Summer term 2025 June 4, 2025

Algebraic K-theory

Sheet 5

Exercise 1. Let $f\colon X \to Y$ be a map of anima. Show that the following conditions are equivalent.

1. For every local coefficient system $\mathcal{L}$ on Y, the map $f_*\colon H_*(X; f^*\mathcal{L}) \to H_*(Y;\mathcal{L})$ is an isomorphism,
2. the map $X \times_Y \widetilde{Y} \to \widetilde{Y}$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$; here $\widetilde{Y} \to Y$ denotes the universal cover, and
3. for every point $y \in Y$, the map $\mathrm{fib}_y(f) \to *$ induces an isomorphism upon applying $H_*(-;\mathbb{Z})$.

Exercise 2. Without using the construction, show that if the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ admits a left adjoint L, then the unit map $X \to LX$ induces an isomorphism on $H_*(-;\mathbb{Z})$. Hint: Show that $X \to LX$ induces an isomorphism on $H^*(-;\mathbb{Z})$ and show that a map which induces an isomorphism on $H^*(-;\mathbb{Z})$ in fact also induces an isomorphism on $H_*(-;\mathbb{Z})$.

Exercise 3. Show that the canonical map $(X \times Y)^+ \to X^+ \times Y^+$ is an equivalence for all anima X and Y.

Exercise 4. Let $f\colon X \to Y$ be acyclic. Show that for all $x \in X$, the map $\pi_1(X) \to \pi_1(Y)$ is surjective with perfect kernel. Moreover, show that if f induces an isomorphism $\pi_1(X,x) \to \pi_1(Y, f(x))$ for all $x \in X$, then f is an equivalence.

Exercise 5. Let $F \to E \to B$ be a fibre sequence with B hypoabelian. Show that $F^+ \to E^+ \to B$ is again a fibre sequence.

This sheet will be discussed on 12 June 2025.