MATHEMATISCHES INSTITUT
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Übungen zur Vorlesung Differential– und Integralrechnung I (NV) Lösungsvorschlag

45. a) i.

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)}$$
$$= \frac{\frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)}}{1-\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)},$$

ii.

$$\arcsin(x)+\arcsin(y) = \arcsin(\sin(\arcsin(x)+\arcsin(y)))$$
$$= \arcsin(\sin(\arcsin(x))\cos(\arcsin(y)) + \sin(\arcsin(y))\cos(\arcsin(x)))$$
$$= \arcsin(x\sqrt{1-\sin^2(\arcsin(y))} + y\sqrt{1-\sin^2(\arcsin(x))})$$
$$= \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2}).$$

b) i.

$$\sin(2x) = \sin(x+x) = \sin(x)\cos(x)+\cos(x)\sin(x) = 2\sin(x)\cos(x),$$

ii.

$$\sin(x)+\sin(y) = \sin\left(2\frac{x}{2}\right) + \sin\left(2\frac{y}{2}\right) = 2(\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + \sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right))$$
$$= 2(\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)(\sin^2\left(\frac{y}{2}\right)+\cos^2\left(\frac{y}{2}\right)) + \sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)(\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)))$$
$$= 2(\sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)+\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right))(\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)+\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right))$$
$$= 2(\sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)+\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right))(\cos\left(\frac{x}{2}\right)\cos\left(\frac{-y}{2}\right)-\sin\left(\frac{x}{2}\right)\sin\left(\frac{-y}{2}\right))$$
$$= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

46. a)

$$\begin{aligned}
&\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\\
&1 = \sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 2\sin^2\left(\frac{\pi}{4}\right)\\
&\frac{1}{2} = \sin^2\left(\frac{\pi}{4}\right)\\
&\sqrt{\frac{1}{2}} = \sin\left(\frac{\pi}{4}\right),
\end{aligned}$$

b)

$$= \cos\left(\frac{\pi}{4}\right),$$

c)

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = 1.$$

47. Leider falsche Angabe, die Lösungen heißen $\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$.

a) Weil $x^3 = 1 \Leftrightarrow (-x)^3 = -1$, sind die Lösungen $\{-1, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}\}$,

b) i.

$$\begin{aligned}
\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) &= e^{i\frac{\pi}{3}}\\
\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^3 &= \left(e^{i\frac{\pi}{3}}\right)^3 = e^{i\pi} = -1
\end{aligned}$$

Daher steht oben links eine der Lösungen von $x^3 = -1$. Die 1. kann nicht sein, weil der $\sin\left(\frac{\pi}{3}\right) \neq 0$, die 2. nicht, weil er nicht < 0 ist, also folgt

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

ii.

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

iii.

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

48. a)

$$\begin{aligned}
\left|\frac{\frac{(n+1)^2}{(1+2i)^{n+1}}}{\frac{n^2}{(1+2i)^n}}\right| &= \left|\left(\frac{n+1}{n}\right)^2 \frac{1}{1+2i}\right| = \left(\frac{n+1}{n}\right)^2 \frac{1}{|1+2i|} = \left(\frac{n+1}{n}\right)^2 \frac{1}{\sqrt{5}}\\
&\leq \frac{16}{9}\frac{1}{\sqrt{5}} = \sqrt{\frac{256}{81}\frac{1}{5}} = \sqrt{\frac{256}{405}} < 1
\end{aligned}$$

für $n \geq 3$.

b) ii.

$$\left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1,$$

iii.

$$\frac{1}{1-\frac{1+i}{2}} = \frac{1}{\frac{1-i}{2}} = 1+i.$$