cadmo
center for algorithms, discrete mathematics & optimization

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Geometric Inhomogeneous Random Graphs (GIRGs)

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joint work with K. Bringmann, R. Keusch, C. Koch

Motivation: Network Models

- want to develop good algorithms for large *real-world networks*
  want to have asymptotic statements, benchmarks, …

- real network data is *scarce* and *hard to obtain*
  social: facebook, twitter, mobile phone, friendship, collaboration..
  technological: internet, www, web of things,…

- these networks *share many properties*
  power law degrees, (ultra-)small world, strong clustering, small separators,…

Motivation: Network Models

properties / models	power law degree?	small world?	cluster-ing?	non-rigid clust.?	easy to analyze?
Kleinberg	✗	✓	(✓)	(✓)	✓
pref. attachm.	✓	✓	✗	✗	✗
Chung-Lu	✓	✓	✗	✗	✓
geom. random	✗	✗	✓	✗	✓
hyperbolic	✓	✓	✓	✗	✗
spatial pref. att.	✓	✓	✓	✗	✗
GIRGs	✓	✓	✓	✓	✓

Motivation: Hyperbolic Random Graphs

- best model so far: *hyperbolic random graphs*
- each vertex draws a random position in a hyperbolic disc of radius R.
- two points connect if and only if their distance is at most R.
- has many nice properties:
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BUT: kind of complicated…. 😧

$$\bar{k}(r) = \frac{N}{2\pi(\cosh R - 1)} \left\{ 2\pi(\cosh R - 1) - 2\cosh R \left(\arcsin \frac{\tanh(r/2)}{\tanh R} + \arctan \frac{\cosh R \sinh(r/2)}{\sqrt{\sinh(R + r/2)\sinh(R - r/2)}} \right) \right.$$
$$\left. + \arctan \frac{(\cosh R + \cosh r)\sqrt{\cosh 2R - \cosh r}}{\sqrt{2}(\sinh^2 R - \cosh R - \cosh r)\sinh(r/2)} - \arctan \frac{(\cosh R - \cosh r)\sqrt{\cosh 2R - \cosh r}}{\sqrt{2}(\sinh^2 R + \cosh R - \cosh r)\sinh(r/2)} \right\}, \quad (11)$$

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GIRGs (Geometric Inhomogeneous Random Graphs)

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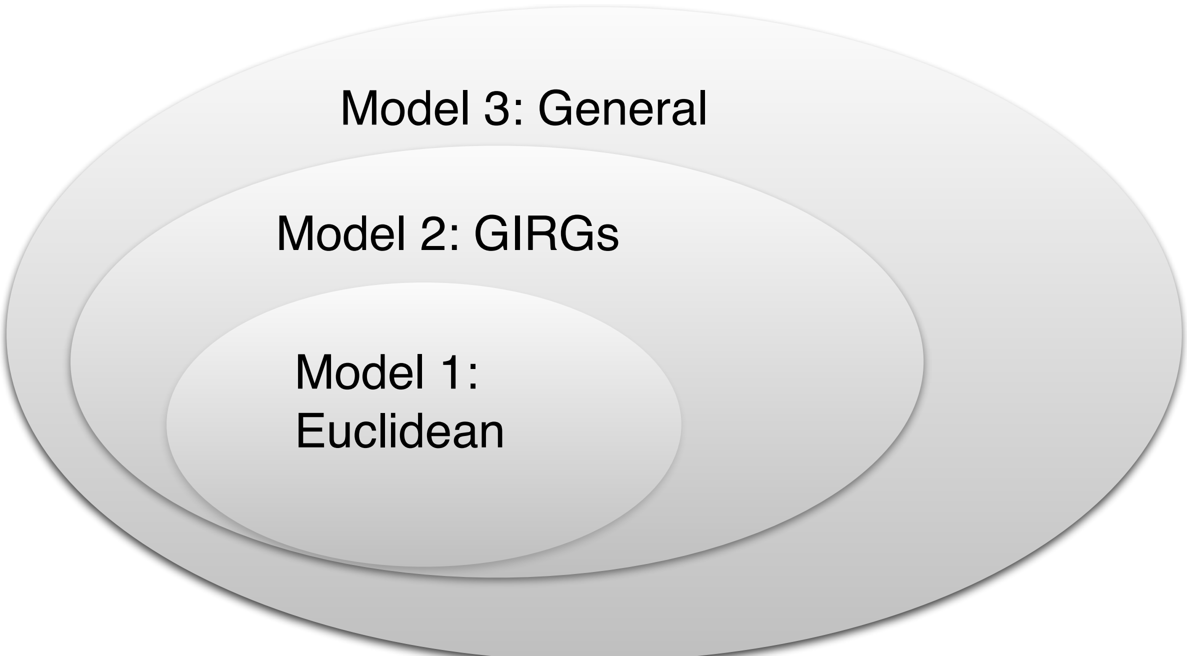

