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THE DIAMETER OF A CYCLE PLUS A RANDOM MATCHING*

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Abstract. How small can the diameter be made by adding a matching to an n-cycle? In this paper this question is answered by showing that the graph consisting of an n-cycle and a random matching has diameter about $\log_2 n$, which is very close to the best possible value. It is also shown that by adding a random matching to graphs with certain expanding properties such as expanders or Ramanujan graphs, the resulting graphs have near optimum diameters.

Key words. diameter, random graphs, expanders

AMS(MOS) subject classification. 05C

1. Introduction. The following problem frequently comes up in connection with network optimization: For given integers n and k, find a graph on n vertices with maximum degree k, having diameter as small as possible.

The complementary problem of the preceding problem can be described as follows:

For given integers k and D, find a graph, with bounded degree k and diameter at most D, having as many vertices as possible.

Known solutions for these problems fall into two types; one type is to construct explicitly such good graphs [3], [4], [8], [17] and the other is to take the probabilistic approach by analyzing the random regular graphs [6], [7], [10]. As it turns out, random graphs generally outperform those graphs constructed by various methods (with a few exceptions when n or D is small). In fact, random graphs have diameters very close to the optimum value, while the best constructions have much larger diameters [1], [10].

Although random graphs are easy to analyze probabilistically, the memory required for storing the edges is proportional to n^2, whereas systematic constructions tend to use much less memory. For instance, some r-regular expander graphs only need to store k numbers [11]. This is particularly crucial in certain network problems involving routing and distributing computing. The result in this paper can be viewed as a "halfway" solution which blends a good construction with a small amount of randomness. In particular, our "hybrid" graphs only require cn memory to record all the edges (instead of cn^2 for random graphs), while the diameters are near optimal.

We will show that the graph obtained by adding a random matching to the n-cycle C_n has diameter very close to the optimum value, thus settling a problem of Farley and Hedetniemi [13]. We also prove a general theorem which asserts that by adding a random matching to k-regular graphs with certain expanding properties (detailed in § 3) the resulting graphs have diameter about $\log_k n$, which is the order of the best possible value. The best-known expanding graphs were constructed by Lubotzky, Phillips, and Sarnak [18] and have diameter $2 \log_{k-1} n$. Adding a matching result, with high probability, reduces the diameter by a factor of 2.

The paper is organized as follows. In § 2 we obtain sharp diameter bounds for a cycle plus a random matching. In § 3, we discuss several generalizations, including a general theorem which implies that adding a random matching to a k-regular graph with certain expanding properties results in a graph with diameter very close to the optimum value. In § 4, we conclude with various remarks and questions.

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2. The diameter of a cycle plus a random matching. Let C_n denote an n-cycle with vertices $v_1, v_2, \cdots, v_n$, where v_i is adjacent to v_{i+1} and all indices are then modulo n. A matching on C_n is just a partition of $\{1, \cdots, n\}$ into disjoint pairs (plus a singleton if n is odd). It is clear that C_n has diameter $\lfloor n/2 \rfloor$. We will show that by adding a matching to C_n the diameter can be reduced dramatically. We remark that it is known that any graph on n vertices with maximum degree 3 has diameter at least $\log_2 n - 2$ (see [2], [10]) and the best constructions for degree 3 graphs has diameter $1.47 \log_2 n$ (see [12], [15]). First, we consider the probability space of all graphs formed by adding a matching to C_n. We assume that any two such graphs have the same probability. We will prove such hybrid graphs have diameter about the same as random graphs of degree 3.

THEOREM 1. *Let G be a graph formed by adding a random matching to an n-cycle. Then with probability tending to* 1 *as n goes to infinity, G has diameter $D(G)$ satisfying*

$$\log_2 n - c \leq D(G) \leqq \log_2 n + \log_2 \log n + c$$

where c is a small constant (*at most* 10).

Proof. Let C denote the n-cycle in G and M denote a random matching in G. For a vertex x, we define

$$S_i(x) = \{y: d(x,y) = d_G(x,y) = i\},$$

$$B_l(x) = \bigcup_{i \leq l} S_i(x).$$

Now pick a fixed vertex x. Draw the chord (an edge in M) incident to x. This determines $S_1(x)$. Then we add the neighbors of $S_1(x)$ one by one to determine $S_2(x)$ and proceed to determine $S_i(x)$. Call a chord incident to a vertex in $S_i(x)$ *inessential* if the other vertex in $S_i(x)$ is within distance $3 \log_2 n$ (in C) of the vertices determined so far. Since $|B_i(x)| \leqq 3 \cdot 2^i$, the probability of an edge being inessential (at level i) is at most

$$\frac{18 \cdot 2^{i+1} \log_2 n}{n}.$$

Hence the probability that in the sequence of chords chosen in $B_l(x)$ at least two chords are inessential is at most

$$\binom{3 \cdot 2^{l+1}}{2} \left(\frac{18 \cdot 2^{l+1} \log_2 n}{n} \right)^2 = O(n^{-6/5} (\log_2 n)^2)$$

for $l = [\log_2 n/5]$. Therefore the probability that for every vertex x at most one of the chords in $B_l(x)$ is inessential is at least $1 - O(n^{-1/5} (\log n)^2)$. From now on we will be only interested in graphs satisfying the property mentioned above, and will only consider conditional probabilities on this event, say, event A.

