Order-preserving Random Dynamical Systems

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Summary. This paper is concerned with stochastic recursions $X_n = H_n(X_{n-1})$ for $n \in \mathbf{N}$, where $(H_n)_{n \in \mathbf{N}}$ is a sequence of i.i.d. random transformations in $\mathbf{R}_+$ independent of the initial variable X_0. If the transformations are supposed to be continuous and order-preserving, this results in quite natural notions of (topological) recurrence and transience. In the recurrent case existence and uniqueness of an invariant measure as well as mean and pointwise ergodic theorems are established. Moreover, the associated attractor is investigated. The classification is completed by distinguishing positive and null recurrence corresponding, respectively, to the case of a finite or infinite invariant measure. Equivalently, this amounts to finite or infinite mean passage times.

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Introduction

In this paper random dynamical systems are studied under the following assumptions on space and time: the evolution is supposed to develop in the space of reals and to proceed in discrete time starting at zero. Thus the process is given by a sequence $(X_n)_{n\geq 0}$ satisfying the recursion

$$X_n = H_n(X_{n-1}) \quad \text{for} \quad n \in \mathbf{N} \,,$$

where X_0 is a real-valued random variable and $H_n, n \in \mathbf{N}$ are random transformations from $\mathbf{R}$ to $\mathbf{R}$. To obtain the Markov property the variables X_0, $H_1, H_2, \ldots$ are supposed to be independent, to obtain a stationary transition kernel the variables $H_1, H_2, \ldots$ are supposed to be identically distributed. If their common distribution ν is carried by a finite family of transformations, this results in an "iterated function system", made popular during the past decade by Barnsley, Elton and others (see e.g. [2, 3, 8, 9]). On the other hand, it is shown, for instance, by Kifer in Theorem 1.1 of [18] that, without restricting the support of ν, the model above provides just another possibility to introduce any Markov chain with state space $\mathbf{R}$.

In applications, such a representation arises often quite naturally as is exhibited by the following examples (see also [1, 24, 25]):
(1) The simplest queuing process is given by

$$X_n = (X_{n-1} + U_n)^+ \quad \text{for} \quad n \in \mathbf{N}$$

with i.i.d. variables $U_n, n \in \mathbf{N}$. Here, X_{n-1} denotes the waiting time of customer $n-1$ and $U_n = S_{n-1} - (T_n - T_{n-1})$ is the balance between his service time and the arrival times of customers $n-1$ and n.
(2) A "savings process" is defined by

$$X_n = U_n X_{n-1} + V_n \quad \text{for} \quad n \in \mathbf{N}$$

with i.i.d. variables $(U_n, V_n), n \in \mathbf{N}$. Here, X_{n-1} denotes the balance of a savings account at time $n-1$, U_n the interest / inflation rate during period n and V_n the deposit made at time n (this and a survey of other affine recursions in biology, economics, physics etc. can be found in Vervaat [29]).
(3) An "exchange process" is defined by

$$X_n = (X_{n-1} - U_n) \vee V_n \quad \text{for} \quad n \in \mathbf{N}$$

with i.i.d. variables $(U_n, V_n), n \in \mathbf{N}$. Here, X_{n-1} is the utility of some equipment in use at time $n-1$, U_n its loss in utility during period n, and V_n the utility of a new equipment available at time n (for this and related examples see, for instance, Helland and Nilsen [13]).

The first example differs from the second and the third one by a particular characteristic: the state 0 is a regeneration point, and thus the queuing process fits into the Doeblin-Harris theory for not necessarily discrete Markov chains.

Since the dominating measure, whose existence is postulated in this theory, fails to exist in general, there are attempts, for instance by Rosenblatt [27] or Tweedie [28], to classify Markov chains by stressing the topological structure of the state space. Even under continuity assumptions on the underlying kernel, however, this results in a variety of notions of transience and null or positive recurrence, being thus less convincing than the classical notions for discrete Markov chains. The present paper, therefore, emphasizes the order structure of the state space, motivated by two observations holding for various applications. First, the transformations $h : x \to (x+u)^+$, $h : x \to ux + v$, and $h : x \to (x - u) \vee v$ appearing in the examples above are all order-preserving (observing $U_n \geq 0$ in (2)). Second, in these examples the proper state space is $\mathbf{R}_+$ (observing $V_n \geq 0$ in (2) and (3)). As it turns out, accepting these two restrictions presents an appropriate compromise in order to obtain a satisfying theory as well as substantial applications.

While there exists an extensive literature based on the metric structure of the state space by requiring, for instance, an "average contractivity" of the underlying transformations, there are only a few papers with special emphasis on the order structure. The earliest one dates back to Dubins and Freedman [7], who limit, however, their investigations to the compact state space [0,1]. This restriction is given up in Yahav [30], who studies concave increasing mappings from $\mathbf{R}_+$ to $\mathbf{R}_+$. Extending the state space further to $\mathbf{R}$, Bhattacharaya and Waymire in Section II.14 of [4] take up a "splitting" condition from [7]. In all this treatments the main interest concerns the existence and uniqueness of stationary distributions. This holds as well for Brandt et al., who consider in Section 1.3 of [5] order-preserving mappings in partially ordered Polish spaces requiring, however, appropriate compactness and contraction properties.

To see fluctuation aspects to be as interesting as equilibrium results, consider an exchange process with deterministic loss of utility, say $U_n = 1$. Then it is easily established that a (unique) stationary distribution exists if and only if the utility V_n of the substitute has a finite expectation. This fails, for instance, if V_n has the density

$$f_1(x) = (x+1)^{-2} \quad \text{or} \quad f_2(x) = 2x(x+1)^{-3} \quad \text{for} \quad x \geq 0\,.$$

Due to $f_1(x) \leq f_2(x) \leq 2f_1(x)$ for $x \geq 1$, in both cases V_n behaves similarly as far as it concerns the existence of moments. Nevertheless, there is a significant difference: while in the second case $(X_n)_{n\geq 0}$ escapes to infinity, the process is uniformly distributed on $\mathbf{R}_+$ in the first case (this example appears in the context of (2.4), (6.3)−(6.4), (9.7)).

A detailed survey of the principal results of the present paper appears dispensable, because the headings of the different sections provide a first information about the contents. Instead, the main feature will be summarized as follows: the order-preserving random dynamical systems as considered here represent one of the best suited models for extending discrete Markov chain theory to an uncountable state space. That is because the fundamental criteria for positive / null recurrence or transience − by means of the n-step transition kernels resp. the potential kernel, by means of hitting probabilities resp. mean passage times, or by means of a unique invariant measure – all find their counterpart in the present paper (for a first orientation see the remarks following (6.5), (9.8), (10.1) and preceding (11.4)).

Finally, a historical remark is in order. This work originated from a three-part paper devoted to the special case of recursions

$$X_n = U_n X_{n-1} + V_n \quad \text{with} \quad U_n, V_n \geq 0 \quad \text{for} \quad n \in \mathbf{N}.$$

This affine model, which for constant U_n contains in particular first-order autoregressive processes, is of special importance, because it may serve to approximate more complex situations by linearization. During the refereeing process of [16], however, it became clear that most results rely on the topological and order structure of the state space only and make no use of the linear structure. Moreover, it turned out that within a more general framework not only several proofs could be simplified but also several theorems could be strengthened. This led to the decision to develop first the general theory in the present paper and to deal with the special features of affine recursions in a subsequent paper. As it is to be expected, there are, for instance, stronger criteria for positive / null recurrence or transience, if the underlying mappings are compatible with the linear structure. To state just one of the main results of [17], let $(S_n)_{n\geq 0}$ be the random walk with increments $\log U_n$ ($\geq -\infty$). Then, under a weak boundedness condition on V_n (and excluding the degenerate case of a common fixed point of the underlying mappings), the following trichotomy holds: the process $(X_n)_{n\geq 0}$ is positive recurrent resp. null recurrent resp. transient, if the associated random walk $(S_n)_{n\geq 0}$ diverges to $-\infty$ resp. oscillates between $-\infty$ and $+\infty$ resp. diverges to $+\infty$.

Preliminaries

Throughout the paper the state space E is a subinterval of $\mathbf{R}$ satisfying $\min E = 0$ and endowed with its Borel σ-algebra $\mathcal{B}(E)$. As usual, $\mathcal{C}(E)$ denotes the space of bounded continuous functions $f : E \to \mathbf{R}$ and $\mathcal{K}(E)$ the subspace consisting of functions with compact support. Employing the order structure, $\mathcal{R}(E)$ denotes the space of ("regular") functions $f : E \to \mathbf{R}$ having limits within $\mathbf{R}$ from the right and from the left everywhere (including $\sup E$) and $\mathcal{V}(E)$ the subspace consisting of functions of bounded variation.

Let the space of continuous mappings from E to E be endowed with the compact open topology. Then the closed subspace $\mathcal{H}[E]$ of order-preserving mappings inherits a Polish topology, which is also the initial topology with respect to the evaluation maps $h \to h(x)$, $x \in E$. Since $\mathcal{H}[E]$ is stable with respect to composition and composition is continuous, $\mathcal{H}[E]$ is a topological semigroup. Since E can be embedded into $\mathcal{H}[E]$, the mapping $(x, h) \to h(x)$ is continuous, too.

