LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Summer term 2025 12. Juni 2025

Algebraic K-theory

Sheet 6

We are in the situation of Milnor patching in the lecture.

Exercise 1. Assume that $S \in M_{n,m}(B')$ is the image of an invertible matrix $T \in M_{n,m}(B)$. Then $M(A'^n, B^m, S)$ is finite free and the A-linear maps $A'^n \leftarrow M(A'^n, B^m, S) \to B^m$ induce isomorphisms

$$M(A'^n, B^m, S) \otimes_A A' \cong A'^n \quad \text{and} \quad M(A'^n, B^m, S) \otimes_A B \cong B^m.$$

The resulting isomorphism

$$B'^n \cong A'^n \otimes_{A'} B' \cong M(A'^n, B^m, S) \otimes_A B' \cong B^m \otimes_B B' \cong B'^m$$

is S.

Exercise 2. Let $S \in M_{n,m}(B')$ be an invertible matrix and assume that $B \to B'$ is surjective. Then the invertible matrix

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \in M_{m+n,m+n}(B')$$

is the image of an invertible matrix in $M_{m+n,m+n}(B)$.

Exercise 3. Assume P is a finite free A'-module, Q is a finite free B-module, and $\alpha \colon P \otimes_{A'} B' \cong Q \otimes_B B'$ is an isomorphism of B'-modules. Then $M(P, Q, \alpha)$ is finite projective and the tautological maps $M(P, Q, \alpha) \otimes_A A' \to P$ and $M(P, Q, \alpha) \otimes_A B \to Q$ are isomorphisms and the resulting composite isomorphism

$$P \otimes_A' B' \cong M(P, Q, \alpha) \otimes_A B' \cong Q \otimes_B B'$$

is α.

Exercise 4. Let now (P, Q, α) be a general object of $\mathrm{Proj}(A') \times_{\mathrm{Proj}(B')} \mathrm{Proj}(B)$. Show that there exists (P', Q', α') such that $P \oplus P'$ and $Q \oplus Q'$ are free and finish the proof of Milnor's patching theorem.

Exercise 5. Show that the map $\partial \colon \mathrm{GL}(B') \to K_0(A)$ defined in the lecture is a monoid homomorphism.

This sheet will be discussed on 3 July 2025.