}}}$.

Exercise 3. Let $\mathcal{C}$ be a non-empty ∞-category which admits binary products. Show that $\mathcal{C}$ is contractible, that is, $|\mathcal{C}| \simeq *$.

Exercise 4. Let $\mathcal{E}$ be an exact ∞-category and $\mathcal{E}^{\mathrm{op}}$ its opposite exact ∞-category. Show that there is a canonical equivalence $K(\mathcal{E}) \simeq K(\mathcal{E}^{\mathrm{op}})$.

Exercise 5. Let $\mathcal{E}\colon I \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ be a filtered diagram of exact ∞-categories and exact functors. Show that $\operatorname{colim}_I \mathrm{in}\mathcal{E}_i$ and $\operatorname{colim}_I \mathrm{pr}\mathcal{E}_i$ define an exact structure on $\operatorname{colim}_I \mathcal{E}_i$ and that the canonical map $\operatorname{colim}_i K(\mathcal{E}_i) \to K(\operatorname{colim}_i \mathcal{E}_i)$ is an equivalence.

This sheet will be discussed on 10 July 2025.