For a fixed vertex x, consider those vertices y in $S_i(x)$ for which there is a unique path from x to y of length i, say $x_0 = x, x_1, \cdots, x_i = y$, such that (i) if x_{i-1} is adjacent to y on the cycle C then $B_i(x)$ has no vertex within $3 \log_2 n$ of x_i on the side opposite to x_{i-1} (see Fig. 1); (ii) if $x_{i-1}y$ is a chord then $B_i(x) - \{x_i\}$ has no vertex on C within distance $3 \log_2 n$ of x_i (see Fig. 2).

Denote the set of y's in (i) by $C_i(x)$ and the set of y's in (ii) by $D_i(x)$. Thus $C_i(x) \cup D_i(x) \subseteqq S_i(x)$. Now, if A holds then

$$|C_i(x)| \geqq 2^{i-2} \quad \text{and} \quad |D_i(x)| \geqq 2^{i-3} \quad \text{for } i \leqq \tfrac{1}{5} \log_2 n.$$

Now we consider i between $\frac{1}{5} \log_2 n$ and $\frac{3}{5} \log_2 n$. The probability of a chord being inessential is at most

$$\frac{18 \cdot 2^{i+1} \log_2 n}{n} < n^{-1/6}$$

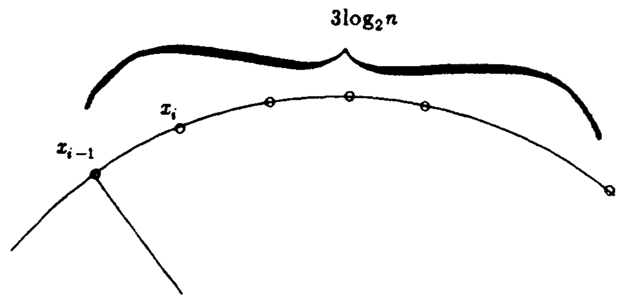

FIG. 1

for n large. Since there are at most 2^i chords leaving $S_i(x)$ for $\frac{1}{5}\log_2 n \leqq i \leqq \frac{3}{5}\log_2 n$, the probability that there are at least $2^i n^{-1/10}$ inessential chords leaving $S_i(x)$ is at most

$$\binom{2^i}{2^i n^{-1/10}} n^{-1/6 \cdot 2^i n^{-1/10}} \leqq (n^{1/10} n^{-1/6})^{2^i n^{-1/10}} \leqq (n^{-1/20})^{n^{1/10}} < n^{-5}$$

for large n. Therefore, with probability $1 - O(n^{-2})$ for every x and every i satisfying $\frac{1}{5}\log_2 n \leqq i \leqq \frac{3}{5}\log_2 n$, at most $2^i n^{-1/10}$ inessential chords leave $S_i(x)$. Call this event B.

For y in $C_i(x)$, a new neighbor of y in C is a "potential" element of $C_{i+1}(x)$ and a new neighbor, which is the end-vertex of the chord from y, is a "potential" element of $D_{i+1}(x)$. (Here "potential" means that the vertices in question become elements of $C_{i+1}(x)$ or $D_{i+1}(x)$, unless the corresponding edge is inessential.) Also, if $y \in D_i(x)$, then the two new neighbors of y on C are potential elements of $C_{i+1}(x)$. Hence if A and B both hold, then for $3 \leqq i \leqq \frac{1}{5}\log_2 n$ and for any x we have