$\mathcal{M}(E)$ denotes the class of locally finite measures on E and $\mathcal{M}_1(E)$ the subclass consisting of probability measures. If μf denotes the μ-integral of a function f, then $\mathcal{M}(E)$ is endowed with the vague (weak*) topology, i.e. the initial topology with respect to the mappings $\mu \to \mu f$, $f \in \mathcal{K}(E)$; the corresponding convergence is denoted by $\underset{v}{\to}$. On the subspace $\mathcal{M}_1(E)$ this induces the weak (narrow) topology, i.e. the initial topology with respect to the mappings $\mu \to \mu f$, $f \in \mathcal{C}(E)$; the corresponding convergence is denoted by $\underset{w}{\to}$.

The main object of this paper is the space $\mathcal{N}[E]$ of distributions ν on $\mathcal{H}[E]$. The semigroup structure of $\mathcal{H}[E]$ induces a convolution in $\mathcal{N}[E]$ and makes this space a topological semigroup itself. Corresponding powers are simply denoted by ν^n, i.e.

$$\int f(h(x))\nu^n(dh) = \int \ldots \int f(h_1 \circ \ldots \circ h_n(x))\, \nu(dh_1) \ldots \nu(dh_n)$$

for $x \in E$, $f \in \mathcal{C}(E)$ and $n \in \mathbf{N}$, while ν^0 is the unit measure ε_h with h being the identity map. Since $\mathcal{H}[E]$ is again a Polish space, in particular the support N is well-defined for $\nu \in \mathcal{N}[E]$.

Now the stochastic model can be precisely introduced. Let be given, on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$,

(1) a sequence of independent random variables $H_n : \Omega \to \mathcal{H}[E]$ with identical distribution $\nu \in \mathcal{N}[E]$,

(2) a random variable $X_0 : \Omega \to E$ that is independent of $(H_n)_{n \in \mathbf{N}}$.

This defines an "order-preserving random dynamical system" by

$$X_n = \psi(X_{n-1}, H_n) \quad \text{with} \quad \psi(x,h) = h(x)\,,$$

which in the sequel will be briefly written as

$$X_n = H_n(X_{n-1}) \quad \text{for} \quad n \in \mathbf{N}\,.$$

Thus the distribution of $(X_n)_{n \geq 0}$ is completely determined by $\nu \in \mathcal{N}[E]$ and the initial law $\mu_0 = \mathcal{L}(X_0)$. Here, the primary component is ν, and all notions to be defined will depend on that distribution. This dependence will, however, be suppressed in the related notations, because ν is supposed to be fixed.

As usual, the initial law is largely of secondary importance only. If in particular $X_0 = x$, this will be expressed by the notation $(X_n^x)_{n \geq 0}$, i.e.

$$X_n^x = H_n \circ \ldots \circ H_1(x) \quad \text{for} \quad x \in E \quad \text{and} \quad n \geq 0\,.$$

Thus for general μ_0 conditional probabilities are given by

$$\mathbf{P}^x((X_n, n \geq 0) \in B) = \mathbf{P}((X_n^x, n \geq 0) \in B)$$

with an anologous equation for conditional expectations.

Clearly, $(X_n)_{n \geq 0}$ is a homogeneous Markov process. Its transition kernel, always denoted by P, transforms a function f on E into Pf given by

$$Pf(x) = \int_{\mathcal{H}[E]} f(h(x))\, \nu(dh) \quad \text{for} \quad x \in E$$

and a measure μ on E into μP given by

$$\mu P(B) = \int_E \nu(h(x) \in B)\, \mu(dx) \quad \text{for} \quad B \in \mathcal{B}(E)\,,$$

which for a $\sigma-$finite measure μ by Fubini equals

$$\mu P(B) = \int_{\mathcal{H}[E]} \mu(h(x) \in B)\, \nu(dh) \quad \text{for} \quad B \in \mathcal{B}(E)\,.$$

The kernel P belongs to the class $\mathcal{P}[E]$ of Markov kernels from E to E satisfying the following two conditions:

(1) P transforms $\mathcal{C}(E)$ into itself,

(2) P transforms bounded increasing functions into functions of the same type.

It has to be mentioned here that the mapping $\nu \to P$ from $\mathcal{N}[E]$ to $\mathcal{P}[E]$ is neither injective nor surjective.

Finally, it has to be pointed out that, in contrast to topology and order, the algebraic structure of the state space is not taken into account. Thus distributions $\nu \in \mathcal{N}[E]$ and $\nu' \in \mathcal{N}[E']$ are called "conjugate" and have to be classified in the same way, if there is an order-preserving homeomorphism $g : E' \to E$ such that ν' is the image of ν under the mapping $h \to g^{-1} \circ h \circ g$.

For an example in the case $E = \mathbf{R}_+ = E'$ let $h_+ \in \mathcal{H}[E]$ be strictly increasing with $h_+(x) > x$ for all $x \in E$ and define $h_- \in \mathcal{H}[E]$ by $h_-(x) = 0$ for $x \leq h_+(0)$ and $h_-(x) = h_+^{-1}(x)$ otherwise. Then a distribution $\nu \in \mathcal{N}[E]$ with support $N = \{h_-, h_+\}$ is conjugate to the distribution $\nu' \in \mathcal{N}[E']$ belonging to the queuing process

$$X_n = (X_{n-1} + U_n)^+ \quad \text{for} \quad n \in \mathbf{N}$$

with independent variables $U_n, n \in \mathbf{N}$, satisfying

$$\mathbf{P}(U_n = -1) = \nu(\{h_-\}) \quad \text{and} \quad \mathbf{P}(U_n = +1) = \nu(\{h_+\}).$$

Indeed, it is easily checked that any strictly increasing function $g \in \mathcal{C}([0,1])$ with $g(0) = 0$ and $g(1) = h_+(0)$ can be extended to an appropriate homeomorphism by the definition

$$g(x+n) = h_+^n(g(x)) \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad n \in \mathbf{N}.$$

1. Lower and upper limit

As in discrete Markov chain theory, the first question to be settled concerns the appropriate decomposition of the state space E. For some $t \in E$ it may split into two intervals $E_1 = [0, t[$ and $E_2 = E \setminus E_1$ such that $h[E_i] \subset E_i$ for $\nu-$almost all $h \in \mathcal{H}[E]$ and $i = 1, 2$. With $0 = \min E$ as reference state the corresponding "class" can be characterized explitely as well as implicitly:

(1.1) Proposition *For $\nu \in \mathcal{N}[E]$ the set*

$$E_0 := \bigcup_{n \in \mathbf{N}} \{x \in E : \mathbf{P}(X_n^0 \geq x) > 0\}$$

is the smallest subinterval I of E satisfying the conditions

(a) $$0 \in I\,,$$

(b) $$\nu(h[I] \subset I) = 1\,.$$

Proof. 1. Since any – open or closed – interval I in E containing 0 can be represented as a countable union of intervals $[0, x_k]$, the subset of $\mathcal{H}[E]$ occurring in (b) is apparently of type G_δ (or even closed) and thus in particular measurable. Since this representation applies to $I = E_0$ itself, in establishing (b) it suffices to show that, $x = x_k$ being fixed, $h(x) \in E_0$ for $\nu-$almost all $h \in \mathcal{H}[E]$.

2. To this end choose $n \in \mathbf{N}$ with $\mathbf{P}(X_n^0 \geq x) > 0$ and define

$$\mathcal{H}_t := \{h \in \mathcal{H}[E] : h(x) \geq t\} \quad \text{for} \quad t \in E\,.$$

Then the independence of X_n^0 and H_{n+1} yields

$$\begin{aligned}\{h \in \mathcal{H}[E] : h(x) \in E_0\} &\supset \{h \in \mathcal{H}[E] : \mathbf{P}(X_{n+1}^0 \geq h(x)) > 0\} \\ &\supset \{h \in \mathcal{H}[E] : \mathbf{P}(X_n^0 \geq x, H_{n+1}(x) \geq h(x)) > 0\} \\ &= \{h \in \mathcal{H}[E] : \mathbf{P}(H_{n+1}(x)) \geq h(x)) > 0\} \\ &= \{h \in \mathcal{H}[E] : \nu(\mathcal{H}_{h(x)}) > 0\}\,.\end{aligned}$$

Due to $h \in \mathcal{H}_{h(x)}$ this implies

$$\begin{aligned}\{h \in \mathcal{H}[E] : h(x) \notin E_0\} &\subset \{h \in \mathcal{H}[E] : \nu(\mathcal{H}_{h(x)}) = 0\} \\ &\subset \bigcup \{\mathcal{H}_{h(x)} : h \in \mathcal{H}[E] \text{ with } \nu(\mathcal{H}_{h(x)}) = 0\}\,.\end{aligned}$$

Since the sets $\mathcal{H}_t$ decrease for increasing t, this union can be replaced by a countable one and is thus itself a null set with respect to ν, as had to be shown.

