Summer term 2025 May 28, 2025

Algebraic K-theory

Sheet 4

Exercise 1. In this exercise, you may use the following fact about colimits in anima. Let I be a small ∞-category and and let $\bar{\tau}\colon \bar{X} \to \bar{Y}$ a natural transformation of functors $I^{\triangleright} \to \mathrm{An}$. Suppose $\bar{Y}$ is a colimit cone and that $\tau = \bar{\tau}_{|I}\colon X \to Y$ is a cartesian transformation of functors $I \to \mathrm{An}$, where $X = \bar{X}_{|I}$ and $Y = \bar{Y}_{|I}$. That is, for all morphisms $i \to j$ in I, the square

$$\begin{tikzcd} X(i) \arrow[r] \arrow[d] & Y(i) \arrow[d] \\ X(j) \arrow[r] & Y(j) \end{tikzcd}$$

is cartesian. Then $\bar{X}$ is a colimit cone if and only if $\bar{\tau}\colon \bar{X} \to \bar{Y}$ is a cartesian transformation.

Now, show the following result (called Rezk's equifibrancy criterion). Given a pullback diagram of functors $I \to \mathrm{An}$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "\tau"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

where τ is a cartesian transformation. Then the square of colimits

$$\begin{tikzcd} \mathrm{colim}_I X' \arrow[r] \arrow[d] & \mathrm{colim}_I X \arrow[d] \\ \mathrm{colim}_I Y' \arrow[r] & \mathrm{colim}_I Y \end{tikzcd}$$

is again a pullback diagram.

Exercise 2. Let X be a Segal anima. Show that its décalage $\mathrm{dec}(X)$ is again a Segal anima and participates in a pullback diagram

$$\begin{tikzcd} \mathrm{const}(X_1) \arrow[r] \arrow[d] & \mathrm{dec}(X) \arrow[d] \\ \mathrm{const}(X_0) \arrow[r] & X \end{tikzcd}$$

as in the lecture.

Exercise 3. In this exercise you may use that $|\mathrm{dec}(X)| \simeq X_0$ for any simplicial anima X. Let $G \in \mathrm{Grp}(\mathrm{An})$ be a group in anima. Show that the square

$$\begin{tikzcd} G \arrow[r] \arrow[d] & |\mathrm{dec}(\mathrm{Bar}(G))| \arrow[d] \\ * \arrow[r] & |\mathrm{Bar}(G)| \end{tikzcd}$$

is a pullback and deduce that $G \simeq \Omega|\mathrm{Bar}(G)|$, where $|-| = \mathrm{colim}_{\Delta^{\mathrm{op}}}$. Why does the proof not apply in case M is a monoid rather than a group?

Exercise 4. Let $M \in \mathrm{CMon}(\mathrm{An})$ be a commutative monoid. Show that $\mathrm{Bar}(M)$ is indeed left Kan extended from both its restrictions to $\Delta^{\mathrm{op}}_{\leq 1}$ and $\Delta^{\mathrm{op}}_{\leq 1,\mathrm{inj}}$.

This sheet will be discussed on 5 June 2025.