$$|C_i(x)| \geqq 2^{i-2} \quad \text{and} \quad |D_i(x)| \geqq 2^{i-3}$$

and for $\frac{1}{5}\log_2 n \leqq i \leqq \frac{3}{5}\log_2 n$ we have

$$|C_{i+1}(x)| \geqq |C_i(x)| + 2|D_i(x)| - 2^{i+1}n^{-1/10},$$

$$|D_{i+1}(x)| \geqq |C_i(x)| - 2^{i+1}n^{-1/10}.$$

Therefore, for $3 \leqq i \leqq \frac{3}{5}\log_2 n$, we have

$$|C_i(x)| \geqq 2^{i-3} \quad \text{and} \quad |D_i(x)| \geqq 2^{i-4}.$$

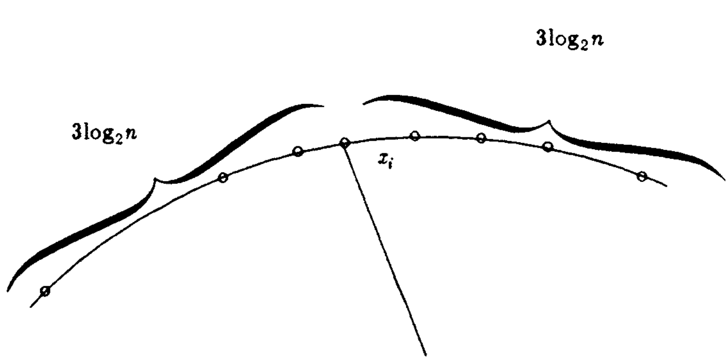

FIG. 2

Now set $i_0 = \lceil \frac{1}{2}(\log_2 n + \log_2 \log n + c)\rceil$. We want to estimate the conditional probability (on A and B) for two points x and y having distance at least $2i_0 + 1$ in G. Let us choose chords leaving $C_{i_0}(x)$ one by one. At each choice the probability of not choosing the other end vertex in $C_{i_0}(y)$ is at most $1 - (2^{i_0-3}/n)$.

Since we have to make at least $|C_{i_0}(x)|/2 \geqq 2^{i_0-4}$ such choices, we have

$$\begin{aligned}\text{Prob}(d(x,y) > 2i_0 + 1 \mid A \cap B) &\leqq \left(1 - \frac{2^{i_0-3}}{n}\right)^{2^{i_0-4}}\\ &\leqq \exp(-2^{2i_0-7}/n)\\ &\leqq \exp(-(\log n)2^{c-7})\\ &\leqq n^{-4}\end{aligned}$$

if $c \geqq 9$.

We are now ready to consider the probability that $D(G) > 2i_0 + 1$.

$$\begin{aligned}\text{Prob}(D(G) > 2i_0 + 1) &\leqq (1 - P(A)) + (1 - P(B)) + \sum_{x,y} \text{Prob}(d(x,y) > 2i_0 + 1 \mid A \cap B)\\ &\leqq c_1(n^{-1/5}(\log n)^2) + c_2(n^{-2}) + n^{-2} = o(1).\end{aligned}$$

Therefore almost all G have diameter at most

$$2\lceil \tfrac{1}{2}(\log_2 n + \log_2 \log n + 9)\rceil \leqq \log_2 n + \log_2 \log n + 10.$$

The proof of Theorem 1 is complete.

3. Several generalizations. The proof in § 2 can be easily carried over to the following generalizations or variations of Theorem 1.

PROPOSITION 1. *If we add a random matching to a graph on n vertices which is a disjoint union of large cycles (say at least* $100\sqrt{n}$ *each), the resulting graph has diameter D satisfying*

$$\log_2 n - c \leqq D(G) \leqq \log_2 n + \log_2 \log n + c$$

with probability tending to 1 *as n goes to infinity, where c is a small constant (at most* 10).

PROPOSITION 2. *Suppose T is a complete binary tree on* $2^k - 1$ *vertices. If we add two random matchings of size* 2^{k-1} *to the leaves of T, then the resulting graph has diameter D satisfying*

$$\log_2 n - c' \leqq D(G) \leqq \log_2 n + \log_2 \log n + c$$

with probability tending to 1 *as n goes to infinity, where c and c' are small constants at most* 10.

All the results in this paper are included in the following general version.

THEOREM 2. *Suppose H is a graph on n vertices with bounded degree k satisfying the property that for any* $x \in V(H)$, *the ith neighborhood* $N(x)$ of x *(i.e.,* $N_i(x) = \{y: d_H(x, y) = i\}$) *contains at least* $c_1 k(k-1)^{i-2}$ *vertices for* $i \leqq (\frac{1}{2} + \varepsilon)\log_{k-1} n$, *where* ε *and* c_1 *denote some fixed positive values. Then by adding a random matching to H the resulting graph G has diameter D(G) satisfying*

$$\log_k n - c \leqq D(G) \leqq \log_k n + \log_k \log n + c$$

with probability tending to 1 *as n goes to infinity, where c is a constant depending on* ε *and* c_1.

Proof. The proof is very similar to the proof of Theorem 1. We will sketch the idea without giving all the details. Let G denote the graph formed by adding a random matching M to H. We define, for each vertex x, $S_i(x)$ and $B_l(x)$ as before (in the proof of Theorem 1). The definition of a chord being *inessential* stays the same except that $\log_2$ is replaced by $\log_k$ and 3 is replaced by $k + 1$.