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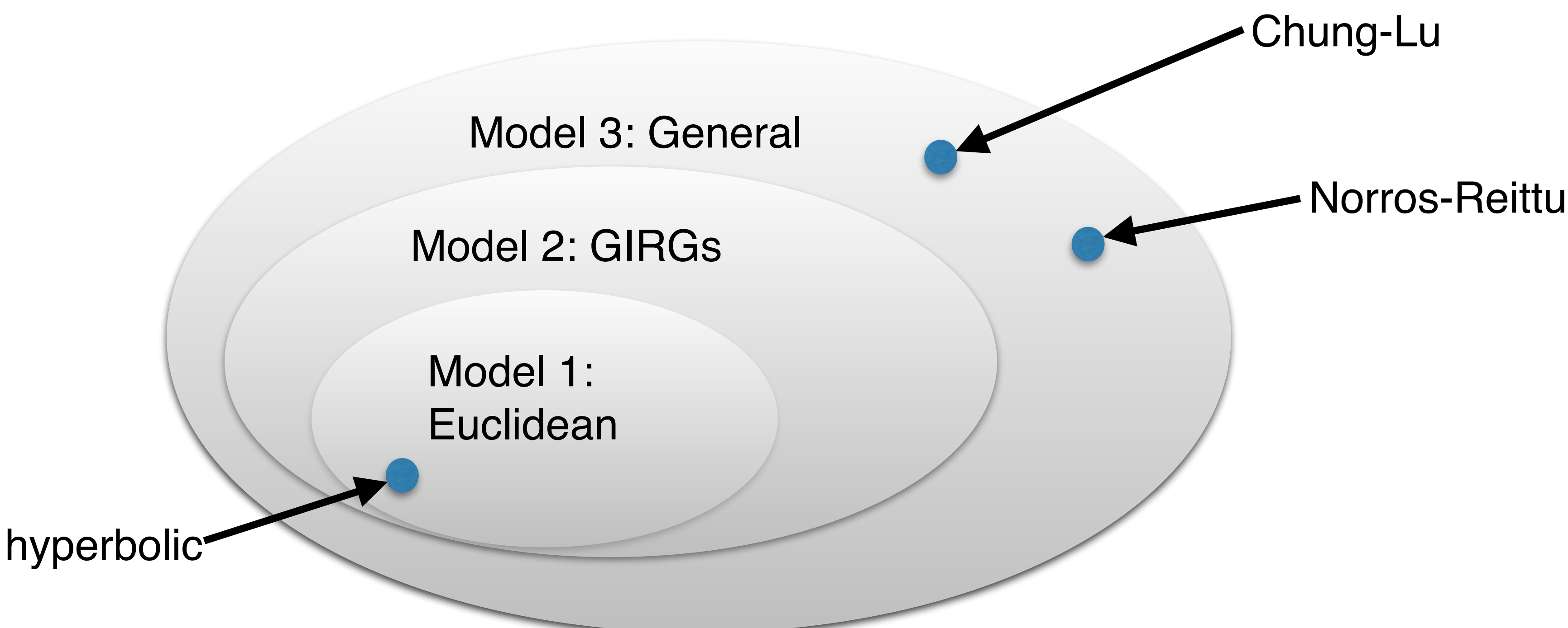