3. Finally, let the subinterval I of E satisfy (a) and (b). Then iteration yields the equation $\nu^n(h[I] \subset I) = 1$ and thus in particular

$$\mathbf{P}(X_n^0 \in I) = \nu^n(h(0) \in I) = 1 \quad \text{for all} \quad n \in \mathbf{N}\,.$$

For $x \in E$ satisfying the condition $\mathbf{P}(X_n^0 \geq x) > 0$ this implies $x \in I$, thus proving E_0 to be minimal. □

The interval E_0 can be an open or closed subset of E as it may happen already in the following two trivial examples:

(1) the "deterministic system" (E, h), where $\nu = \varepsilon_h$ for some mapping $h \in \mathcal{H}[E]$, yielding deterministic variables $X_n, n \in \mathbf{N}$, whenever X_0 is a constant;
(2) the "independent system" (E, μ), where ν is the image of some measure $\mu \in \mathcal{M}_1(E)$ with respect to the canonical injection j of E into $\mathcal{H}[E]$, yielding independent variables $X_n, n \in \mathbf{N}$, with distribution μ.

If the interval E_0 is a proper subset of E, its upper endpoint is easily identified:

(1.2) Proposition *The supremum $\overline{x}$ of the set E_0 defined by* (1.1) *is given by*

$$\overline{x} = \min\{x \in E : h(x) \underset{\nu}{\leq} x\} \quad \textit{whenever} \quad \overline{x} \in E\,.$$

Proof. For each $x \in E$ with $h(x) \underset{\nu}{\leq} x$ the interval $I = [0, x]$ satisfies the condition in (1.1) and thus contains E_0, which implies $\overline{x} \leq x$. On the other hand, the definition of E_0 yields $h(x) \underset{\nu}{\leq} \overline{x}$ for $x < \overline{x}$, which by the continuity of $h \in \mathcal{H}[E]$ implies $h(\overline{x}) \underset{\nu}{\leq} \overline{x}$ whenever the condition $\overline{x} \in E$ is satisfied. □

The assumption $\overline{x} \in E$ is no real restriction, because E being replaced by $\overline{E} := E \cup \{\overline{x}\}$ mappings $h \in \mathcal{H}[E]$ have unique extensions $\overline{h} \in \mathcal{H}[\overline{E}]$, where possibly ∞ has to be adjoined to $\mathbf{R}_+$. Moreover, if $\overline{x} \in E$ and the support N of ν is finite, $\overline{x}$ clearly is a fixed point of some $h \in N$.

The implicit description of E_0 suggests the following notion:

(1.3) Definition *The distribution $\nu \in \mathcal{N}[E]$ is called "irreducible", if the set E_0 defined by* (1.1) *coincides with the state space E.*

Irreducibility apparently can always be achieved, replacing mappings $h \in \mathcal{H}[E]$ by their restrictions $h_0 \in \mathcal{H}[E_0]$. Moreover, since irreducibility is clearly invariant under conjugation, it suffices in principle, neglecting the trivial case $E = \{0\}$, to treat exclusively the two cases $E = [0, 1]$ and $E = \mathbf{R}_+$ – at the expense, however, of permanent repetitions.

The notion of irreducibility is also compatible with convolution powers:

(1.4) Proposition *If $\nu^k \in \mathcal{N}[E]$ is irreducible for one $k \in \mathbf{N}$, this holds for all $k \in \mathbf{N}$.*

Proof. The assertion is an immediate consequence of the fact that, for arbitrary $x \in E$, the sequence $(\mathbf{P}(X_n^0 \geq x))_{n \geq 0}$ increases due to

$$\begin{aligned} \mathbf{P}(X_n^0 \geq x) &= \mathbf{P}(H_n \circ \ldots \circ H_1(0) \geq x) \\ &= \mathbf{P}(H_1 \circ \ldots \circ H_n(0) \geq x)\,. \quad \square \end{aligned}$$

The following two theorems are of central importance in the sequel. Due to the assumptions concerning the consistency with the order structure, lower and upper limit of the process $(X_n)_{n \geq 0}$ turn out to be constants, which in addition do not depend on the initial variable X_0. The first result is easily established:

(1.5) Theorem *If $\nu \in \mathcal{N}[E]$ is irreducible, the constant $\overline{x} = \sup E$ satisfies*

$$\limsup_{n\to\infty} X_n = \overline{x} \quad a.s.\,,$$

regardless of the initial law.

Proof. For $x \in E$ there exists $l \in \mathbf{N}$ such that

$$\mathbf{P}(H_l \circ \ldots \circ H_1(0) \geq x) > 0\,.$$

Then by monotonicity and independence

$$\begin{aligned} \mathbf{P}(X_n \geq x \text{ infinitely often}) &\geq \mathbf{P}(\limsup_{k\to\infty} \{H_{(k+1)l} \circ \ldots \circ H_1(0) \geq x\}) \\ &\geq \mathbf{P}(\limsup_{k\to\infty} \{H_{(k+1)l} \circ \ldots \circ H_{kl+1}(0) \geq x\}) \\ &= 1. \end{aligned}$$

Since $x \in E$ is arbitrary, $\limsup_{n\to\infty} X_n \geq \overline{x}$ holds almost surely, while the inverse inequality is obvious. □

The second result is less immediate:

(1.6) Theorem *If $\nu \in \mathcal{N}[E]$ is irreducible, there is a constant $\underline{x} \leq \overline{x}$ satisfying*

$$\liminf_{n\to\infty} X_n = \underline{x} \quad a.s.\,,$$

regardless of the initial law.

Proof. 1. For $x \in E$ and $n \geq 0$ define

$$\underline{X}_n^x := \liminf_{k\to\infty} H_{n+k} \circ \ldots \circ H_{n+1}(x)\,.$$

Then the inequality

$$\begin{aligned} \underline{X}_0^0 &= \liminf_{k\to\infty} H_{n+k} \circ \ldots \circ H_{n+1}(X_n^0) \\ &\geq \liminf_{k\to\infty} H_{n+k} \circ \ldots \circ H_{n+1}(0) \;=\; \underline{X}_n^0 \end{aligned}$$

combined with the equation $\mathcal{L}(\underline{X}_0^0) = \mathcal{L}(\underline{X}_n^0)$ yields

$$\underline{X}_0^0 = \underline{X}_n^0 \quad \text{a.s.} \quad \text{for all} \quad n \geq 0\,,$$

hence $\underline{X}_0^0$ is measurable with respect to the completed tail σ-field of $(H_n)_{n\in\mathbf{N}}$. Thus there is a constant $\underline{x}$ satisfying

(1) $$\underline{X}_0^0 = \underline{x} \quad \text{a.s.}\,.$$

2. For fixed $x \in E$ choose $n \in \mathbf{N}$ such that $\mathbf{P}(A) > 0$ for $A := \{X_n^0 \geq x\}$. With the notation $\mu_n := \mathcal{L}(X_n^0)$ it follows from (1) that

$$\begin{aligned} 1 &= \mathbf{P}((\underline{X}_0^0 \leq \underline{x}) \\ &= \int_E \mathbf{P}(\underline{X}_n^y \leq \underline{x})\, \mu_n(dy) \\ &\leq \int_{y \geq x} \mathbf{P}(\underline{X}_n^x \leq \underline{x})\, \mu_n(dy) \,+\, \int_{y<x} \mathbf{P}(\underline{X}_n^x \leq \underline{x})\, \mu_n(dy) \\ &\leq \mathbf{P}(A)\, \mathbf{P}(\underline{X}_n^x \leq \underline{x}) \,+\, \mathbf{P}(\Omega \setminus A)\,. \end{aligned}$$

In view of $\mathbf{P}(A) > 0$ and the equation $\mathcal{L}(\underline{X}_0^x) = \mathcal{L}(\underline{X}_n^x)$ this implies

$$\underline{X}_0^x \leq \underline{x} \quad \text{a.s.} \qquad \text{for all} \quad x \in E\,. \tag{2}$$

3. Together, (1) and (2) yield

$$\underline{x} = \underline{X}_0^0 \leq \underline{X}_0^x \leq \underline{x} \quad \text{a.s.} \qquad \text{for all} \quad x \in E\,,$$

and the assertion follows by applying Fubini. □

The preceding results suggest the following terminology:

(1.7) Definition *If $\nu \in \mathcal{N}[E]$ is irreducible, the constants $\underline{x}$ and $\overline{x}$ in* (1.5) *and* (1.6) *are called "lower limit" and "upper limit" of ν, respectively.*

Now a counterpart of (1.2) can be derived:

(1.8) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible, its lower limit $\underline{x}$ is given by*

$$\underline{x} = \max\{x \in E : h(x) \underset{\nu}{\geq} x\} \qquad \textit{whenever} \quad \underline{x} \in E\,.$$

Proof. For each $x \in E$ with $h(x) \underset{\nu}{\geq} x$ iteration yields $X_n^x \geq x$ a.s. for all $n \in \mathbf{N}$, which by (1.6) implies $\underline{x} \geq x$. On the other hand, whenever $\underline{x} \in E$, choose $x > \underline{x}$ in the case $\underline{x} < \overline{x}$ and $x = \underline{x}$ in the case $\underline{x} = \overline{x}$. In both cases the hitting times $T_1 < T_2 < \ldots$ of $[0, x]$ by $(X_n^0)_{n \geq 0}$ are defined almost surely, where again by (1.6)

$$\begin{aligned} \underline{x} &= \liminf_{n \to \infty} X_n^0 \\ &\leq \liminf_{k \to \infty} X_{T_k+1}^0 \\ &\leq \liminf_{k \to \infty} H_{T_k+1}(x)\,. \end{aligned}$$

Since $T_k, k \in \mathbf{N}$, are stopping times with respect to $(H_n)_{n \in \mathbf{N}}$, the variables $H_{T_k+1}, k \in \mathbf{N}$, are again independent with distribution ν. Thus $H_{T_k+1}(x) \geq \underline{x}$ a.s., or equivalently, $h(x) \underset{\nu}{\geq} \underline{x}$, which for $x \downarrow \underline{x}$ (if necessary) implies $h(\underline{x}) \underset{\nu}{\geq} \underline{x}$ by the continuity of $h \in \mathcal{H}[E]$. □

As stated for the upper limit, if $\underline{x} \in E$ and the support N of ν is finite,

$\underline{x}$ clearly is a fixed point of some $h \in N$.