It is easy to see that for $l = \lfloor \log_k n/5 \rfloor$, the probability that, for every vertex x, at most one of the chords in $B_l(x)$ is inessential is at least $1 - O(n^{-1/5}(\log_k n)^2)$. Similarly, for $\frac{1}{5} \log_k n \leqq i \leqq (\frac{1}{2} + \varepsilon) \log_k n$, the probability that at least $k^i n^{-1/10}$ inessential chords leaving $S_i(x)$ is at most n^{-5}. Now we bound the conditional probability (on A and B). We define $C_i(x)$ and $D_i(x)$ the same way, except that we replace C by H and require that $B_i(x)$ have no vertex with distance $(k + 1) \log_k n$ of x_i in $G - \{x_i, y\}$. Again we have $|C_i(x)| \geqq c_1 k^{i-2}$ and $|D_i(x)| \geqq c_1 k^{i-3}$ for $i \leqq \frac{1}{5} \log_k n$.

For i between $\frac{1}{5} \log_k n$ and $(\frac{1}{2} + \varepsilon) \log_k n$, we have

$$|C_{i+1}(x)| \geqq (k-1)|C_i(x)| + k|D_i(x)| - k^{i+1} n^{-1/10},$$

$$|D_{i+1}(x)| \geqq |C_i(x)| - k^{i+1} n^{-1/10}.$$

Therefore, for $3 \leqq i \leqq (\frac{1}{2} + \varepsilon) \log_k n$ we have

$$|C_i(x)| \geqq c_1 k^{i-3} \quad \text{and} \quad |D_i(x)| \geqq c_1 k^{i-4}.$$

Now choose $i_0 = \lceil \frac{1}{2} \log_k n + \log_k \log n + c \rceil$. The probability of two vertices x and y of distance $> 2i_0 + 1$ is at most

$$\left(1 - \frac{c_2 k^{i_0 - 3}}{n}\right)^{k^{i_0 - 4}} \leqq n^{-4}.$$

Thus the probability that $D(G) > 2i_0 + 1$ is no more than $O(n^{-1/5}(\log n)^2) + O(n^{-2}) + n^{-2}$. Therefore almost all G have diameter

$$\log_k n + \log_k \log n + 10.$$

This concludes the proof of Theorem 2.

One natural question is which k-regular graphs satisfy the expanding property $N_i(x) \geqq ck^{i-1}$ for every vertex x (as described in Theorem 2)? Of course, random graphs have such an expanding property. In the past few years much progress has been made on various explicit constructions of so-called expander graphs [1], [14], [16], [18], [19]. All these expander graphs have various expanding properties for different applications. In particular, a graph is an expander graph if it has relatively small second largest eigenvalues for its adjacency matrix [18].

Let us denote by λ the second largest (in absolute value) eigenvalue of the adjacency matrix of a k-regular graph G. (Of course, the largest eigenvalue is k.) Tanner [20] proved that for any set X of vertices, the number of neighbors $N(X)$ of X is at least

$$N(X) \geqq \frac{k^2|X|}{(k^2 - \lambda^2)|X|/n + \lambda^2}.$$

Clearly if $|\lambda| < k - \varepsilon$, then G satisfies the expanding property required in Theorem 2.

Recently, Lubotzky, Phillips, and Sarnak [17] constructed graphs with λ satisfying $|\lambda| \leqq 2\sqrt{k-1}$, which is the best possible value. They call these graphs Ramanujan graphs. It is easy to see that Ramanujan graphs satisfy $N_i(x) \geqq (k - 1)^{i-1}/2$ for each vertex x. Ramanujan graphs have diameter $2 \log_{k-1} n + c$, while the lower bound for the diameter is $\log_{k-1} n$. By adding a matching to a Ramanujan graph, the resulting graph has diameter $(1 + o(1)) \log_k n$.

4. Concluding remarks. Many problems concerning the diameter of graphs remain open. We mention several of them here.

(1) Find explicit constructions for graphs with n vertices and degree at most k having diameter $(1 + o(1)) \log_{k-1} n$.

(2) For given integers n, k, t, let $f(n, k, t)$ denote the minimum value over all diameters of graphs which are formed by deleting (any choice of) t edges from a graph with n vertices and degree at most k. The problem is to determine $f(n, k, t)$ and to characterize the optimal graphs.

(3) Find efficient algorithms for determining the diameter of a graph. The best-known algorithms require $O(n^{2.38})$ time or $O(ne)$ time (see [10]). In particular, for planar graphs is there an $o(n^2)$ algorithm?

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