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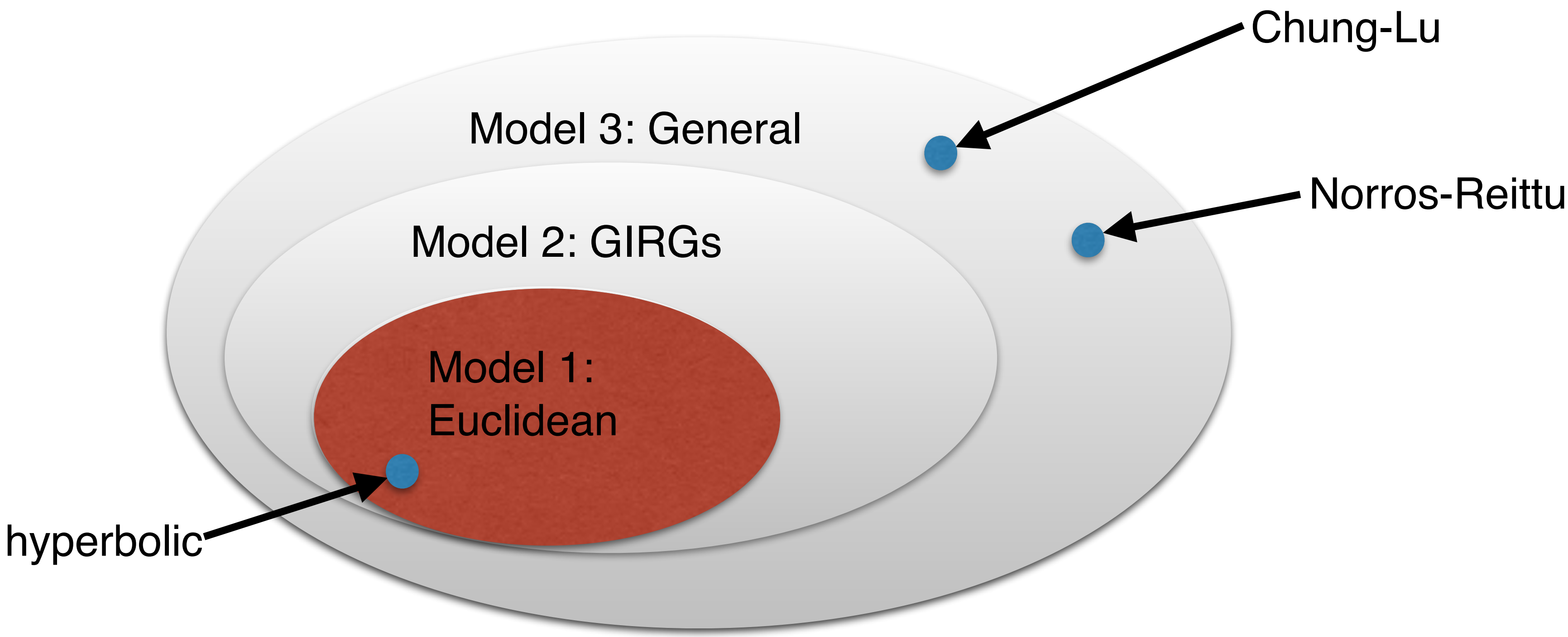

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where $\alpha > 1$ is a parameter.

Basic Properties

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Lemma: For any fixed x_i, w_i, w_j,

1. $\Pr_{x_j}[v_i \sim v_j] = \Theta\left(\min\left\{1, \frac{w_i w_j}{n}\right\}\right)$.
2. $\mathbb{E}[\deg(v_i)] = \Theta(w_i)$.