The upper limit of an irreducible distribution ν is always uniquely determined by its support N; indeed:

$$\overline{x} = \sup\{h_n \circ \ldots \circ h_1(0) : n \in \mathbf{N} \text{ and } h_i \in N\}\,.$$

A corresponding result for the lower limit fails to hold; in fact, even the alternative $\underline{x} \in E$ or $\underline{x} \notin E$ is not a question on N alone, but will lead to the basic distinction between recurrence and transience of ν.

Proper convergence of the process $(X_n)_{n\geq 0}$, even if weakened to convergence in probability, is limited to a degenerate case:

(1.9) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible, then for arbitrary initial law the following assertions are equivalent:*

(a) $$\underline{x} = x = \overline{x} \quad \textit{for some } x \in E\,,$$

(b) $$(X_n)_{n\geq 0} \textit{ converges in } E \textit{ in probability}\,,$$

(c) $$h(x) \underset{\nu}{=} x \quad \textit{for some } x \in E\,.$$

Proof. 1. The implication (a) $\Rightarrow$ (b) is immediate from (1.5) and (1.6).

2. Assume now $X_n \to X$ in probability with $\mu = \mathcal{L}(X) \in \mathcal{M}_1(E)$ and let d be a bounded metric inducing the topology of E. Then integration over $\Omega \times \mathcal{H}[E]$ by $\mathbf{P} \otimes \nu$ yields

$$\begin{aligned} \iint d(X, h(X))\, d\mathbf{P}\, d\nu \;\;\leq\;\; & \iint d(X, X_n)\, d\mathbf{P}\, d\nu \\ + \; & \iint d(X_n, h(X_n))\, d\mathbf{P}\, d\nu \\ + \; & \iint d(h(X_n), h(X))\, d\mathbf{P}\, d\nu\,. \end{aligned}$$

For $n \to \infty$ the three summands on the right-hand side tend to 0: the first one because of the assumption; the second one, because it equals $\mathbf{E}(d(X_n, X_{n+1}))$ due to the independence of X_n and H_{n+1}; the third one, because $h(X_n) \to h(X)$ in probability due to the continuity of $h \in \mathcal{H}[E]$. Therefore

$$d(x, h(x)) = 0 \quad \text{for} \;\; \mu \otimes \nu\text{-almost all } (x, h)\,,$$

hence by applying Fubini

$$h(x) \underset{\nu}{=} x \quad \text{for} \;\; \mu\text{-almost all } x \in E\,.$$

3. The implication (c) $\Rightarrow$ (a) is a consequence of (1.2), yielding $\overline{x} \leq x$, and (1.8), yielding $\underline{x} \geq x$. $\square$

2. Recurrence and transience

Besides the proper convergence $\underline{x} = \overline{x} \in E$ considered in (1.9) there is an improper convergence $\underline{x} = \overline{x} \notin E$, generalizing the almost sure divergence of the process $(X_n)_{n\geq 0}$ to ∞ in the special case $E = \mathbf{R}_+$. This is a first motivation for the following notion, used similarly in [19] in related context:

(2.1) Definition *If $\nu \in \mathcal{N}[E]$ is irreducible, the distribution ν (or the kernel P or the process $(X_n)_{n\geq 0}$) is called*

(a) *"recurrent" if $\underline{x} \in E$,*

(b) *"transient" if $\underline{x} \notin E$.*

To begin with the simplest example, a deterministic system (E, h) is easily seen to be recurrent if and only if $E = [0, \overline{x}]$ with $\overline{x}$ being the maximum of the increasing sequence $(h^n(0))_{n\geq 0}$. Thus choosing $E = [0, 1[$ and $h(x) = (x+1)/2$ provides an example for transience due to $\underline{x} = 1 = \overline{x}$. This appears only logical, observing that the classification in (2.1) is clearly compatible with conjugacy and (E, h) is conjugate to the deterministic process (E', h') with $E' = \mathbf{R}_+$ and $h'(x') = x' + 1$. Indeed, an appropriate order-preserving homeomorphism $g : E' \to E$ is given by $g(x') = 1 - 2^{-x'}$.

By definition an irreducible distribution $\nu \in \mathcal{N}[E]$ is recurrent whenever $\overline{x} \in E$. This is a special case of a more general sufficient condition that often applies:

(2.2) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible and satisfies*

$$\nu(\sup_{x\in E} h(x) \in E) > 0 \ ,$$

then ν is recurrent.

Proof. First, the subset of $\mathcal{H}[E]$ occurring in the condition is apparently of type F_σ and thus in particular measurable. To prove this condition to imply recurrence, choose $t \in E$ with $\nu(\sup_{x\in E} h(x) \leq t) > 0$. Then for arbitrary initial law by independence

$$\mathbf{P}(X_n \leq t \text{ infinitely often}) \geq \mathbf{P}(\limsup_{n\to\infty} \{\sup_{x\in E} H_n(x) \leq t\}) = 1$$

and thus $\underline{x} \leq t \in E$. □

As a trivial example consider an independent system (E, μ), where by definition $\sup_{x\in E} h(x) \in E$ for $\nu-$almost all $h \in \mathcal{H}[E]$. To see that the condition in (2.2) is far from being necessary, consider the queuing process

$$X_n = (X_{n-1} + U_n)^+ \quad \text{for} \quad n \in \mathbf{N}\,,$$

where the i.i.d. variables $U_n, n \in \mathbf{N}$, are arbitrary. With state space $E = \mathbf{R}_+$

the corresponding distribution ν is carried by the mappings $h : x \to (x+u)^+$, $u \in \mathbf{R}$, and thus irreducible whenever $\mathbf{P}(U_n > 0) > 0$. In this case the condition in (2.2) is not satisfied, while it is well-known that $X_n \to \infty$ a.s. if and only if the random walk with increments U_n does not diverge to $+\infty$.

In the case of transience the process $(X_n)_{n\geq 0}$ diverges exponentially fast in the following sense:

(2.3) Proposition *If $\nu \in \mathcal{N}[E]$ is transient, the random cardinality*

$$Z := |\{n \geq 0 : X_n \leq t\}|$$

for arbitrary initial law and every $t \in E$ satisfies

$$\mathbf{E}(\exp(uZ)) < \infty \quad \textit{for some } u > 0\,.$$

Proof. Define recursively

$$T_0 := 0 \quad \text{and} \quad T_k := \inf\{n > T_{k-1} : X_n \leq t\} \ (\leq \infty)\,.$$

Then by transience

$$0 = \mathbf{P}^0(X_n \leq t \text{ infinitely often}) = \lim_{k\to\infty} \mathbf{P}^0(T_k < \infty)\,,$$

hence there exists $l \in \mathbf{N}$ such that $\vartheta := \mathbf{P}^0(T_l < \infty) < 1$. With the decreasing function

$$g(x) := \mathbf{P}^x(T_l < \infty) \quad \text{for} \quad x \in E$$

the Markov property implies

$$\begin{aligned} \mathbf{P}^0(T_{(k+1)l} < \infty) &= \int_{\{T_{kl}<\infty\}} g(X_{T_{kl}})\ d\mathbf{P}^0 \\ &\leq \int_{\{T_{kl}<\infty\}} g(0)\ d\mathbf{P}^0 \\ &= \vartheta\ \mathbf{P}^0(T_{kl} < \infty) \quad \text{for} \quad k \geq 0\,. \end{aligned}$$

This yields the bound

$$\mathbf{P}^0(T_{kl} < \infty) \leq \vartheta^k$$

and thus by monotonicity

$$\begin{aligned} \mathbf{P}(Z \geq kl) &\leq \mathbf{P}^0(T_{kl} < \infty) \\ &\leq \vartheta^k \quad \text{for} \quad k \geq 0\,. \end{aligned}$$

Partial integration shows that each $u < -\frac{1}{l}\log\vartheta$ satisfies the assertion. □

It is a fundamental consequence of (2.3) that recurrence and transience can be characterized by applying the potential kernel $G := \sum_{n\geq 0} P^n$ to intervals $[0,t]$ with the right endpoints:

(2.4) Theorem *If $\nu \in \mathcal{N}[E]$ is irreducible, the following dichotomy holds for arbitrary initial law:*

(a) *if ν is recurrent, then*

$$\sum_{n\geq 0} \mathbf{P}(X_n \leq t) = \infty \quad \text{for} \quad t > \underline{x},$$