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Corollary:

- The degree of a vertex v_i of weight w_i is Poisson distributed (in the limit) with mean $\Theta(w_i)$.
- $\mathbb{E}[w_i] = \Theta(1)$ => There are O(n) edges.

(Ultra-)Small World

Theorem: Whp,

1. the graph contains a giant component of linear size.
2. all other components are of polylog size.
3. the diameter of the graph is polylogarithmic.
4. the average distance in the giant is $(2+o(1))\frac{\log\log n}{|\log(\beta-2)|}$.

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- holds in the most general model (including Chung-Lu graphs)
- same is true for other power-law graph models (e.g., preferential attachment)

Clustering

Definition:
The *clustering coefficient* of a graph is

$$\mathrm{cc} := \Pr_{u,v,w}[v \sim w \mid u \in V, v, w \in \Gamma(u)].$$

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- Social (and other) networks have large clustering coefficient.
- most models with power law degrees have $\mathrm{cc} = \Theta(1/n)$.
  (Chung-Lu, preferential attachment, …)
- *exception*: hyperbolic random graphs have $\mathrm{cc} = \Omega(1)$.

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Theorem: GIRGs have $\mathrm{cc} = \Omega(1)$.

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Theorem: GIRGs have *small separators:*

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- was unstudied for hyperbolic random graphs.
- Chung Lu and pref. attachment models are different: Removing $o(n)$ edges or vertices reduces the giant by at most $o(n)$.
- Real-world networks have small separators.

Entropy/Compression

Observation:
The web graph can be stored using bits per edge.

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Theorem: We can store a GIRG with expected $O(n)$ bits, so that we can answer the queries
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- compression algorithm for hyperbolic graphs was known.
- Chung Lu and pref. attachment models have entropy $\Theta(n \log n)$. (I.e., need $\Theta(\log n)$ bits per edge.)

Sampling

Theorem: For every concrete function

$$p(w_i, w_j, x_i, x_j) = \Theta\left(\min\left\{1, \left(|x_i - x_j|^{-d} \cdot \frac{w_i w_j}{n}\right)^{\alpha}\right\}\right),$$

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- Naive sampling needs time $\Theta(n^2)$.
- Efficient algorithms were known for Chung-Lu model and others.
- Best previous algorithm for hyperbolic random graphs had runtime $\Theta(n^{3/2})$.

Greedy Routing

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  REPEAT until we find t:
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Theorem: With probability $\Omega(1)$, greedy routing succeeds

in $(2+o(1))\dfrac{\log\log n}{|\log(\beta-2)|}$ steps.

With small modifications (e.g. backtracking), it succeeds within this time whp and in expectation.

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- We fix a region B of volume .
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- if $p \gg p_0$ then $\Theta(n)$ vertices turn active whp;
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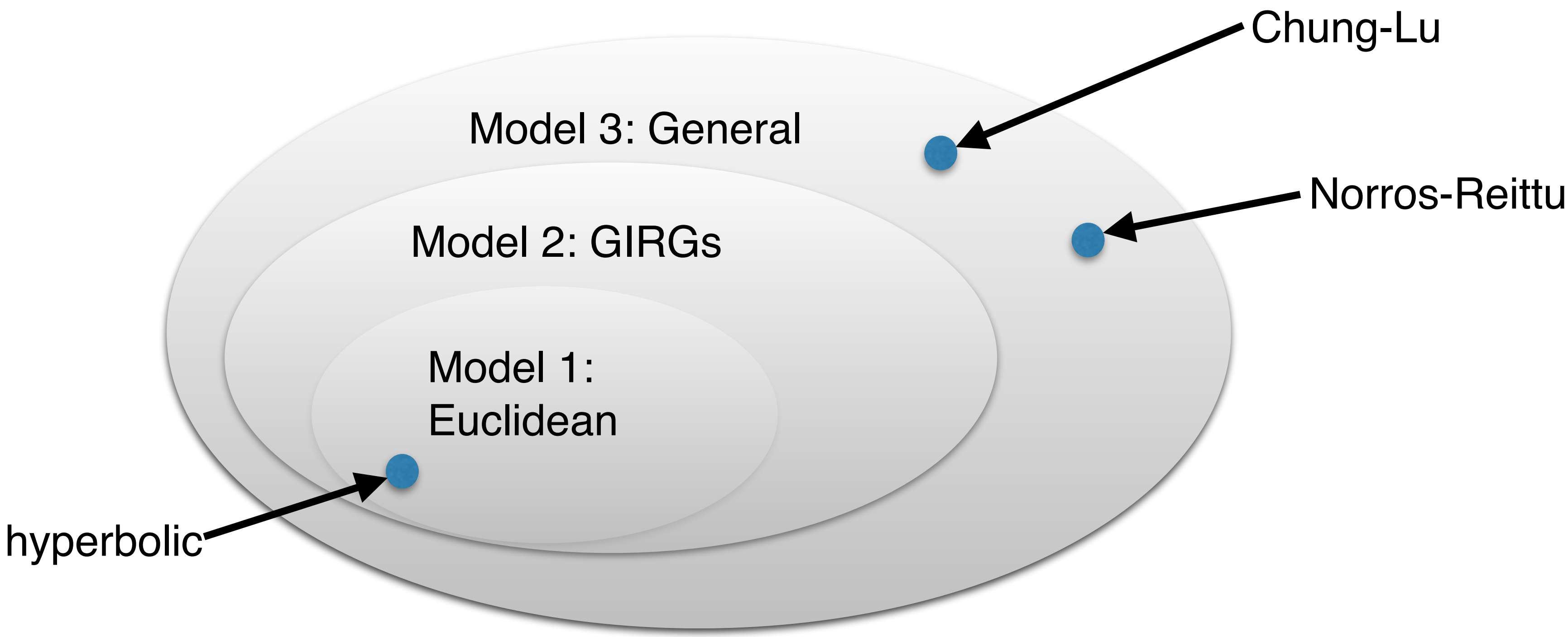
Theorem: Assume $\alpha > \beta - 1$. Let v be a vertex of weight $w \gg 1$ and distance $r \geq \dots$ from B. The whp v turns active in round $(1 \pm o(1))\ell(v) \pm O(1)$, where

$$\ell(v) := \begin{cases} \max\{0,\ \log\log_\nu (\|r^d n/w) /|\log(\beta-2)|\}, & \text{if } w > (r^d n)^{1/(\beta-1)}, \\ (2\log\log_\nu(r^d n) - \log\log_\nu w)/|\log(\beta-2)|, & \text{if } w \leq (r^d n)^{1/(\beta-1)}. \end{cases}$$