(b) *if ν is transient, then*

$$\sum_{n\geq 0} \mathbf{P}(X_n \leq t) < \infty \quad \text{for} \quad t < \overline{x}.$$

Proof. Both assertions follow by taking the expectation of

$$Z = \sum_{n\geq 0} 1_{[0,t]}(X_n),$$

which in case (a) is almost surely infinite and in case (b) is integrable by (2.3). □

For an application consider an exchange process with $U_n = 1$, i.e.

$$X_n = (X_{n-1} - 1) \vee V_n \quad \text{for} \quad n \in \mathbf{N},$$

where the i.i.d. variables $V_n, n \in \mathbf{N}$, are nonnegative. With state space

$$E = \{x \geq 0 : \mathbf{P}(V_n \geq x) > 0\}$$

the corresponding distribution ν is carried by the mappings $h : x \to (x-1)\vee v$, $v \in E$, and thus irreducible by (1.1). Moreover it is recurrent in the case $\overline{x} < \infty$. Indeed, in this case

$$\sup_{x\in E} ((x-1) \vee v) = (\overline{x} - 1) \vee v \in E \quad \text{for} \quad v \in E$$

and thus (2.2) applies. As already indicated in the introduction, recurrence as well as transience can occur in the case $\overline{x} = \infty$. To exhibit appropriate examples, denote by F the common distribution function of $V_n, n \in \mathbf{N}$. Then the explicit representation

$$X_n^0 = (V_1 - (n-1)) \vee \ldots \vee (V_{n-1} - 1) \vee V_n \quad \text{for} \quad n \in \mathbf{N}$$

yields by independence

$$\mathbf{P}(X_n^0 \leq t) = \prod_{0\leq m<n} F(t+m).$$

Now (2.4) applies:

(1) if $V_n, n \in \mathbf{N}$, have the common density $f(x) = (x+1)^{-2}$, then $F(t) = \dfrac{t}{t+1}$ implies

$$\sum_{n\geq 0} \mathbf{P}(X_n^0 \leq t) = \sum_{n\geq 0} \frac{t}{t+n} = \infty \quad \text{for all} \quad t > 0,$$

i.e. the process is recurrent;
(2) if the density is replaced by the function $f(x) = 2x(x+1)^{-3}$, then $F(t) = (\frac{t}{t+1})^2$, and it follows similarly that the process is transient.

More profound criteria for recurrence and transience can be derived by linearization, i.e. comparing the underlying mappings with affine ones. As already mentioned in the introduction, however, this topic is postponed to [17].

This section concludes with a consequence of (2.4) by which some proofs can be simplified:

(2.5) Proposition *If $\nu^k \in \mathcal{N}[E]$ is recurrent for one $k \in \mathbf{N}$, this holds for all $k \in \mathbf{N}$. Moreover, the associated limits $\underline{x}_k$ and $\overline{x}_k$ are independent of k.*

Proof. The case $\underline{x} = \overline{x}$ is settled by (1.4), because $X_n \to X$ a.s. implies $X_{kn} \to X$ a.s. and thus $\underline{x}_k = \underline{x} = \overline{x} = \overline{x}_k$. To settle the case $\underline{x} < \overline{x}$, it is sufficient to apply (2.4) with $X_0 = 0$, because it follows as in the proof of (1.4) that the sequence $(\mathbf{P}(X_n^0 \le t))_{n \ge 0}$ decreases. □

3. A fundamental inequality

The following result will play a key role in deriving the main results in Sections 4 and 6:

(3.1) Lemma *If $\nu \in \mathcal{N}[E]$ is irreducible, there exists an increasing function $c : E \to \mathbf{R}_+$ such that*

$$\sum_{n \ge 0} \mathbf{E}(f(X_n^x) - f(X_n^0)) \le c(x) \sup_{n \ge 0} \mathbf{E}(f(X_n^0))$$

for each increasing function $f : E \to \mathbf{R}_+$ and every $x \in E$.

Proof. First of all, the left-hand side is well-defined, because the differences $f(X_n^x) - f(X_n^0)$ are nonnegative. Now, fix $x \in E$ and choose $k \in \mathbf{N}$ such that

$$\mathcal{H}' := \{h' \in \mathcal{H}[E] : h'(0) \ge x\}$$

satisfies the condition

$$\gamma := \nu^k(\mathcal{H}') = \mathbf{P}(X_k^0 \ge x) > 0\,.$$

With the abbreviation $\mathcal{H} := \mathcal{H}[E]$ this yields the bound

$$\begin{aligned}
\mathbf{E}(f(X_n^x) - f(X_n^0)) &= \int_{\mathcal{H}} [f(h(x)) - f(h(0))]\ \nu^n(dh) \\
&= \gamma^{-1} \int_{h' \in \mathcal{H}'} \int_{h \in \mathcal{H}} [\ldots]\ \nu^n(dh)\, \nu^k(dh') \\
&\le \gamma^{-1} \int_{h' \in \mathcal{H}'} \int_{h \in \mathcal{H}} [f(h \circ h'(0)) - f(h(0))]\ \nu^n(dh)\, \nu^k(dh') \\
&\le \gamma^{-1} \int_{h' \in \mathcal{H}} \int_{h \in \mathcal{H}} [\ldots]\ \nu^n(dh)\, \nu^k(dh') \\
&= \gamma^{-1}\, \mathbf{E}(f(X_{n+k}^0) - f(X_n^0)) \quad \text{for } \ n \ge 0\,,
\end{aligned}$$

where the first inequality follows from

$$f(h \circ h'(0)) \geq f(h(x)) \quad \text{for} \quad h \in \mathcal{H}, h' \in \mathcal{H}'$$

and the second one from

$$f(h \circ h'(0)) \geq f(h(0)) \quad \text{for} \quad h \in \mathcal{H}, h' \in \mathcal{H} \setminus \mathcal{H}' .$$

By summing up, cancelling on the right-hand side, and omitting the term $\sum_{0 \leq l < k} f(X_l^0) \geq 0$ this yields

$$\begin{aligned} \sum_{0 \leq l < m} \mathbf{E}(f(X_l^x) - f(X_l^0)) &\leq \gamma^{-1} \sum_{m \leq l < m+k} \mathbf{E}(f(X_l^0)) \\ &\leq k\gamma^{-1} \sup_{n \geq 0} \mathbf{E}(f(X_n^0)) \quad \text{for all} \quad m \in \mathbf{N} . \end{aligned}$$

According to the choice of k and the definition of γ, therefore,

$$c(x) := (\sup_{k \in \mathbf{N}} \frac{1}{k} \mathbf{P}(X_k^0 \geq x))^{-1} \quad \text{for} \quad x \in E$$

defines an increasing function as desired. □

The next result is an immediate consequence:

(3.2) Lemma *If $\nu \in \mathcal{N}[E]$ is irreducible, then for arbitrary initial law and each increasing function $f : E \to \mathbf{R}_+$ the condition*

$$\sup_{n \geq 0} \mathbf{E}(f(X_n^0)) < \infty$$

implies

(a) $$\sum_{n \geq 0} (f(X_n) - f(X_n^0)) < \infty \quad a.s. ,$$

(b) $$f(X_n) - f(X_n^0) \to 0 \quad a.s. .$$

Proof. The second assertion follows from the first one, which in turn follows in the special case $X_0 = x$ from (3.1), interchanging the order of summation and integration, and in the general case by applying Fubini. □

Actually, the preceding results extend to larger classes of functions:

(3.3) Theorem *If $\nu \in \mathcal{N}[E]$ is irreducible, then for arbitrary initial law*

(a) $$\sum_{n \geq 0} |f(X_n) - f(X_n^0)| < \infty \quad a.s. \quad \textit{for} \quad f \in \mathcal{V}(E) ,$$

(b) $$f(X_n) - f(X_n^0) \to 0 \quad a.s. \quad \textit{for} \quad f \in \mathcal{R}(E) .$$

Proof. (a) A function $f \in \mathcal{V}(E)$ has a representation $f = f_1 - f_2$ with increasing and bounded functions $f_i : E \to \mathbf{R}_+$, for which (3.2a) applies.

(b) The assertion follows for step functions f from (a) and for functions $f \in \mathcal{R}(E)$ by uniform approximation. □

By inspecting the proof it is easily noticed that in both statements the convergence is uniform on compact subsets of E, i.e. for instance

$$\sup_{0 \le x \le t} |f(X_n^x) - f(X_n^0)| \to 0 \quad \text{a.s.} \quad \text{for} \quad f \in \mathcal{R}(E) \quad \text{and} \quad t \in E\,.$$

Assertion (b) cannot be extended to functions $f \in \mathcal{C}(E)$, as is seen in the transient case already by the deterministic system (E, h) with $E = \mathbf{R}_+$ and $h(x) = x + 1$, which is obviously irreducible: if $f(x)$ denotes the euclidean distance of x from the set $\{0, 2, 4, \ldots\}$, then $|f(X_n^1) - f(X_n^0)| = 1$ for all $n \ge 0$.

To exhibit a counterexample in the recurrent case, a more elaborate construction is required. To this end consider the autoregressive process

$$X_n = \frac{1}{2} X_{n-1} + V_n \quad \text{for} \quad n \in \mathbf{N}\,,$$

where the i.i.d. variables $V_n, n \in \mathbf{N}$, attain the values 0 and $\frac{1}{2}$ each with probability $\frac{1}{2}$. With $E = [0, 1[$ as state space the corresponding distribution ν is supported by the two mappings $h_0 : x \to x/2$ and $h_1 : x \to (x+1)/2$. It is clearly irreducible and by (2.2) recurrent, where $\underline{x} = 0$ by (1.8). Denote by $(t_k)_{k \ge 0}$ a strictly increasing sequence in E with $\sup_{k \ge 0} t_k = 1$, to be specified later, and define

$$T_k := \inf\{n \ge 0 : X_n^0 \ge t_k\} \quad \text{for} \quad k \ge 0\,.$$

Since these random times are almost surely finite, there exist $n_k \in \mathbf{N}$ such that

$$\text{(1)} \qquad \limsup_{k \to \infty} \mathbf{P}(T_k \le n_k) = 1\,.$$

Moreover, since the support N of ν is finite, there exist finite sets $B_k \subset [t_k, 1[$ such that

$$\text{(2)} \qquad \mathbf{P}(T_k \le n_k, X_{T_k}^0 \notin B_k) = 0 \quad \text{for} \quad k \ge 0\,.$$

Finally, since the mappings h_i leave the set D of dyadic numbers in E and its complement invariant, for fixed $x \in E \setminus D$ there exist finite sets $C_k \subset [t_k, 1[$ such that

$$\text{(3)} \qquad \mathbf{P}(T_k \le n_k, X_{T_k}^x \notin C_k) = 0 \quad \text{for} \quad k \ge 0\,,$$

satisfying in addition $B_k \cap C_k = \emptyset$. Now, starting with $t_0 = 0$, choose the levels t_k recursively under the constraints $t_k > \max C_{k-1}$ for $k \in \mathbf{N}$. Then the finite sets B_k and C_k are disjoint subsets of the successive intervals $[t_k, t_{k+1}[, k \ge 0$, with union E. Thus there exists a function $f \in \mathcal{C}(E)$ with $0 \le f \le 1$, satisfying

$f(y) = 0$ for $y \in \bigcup_{k\geq 0} B_k$ and $f(y) = 1$ for $y \in \bigcup_{k\geq 0} C_k$. By (2) and (3) this implies

$$\limsup_{k\to\infty} \{T_k \leq n_k\} \subset \{\limsup_{n\to\infty} |f(X_n^x) - f(X_n^0)| = 1\}$$

almost surely, which by (1) yields

$$\limsup_{n\to\infty} |f(X_n^x) - f(X_n^0)| = 1 \quad \text{a.s.}\,,$$

providing the desired counterexample.

As a corollary assertion (b) of (3.3) yields a result on functions f that are regular with respect to P, i.e. satisfy $0 \leq f = Pf$: in the irreducible case such a function, provided it is contained in $\mathcal{R}(E)$, has to be constant. Indeed, the equation $P^n f = f$ for $n \geq 0$ and the boundedness of f combine by (3.3b) to

$$\begin{aligned} f(x) - f(0) &= \mathbf{E}(f(X_n^x)) - \mathbf{E}(f(X_n^0)) \\ &= \mathbf{E}(f(X_n^x) - f(X_n^0)) \\ &\to 0 \quad \text{for all } \ x \in E\,. \end{aligned}$$

To see that outside $\mathcal{R}(E)$ there may exist regular functions that are bounded and not constant, consider the autoregressive process from above. Since in this case the requirement $f = Pf$ amounts to the equation

$$f(x) = \frac{1}{2}\Big(f(\frac{x}{2}) + f(\frac{x+1}{2})\Big) \quad \text{for } \ x \in E\,,$$

for instance $f = 1_D$ with D again denoting the set of dyadic numbers in E is a solution that is not constant.

This section is concluded by a technical result that will be important in the sequel. It implies in particular for intervals $I \subset E$, visited from 0 infinitely often almost surely, and compact subsets K of E the existence of $n \in \mathbf{N}$ such that for $x \in K$ the hitting probabilities $\mathbf{P}(X_n^x \in I)$ are bounded away from 0. Actually, this result can be strengthened and extended to functions:

(3.4) Lemma *Let $\nu \in \mathcal{N}[E]$ be irreducible and $f \in \mathcal{V}(E)$ satisfy*

$$f \geq 0 \quad \textit{and} \quad \sum_{n\geq 0} \mathbf{E}(f(X_n^0)) = \infty\,.$$

Then for every $t \in E$ there exists $m \in \mathbf{N}$ such that

$$\mathbf{E}(\inf_{0\leq x\leq t} f(X_m^x)) > 0\,.$$

Proof. First, measurability is ensured, because it suffices to extend the infimum over a countable dense subset of $[0, t]$, containing the countably many points of discontinuity of f. A representation $f = f_1 - f_2$ with increasing and bounded functions $f_i : E \to \mathbf{R}_+$ yields the estimate

$$\mathbf{E}(\inf_{0\leq x\leq t} f(X_n^x)) \geq \mathbf{E}(f_1(X_n^0)) - \mathbf{E}(f_2(X_n^t)) =: \delta_n\,.$$

Now the identity

$$\delta_n = \mathbf{E}(f(X_n^0)) - (\mathbf{E}(f_2(X_n^t)) - \mathbf{E}(f_2(X_n^0)))$$

ensures the existence of some $m \in \mathbf{N}$ with $\delta_m > 0$, because otherwise by (3.1)

$$\sum_{n \geq 0} \mathbf{E}(f(X_n^0)) \leq \sum_{n \geq 0} (\mathbf{E}(f_2(X_n^t)) - \mathbf{E}(f_2(X_n^0))) < \infty ,$$

contradicting the assumption on f. □

4. Existence and uniqueness of invariant measures

To deal with not necessarily finite invariant measures, the following "localization" is essential:

(4.1) Definition *Let $\nu \in \mathcal{N}[E]$ be recurrent and fix $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$. Then:*

(a) ${}^tP$ denotes the "hitting kernel" belonging to ν and $[0, t]$, i.e.

$${}^tP(x; B) := \mathbf{P}^{\,x}(X_T \in B) \quad \textit{for} \;\; x \in [0, t] \quad \textit{and} \;\; B \in \mathcal{B}([0, t]) ,$$

where

$$T := \inf\{n \in \mathbf{N} : X_n \in [0, t]\} ;$$

(b) for arbitrary initial law $({}^tX_n)_{n \geq 0}$ denotes the "embedded process" belonging to $(X_n)_{n \geq 0}$ and $[0, t]$, i.e.

$${}^tX_n := X_{T_n} \quad \textit{for} \;\; n \geq 0 ,$$

where $T_0 < T_1 < \ldots$ are the random times when $(X_n)_{n \geq 0}$ is in $[0, t]$.

To include the case $t = \overline{x} \in E$, where ${}^tP = P$ and no localization is necessary, is convenient for a unified treatment. To assume $T_n < \infty$ in (b) is no real restriction.

For easy reference the required facts from probabilistic potential theory are stated explicitly:

(4.2) Lemma *Let $\nu \in \mathcal{N}[E]$ be recurrent and $\mu \in \mathcal{M}(E)$ be excessive with respect to P. If ${}^t\mu$ denotes the restriction of μ to $[0, t]$, then*

(a) ${}^t\mu$ is invariant with respect to ${}^tP$ for $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$,

(b) μ is invariant with respect to P.

Proof. (a) If I_A for $A \in \mathcal{B}(E)$ denotes the kernel

$$I_A(x; \cdot) := 1_A(x)\, \varepsilon_x \quad \text{for} \;\; x \in E ,$$

the crucial point is the inequality

$$\mu I_A \sum_{n \geq 0} (P I_{E \setminus A})^n P \leq \mu P$$

for excessive measures, which follows as the dual result for excessive functions (see e.g. Proposition 2.2.6 in [26]). Applied to $A = [0,t]$ this inequality concerns ${}^tP$ and implies

$$(*) \qquad ({}^t\mu\, {}^tP)(B) \le (\mu P)(B) \quad \text{for} \quad B \in \mathcal{B}([0,t])\,.$$

Therefore $\mu P \le \mu$ yields ${}^t\mu\, {}^tP \le {}^t\mu$ proving (a), because ${}^t\mu$ is a finite measure and ${}^tP$ is a stochastic kernel.