Non-Euclidean GIRGs



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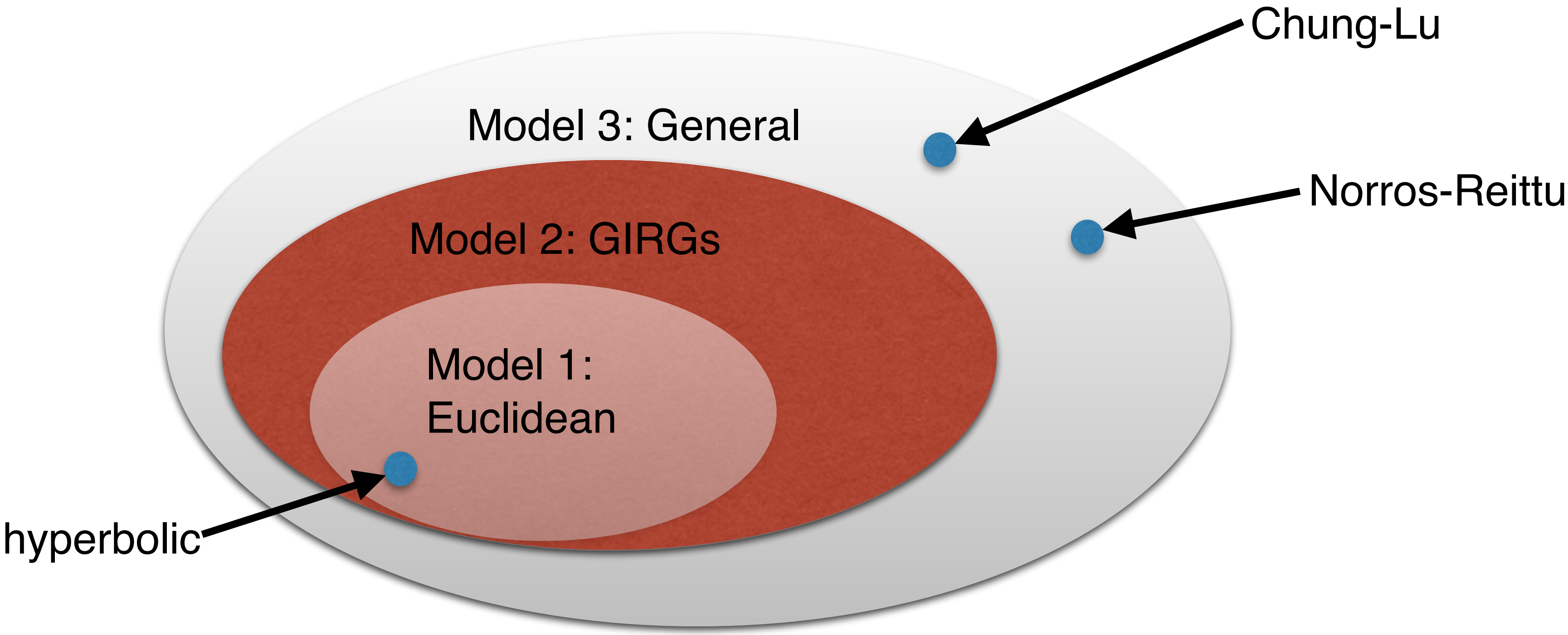

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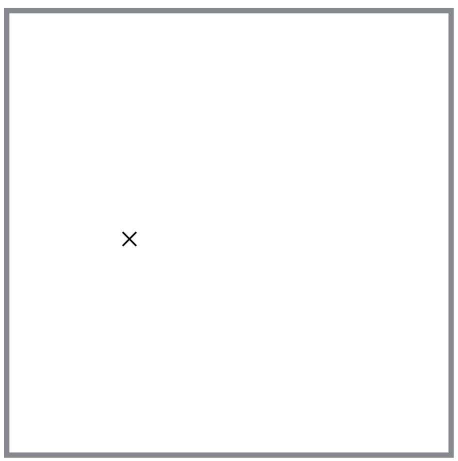

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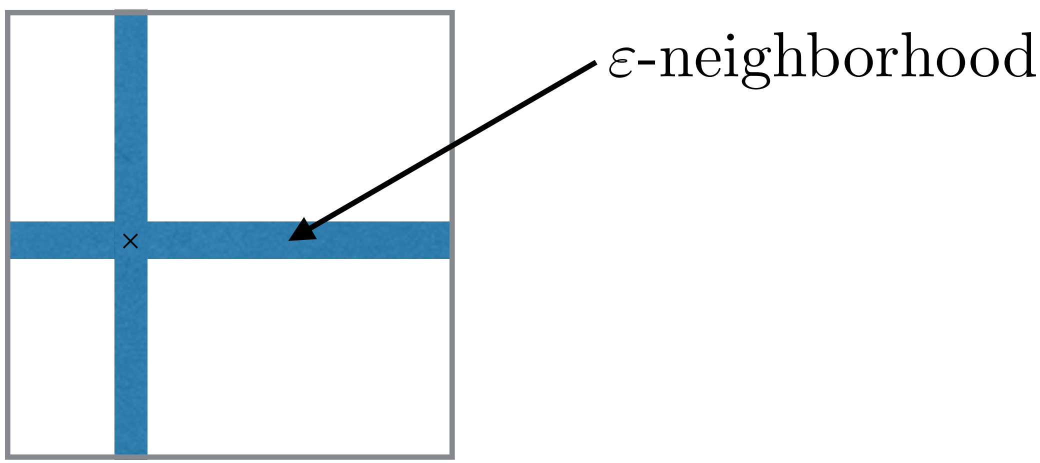