(b) For $0 \le f \in \mathcal{K}(E)$ choose $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$ and supp $f \subset [0,t]$ and denote the restriction of f to $[0,t]$ by ${}^tf$. Then (a) and $(*)$ together imply

$$\mu f = {}^t\mu\, {}^tf = ({}^t\mu\, {}^tP)\, {}^tf \le (\mu P) f\,.$$

By varying f this yields the inequality $\mu \le \mu P$ required for (b). □

Now the existence of an invariant measure for ν (i.e. for P) can be established as in [10], under some simplification due to the monotonicity. More generally, the following version will be needed in Section 6:

(4.3) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent and fix $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$. Then for arbitrary initial law the measures*

$$\varrho_n(B) := \sum_{0 \le m < n} \mathbf{P}(X_m \in B) \Big/ \sum_{0 \le m < n} \mathbf{P}(X_m \le t) \quad \textit{for} \quad B \in \mathcal{B}(E)\,,$$

defined for $n \ge n_0$ according to (2.4a)*, satisfy:*

(a) *$\{\varrho_n : n \ge n_0\}$ is a sequentially compact subset of $\mathcal{M}(E)$,*

(b) *each limit point $\mu \in \mathcal{M}(E)$ of the sequence $(\varrho_n)_{n \ge n_0}$ is a nontrivial invariant measure for ν.*

Proof. (a) For arbitrary $s \in E$ choose $l \in \mathbf{N}$ such that $\vartheta := \mathbf{P}(X_l^s \le t) > 0$, which is possible due to the assumption on t. With $\mu_m := \mathcal{L}(X_m)$ this implies by monotonicity

$$\begin{aligned} \mathbf{P}(X_{m+l} \le t) &\ge \int_{x \le s} \mathbf{P}(X_l^x \le t)\, \mu_m(dx) \\ &\ge \int_{x \le s} \mathbf{P}(X_l^s \le t)\, \mu_m(dx) \\ &= \vartheta\, \mathbf{P}(X_m \le s) \quad \text{for} \quad m \ge 0\,. \end{aligned}$$

With the norming constants

$$r_n := \sum_{0 \le m < n} \mathbf{P}(X_m \le t) \quad \text{for} \quad n \ge 0$$

this provides the estimate

$$\begin{aligned} \sum_{0 \le m < n} \mathbf{P}(X_m \le s) &\le \vartheta^{-1} \sum_{0 \le m < n} \mathbf{P}(X_{m+l} \le t) \\ &\le \vartheta^{-1}\, (r_n + l)\,. \end{aligned}$$

Since $r_n \to \infty$ by (2.4a), this yields

$$\limsup_{n\to\infty} \varrho_n([0,s]) \le \vartheta^{-1} < \infty\,.$$

This being true for all $s \in E$, the measures $\varrho_n, n \ge n_0$, are uniformly locally finite. The asserted compactness follows by considering the associated linear functionals on $\mathcal{K}(E)$ and applying the Riesz representation theorem.

(b) The assumption $\varrho_{n_k} \underset{v}{\to} \mu$ yields $\mu \ne 0$, because

$$\mu([0,t]) \ge \limsup_{k\to\infty} \varrho_{n_k}([0,t]) = 1\,.$$

With $\mu_0 := \mathcal{L}(X_0)$, moreover,

$$\varrho_{n_k} f = r_{n_k}^{-1} \sum_{0\le m<n_k} \mu_0\, P^m f \quad \text{for} \quad 0 \le f \in \mathcal{K}(E)\,.$$

By approximation of $Pf \in \mathcal{C}(E)$ from below this yields

$$\begin{aligned} \mu\, Pf &\le \liminf_{k\to\infty} \varrho_{n_k}\, Pf \\ &= \liminf_{k\to\infty} r_{n_k}^{-1} \sum_{0<m\le n_k} \mu_0 P^m f \\ &= \liminf_{k\to\infty} r_{n_k}^{-1} \sum_{0\le m<n_k} \mu_0 P^m f \\ &= \mu f \quad \text{for} \quad 0 \le f \in \mathcal{K}(E)\,, \end{aligned}$$

because $r_{n_k} \to \infty$ and $\mu_0 P^m f$ is bounded by $\max f$. Therefore μ is excessive and thus invariant by (4.2b). □

The following property of the support of invariant measures will be required before its thorough study in Section 7:

(4.4) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent and $\mu \in \mathcal{M}(E)$ be a nontrivial invariant measure for ν. Then $M := \operatorname{supp} \mu$ satisfies*

$$\inf M = \underline{x} \quad \textit{and} \quad \sup M = \overline{x}\,.$$

Proof. 1. Since $\mu = \mu P^n$ by assumption, $\underline{m} := \inf M$ satisfies

$$0 = \mu([0, \underline{m}[) = \int_E \mathbf{P}(X_n^x < \underline{m})\, \mu(dx)\,,$$

which for fixed $n \ge 0$ implies

$$\mathbf{P}(X_n^x \ge \underline{m}) = 1 \quad \text{for} \quad \mu\text{-almost all } x \in E\,.$$

For any x satisfying this equation simultaneously for all $n \ge 0$, by (1.6)

$$\underline{x} = \liminf_{n\to\infty} X_n^x \ge \underline{m} \text{ (a.s.)}\,.$$

2. To prove the inverse inequality, observe first

$$\mathbf{P}(X_n^x < \underline{x}) = \nu^n(h(x) < \underline{x}) = 0 \quad \text{for} \quad x \geq \underline{x} \quad \text{and} \quad n \in \mathbf{N}$$

according to (2.5) and (1.8). Therefore

$$\begin{aligned} \mu([0,t]) &= \int_E \nu^n(h(x) \leq t)\, \mu(dx) \\ &= \int_{x<\underline{x}} \nu^n(h(x) \leq t)\, \mu(dx) \quad \text{for} \quad t < \underline{x}\,. \end{aligned}$$

Since μ is locally finite, the dual form of Fatou's lemma applies and leads to

$$\begin{aligned} \mu([0,t]) &= (\limsup_{n\to\infty}) \int_{x<\underline{x}} \mathbf{P}(X_n^x \leq t)\, \mu(dx) \\ &\leq \int_{x<\underline{x}} \limsup_{n\to\infty} \mathbf{P}(X_n^x \leq t)\, \mu(dx) \\ &\leq \int_{x<\underline{x}} \mathbf{P}(\limsup_{n\to\infty} \{X_n^x \leq t\})\, \mu(dx) \\ &\leq \int_{x<\underline{x}} \mathbf{P}(\liminf_{n\to\infty} X_n^x \leq t)\, \mu(dx) \\ &= 0 \quad \text{for} \quad t < \underline{x}\,. \end{aligned}$$

For $t \uparrow \underline{x}$ this implies $\underline{m} \geq \underline{x}$.

3. The inequality $\overline{m} \geq \overline{x}$ for $\overline{m} := \sup M$ follows as in part 1 of the proof, replacing (1.6) by (1.5), while the inverse inequality holds by the definition of $\overline{x}$. □

Now uniqueness of the invariant measure can be established. At the first step (3.3) is crucial:

(4.5) Lemma *Let $\nu \in \mathcal{N}[E]$ be recurrent and fix $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$. For $f \in \mathcal{V}(E)$ with supp $f \subset [0,t]$ and $g = 1_{[0,t]}$ define*

$$Q_n^x(f,g) := \sum_{0\leq m<n} f(X_m^x) \Big/ \sum_{0\leq m<n} g(X_m^x) \quad \textit{for} \quad x \in E\,.$$

Then there exist constants $\underline{q}(f,g)$ and $\overline{q}(f,g)$ such that

$$\begin{aligned} \liminf_{n\to\infty} Q_n^x(f,g) &= \underline{q}(f,g) \quad \textit{a.s.} \quad \textit{and} \\ \limsup_{n\to\infty} Q_n^x(f,g) &= \overline{q}(f,g) \quad \textit{a.s.} \quad \textit{for every } x \in E\,. \end{aligned}$$

Proof. 1. The limits on the left-hand side are well-defined according to the choice of t. Moreover, they are independent of x. Indeed:
(1) In the denominator x may be replaced by 0, because the quotient

$$\sum_{0\leq m<n} g(X_m^x) \Big/ \sum_{0\leq m<n} g(X_m^0) = 1 + \Big(\sum_{0\leq m<n} (g(X_m^x) - g(X_m^0)) \Big/ \sum_{0\leq m<n} g(X_m^0)\Big)$$

tends to 1 almost surely by (3.3a) and the assumption on t.
(2) In the nominator x may be replaced by 0, because the difference

$$\sum_{0\le m<n} f(X_m^x) \Big/ \sum_{0\le m<n} g(X_m^0) \;-\; \sum_{0\le m<n} f(X_m^0) \Big/ \sum_{0\le m<n} g(X_m^0)$$
$$= \sum_{0\le m<n} \left(f(X_m^x) - f(X_m^0)\right) \Big/ \sum_{0\le m<n} g(X_m^0)$$

tends to 0 almost surely for the same reason.