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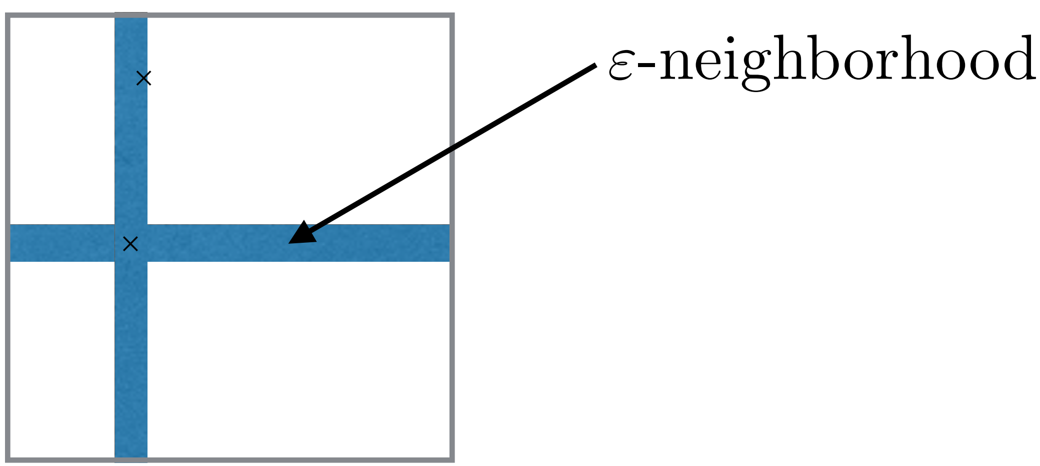

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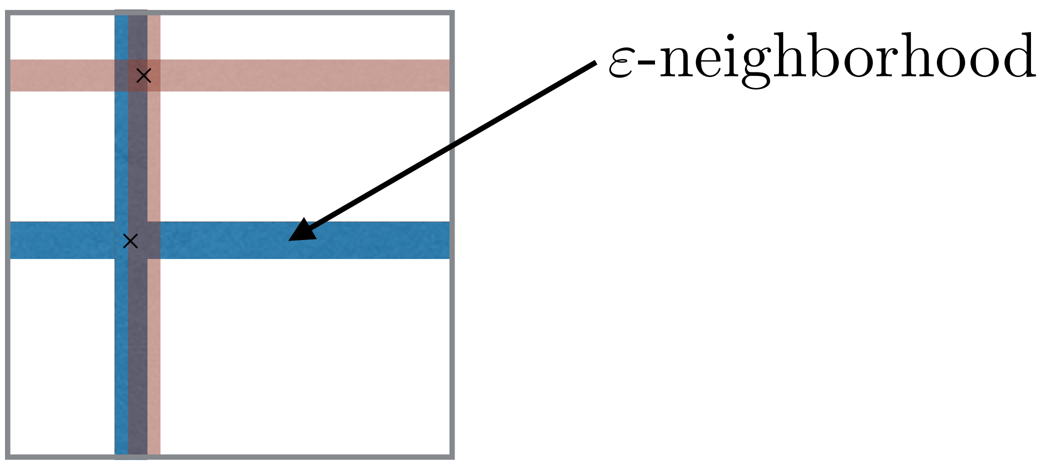

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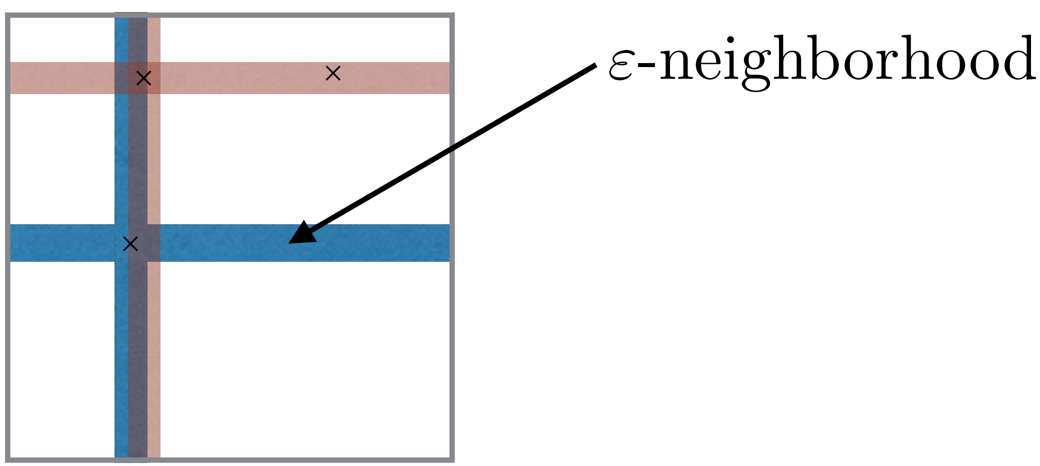

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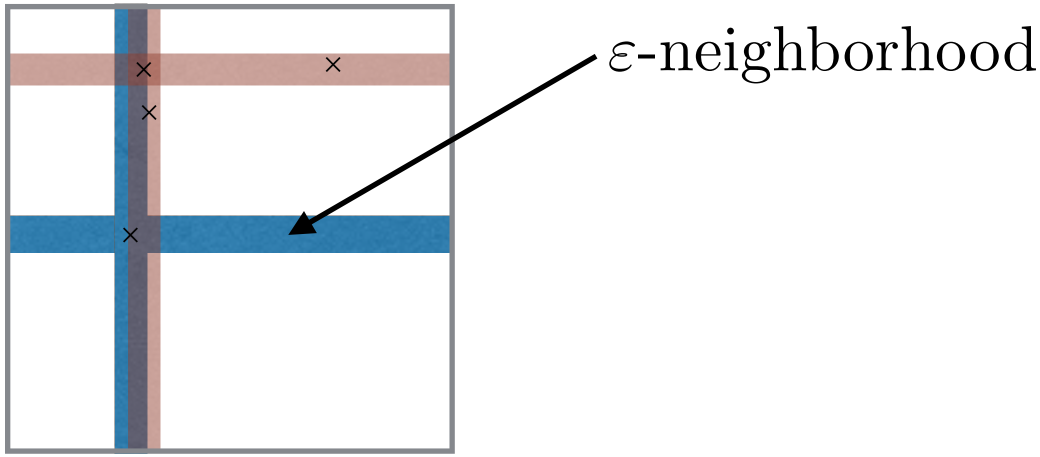

Summary

General Model:
- power law degrees
- small world: components, diameter, average distance

Distance Model:
- strong clustering (if distance function is “nice”)
- may be non-rigid clustering

Euclidean Model (or other norms):
- small separators
- small entropy, efficient compression
- linear time sampling

Future Work

Algorithms
- communication protocols
- de-anonymization

Processes
- infection processes (work in progress)
- information dissemination

Others
- recovering the underlying geometry
- attacks
- dynamic graph problems
- games on graphs

Thank you for your attention!

Questions?