2. Since the summands with $m = 0$ may be neglected, using the abbreviation

$$[\ldots] := H_m \circ \ldots \circ H_2(H_1(0)),$$

the lower limit equals

$$\liminf_{n\to\infty} Q_n^0(f,g) = \liminf_{n\to\infty} \sum_{1\le m<n} f([\ldots]) \Big/ \sum_{1\le m<n} g([\ldots]).$$

Therefore by part 1 of the proof, applying Fubini, the arguments $[\ldots]$ of f and g can be replaced by $H_m \circ \ldots \circ H_2(0)$ up to a set of probability 0. Continuing, it follows that the lower limit is measurable with respect to the completed tail σ-field of $(H_n)_{n\in\mathbf{N}}$. Thus it is almost surely a constant – and the same holds for the upper limit. □

At the next step the pointwise ergodic theorem enters:

(4.6) Lemma *Let $\nu \in \mathcal{N}[E]$ be recurrent and $\mu \in \mathcal{M}(E)$ be a nontrivial invariant measure for ν. Then, in continuation of* (4.5)*,*

$$\underline{q}(f,g) = \int_E f\,d\mu \,/\, \mu([0,t]) = \overline{q}(f,g).$$

Proof. Since $\mu([0,t])$ is finite and by (4.4) strictly positive, the restriction ${}^t\mu$ of μ to $[0,t]$ may be assumed to be normalized, hence by (4.2a) to be a stationary distribution for ${}^tP$. If now X_0 is distributed according to (the trivial extension of) ${}^t\mu$, then $({}^tX_n)_{n\ge 0}$ is a stationary process. Since f is bounded, the classical ergodic theorem therefore ensures that

$$Q(f,g) := \lim_{n\to\infty} \frac{1}{n} \sum_{0\le m<n} f({}^tX_m)$$

exists almost surely. Now on $\{X_0 = x\}$ the sequence $(Q_n^x(f,g))_{n\ge 0}$ arises from the successive means of $(f({}^tX_n))_{n\ge 0}$ through "extension to the right by constancy" in an evident sense. Therefore by (4.5), applying once more Fubini,

$$\underline{q}(f,g) = Q(f,g) = \overline{q}(f,g) \quad \text{a.s.}.$$

Again the stationarity of $({}^tX_n)_{n\ge 0}$ and the boundedness of f imply

$$\mathbf{E}(Q(f,g)) = \int_E f\,d\mu,$$

and the assertion follows. □

Now one of the principal results can be established:

(4.7) Theorem *For each recurrent distribution $\nu \in \mathcal{N}[E]$ there exists a nontrivial invariant measure $\mu \in \mathcal{M}(E)$ such that every excessive measure $\mu' \in \mathcal{M}(E)$ is a multiple of μ.*

Proof. Choose μ according to (4.3) and let $\mu' \in \mathcal{M}(E)$ be another nontrivial excessive, hence by (4.2b) invariant, measure. Then for t and f, g satisfying the assumptions of (4.5), according to (4.6),

$$\int_E f\, d\mu \,/\, \mu([0,t]) = \int_E f\, d\mu' \,/\, \mu'([0,t])\,.$$

Specialized to $f = 1_{[0,s]}\,, s \leq t$, this provides a constant γ_t such that

$$\mu'(B) = \gamma_t\, \mu(B) \quad \text{for} \quad B \in \mathcal{B}([0,t])\,,$$

where in fact γ_t, due to $\mu'([0,t]) \neq 0$, is independent of t. □

The measure μ that this theorem assigns to a recurrent distribution ν actually stands for a one-dimensional family. Nevertheless it will be called "the invariant measure" in the sequel.

At this point it has to be emphasized that the uniqueness statement in (4.7) concerns locally finite measures only. To exhibit an example, choose $E = [0,1[$ and let ν assign mass $\frac{1}{2}$ to the two mappings defined by

$$h_1(x) = \frac{1}{3}\,(2x+1) \quad \text{and} \quad h_2(x) = \frac{1}{2}\,x \vee (2x-1)\,.$$

Then ν is clearly irreducible and the uniform distribution μ on E is easily seen to be a finite invariant measure. With μ as initial law the series $\sum_{n\geq 0} \mathbf{P}(X_n \leq t)$ diverges for all $t > 0$ and thus ν is recurrent by (2.4b). On the other hand, it is not hard to check that

$$\mu' := \sum_{x \in D} x\, \varepsilon_x \quad \text{with } D := E \cap \mathbf{Q}$$

defines another invariant measure, which, however, is σ-finite only.

Finally, it has to be mentioned that – as in discrete Markov chain theory – neither existence nor uniqueness of nontrivial locally finite invariant measures carry over to the transient case.

5. The regenerative case

Clearly, a process $(X_n)_{n\geq 0}$ transient according to (2.1) cannot be recurrent in the sense of Harris, due to $X_n \to \overline{x} \notin E$ a.s.. The counterexample at the end of the preceding section implies, on the other hand, that a process $(X_n)_{n\geq 0}$ recurrent according to (2.1) need not be recurrent in the restricted sense, because in this case there is always an – up to a constant factor – unique σ-finite invariant measure (see e.g. [23, 26]). There is, however, a particular situation that fits into this framework and will be needed in the following section. This "regenerative case" makes no use of the topological structure and arises, if the invariant measure contains atoms. The crucial consequence of this assumption is the following fact:

(5.1) Lemma *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ. Then $\mu(\{z\}) > 0$ implies for arbitrary initial law*

$$\mathbf{P}(X_n = z \text{ infinitely often}) = 1\,.$$

Proof. An application of (4.5) and (4.6) to $t = z$ and $f = 1_{\{z\}}$ yields

$$\sum_{0 \le m < n} 1_{\{z\}}(X_m^x) \Big/ \sum_{0 \le m < n} g(X_m^x) \;\to\; \mu(\{z\})/\mu([0,z]) \quad \text{a.s.}$$

for every $x \in E$, which verifies the assertion by applying Fubini. □

A question left aside in the preceding section is how to determine the invariant measure μ for a recurrent distribution ν. There is, in general, no chance to solve the integral equation

$$\mu(B) = \int_E \mu(h(x) \in B)\, \nu(dh) \quad \text{for} \quad B \in \mathcal{B}(E)$$

explicitly. An exception, however, is provided by the regenerative case as can be deduced from the Harris theory. Since in the order context the proof of the relevant result is especially simple, it is carried out:

(5.2) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ. Then $\mu(\{z\}) > 0$ implies*

$$\mu(B) = \mu(\{z\})\ \mathbf{E}^z\Big(\sum_{0 \le n < T_z} 1_B(X_n)\Big) \quad \textit{for} \quad B \in \mathcal{B}(E)\,,$$

where T_z denotes the hitting time of z by $(X_n)_{n \ge 0}$.

Proof. 1. To establish first the invariance of the measure μ' defined by the expectation on the right-hand side, choose $f = 1_{[0,t]}$ with $t \in E$. Since T_z by (5.1) is almost surely finite, the condition $X_0 = z$ yields

$$\begin{aligned}
\mu' P f &= \mathbf{E}^z\Big(\sum_{0 \le n < T_z} f(X_{n+1})\Big) \\
&= \mathbf{E}^z\Big(\sum_{0 < n \le T_z} f(X_n)\Big) \\
&= \mathbf{E}^z\Big(\sum_{0 \le n < T_z} f(X_n)\Big) \;=\; \mu' f\,.
\end{aligned}$$

Since $t \in E$ is arbitrary, this implies $\mu' P = \mu'$. In view of (4.7) and $\mu'(B) = 1$ for $B = \{z\}$ it remains to show μ' to be locally finite.

2. To this end consider the function $f = 1_{\{z\}}$ and observe that by (5.1) the condition $\sum_{n\geq 0} \mathbf{E}(f(X_n^0)) = \infty$ in (3.4) is satisfied. Thus for $z \leq t \in E$ there exists $n \in \mathbf{N}$ such that

$$\vartheta := \inf_{0\leq x\leq t} \mathbf{P}(X_m^x = z) > 0\,.$$

By the Markov property this implies

$$\begin{aligned}
\mu'([0,t]) &= \mathbf{E}^z\Big(\sum_{n\geq 0} 1_{\{T_z>n\}} 1_{[0,t]}(X_n)\Big) \\
&= \sum_{n\geq 0} \mathbf{P}^{\,z}({}^tX_k \neq z \text{ for } 0<k\leq n) \\
&\leq m\sum_{l\geq 0} \mathbf{P}^{\,z}({}^tX_k \neq z \text{ for } 0<k\leq lm) \\
&\leq m\sum_{l\geq 0} \mathbf{P}^{\,z}({}^tX_{km} \neq z \text{ for } 0<k\leq l) \\
&\leq m\sum_{l\geq 0} (1-\vartheta)^l \ < \ \infty\,,
\end{aligned}$$

i.e. the measure μ' is locally finite. □

In the regenerative case the initial law disappears in a strong sense:

(5.3) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ. Then $\mu(\{z\}) > 0$ implies for arbitrary initial law*

$$\mathbf{P}(X_n = X_n^0 \ \textit{eventually}) = 1\,.$$

Proof. Application of (3.3b) to $f = 1_{\{z\}}$ yields

$$\mathbf{P}(1_{\{z\}}(X_n) = 1_{\{z\}}(X_n^0) \ \text{eventually}) = 1\,.$$

Combined with (5.1) this implies that the random time

$$T := \inf\{n \in \mathbf{N} : X_n = X_n^0 = z\}$$

is almost surely finite, where clearly $X_n = X_n^0$ for $n \geq T$. □

A first consequence of this result concerns the tail events:

(5.4) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and $\mu(\{z\}) > 0$. Then for arbitrary initial law the tail σ-field of $(X_n)_{n\geq 0}$ is trivial.*

Proof. Let B be a Borel subset of $\prod_{n\geq 0} E$, not depending on any finite number of coordinates. Then an application of (5.3) to $n \in \mathbf{N}$ as starting time shows that the two events

$$A := \{(X_0, X_1, \ldots) \in B\} = \{(X_0, \ldots, X_n, H_{n+1}(X_n), \ldots) \in B\}$$

and

$$A' := \{(0, \ldots, 0, H_{n+1}(0), \ldots) \in B\}$$

agree almost surely, hence A is contained in the completed tail σ-field of $(H_n)_{n\in\mathbf{N}}$ as well and thus has probability 0 or 1. □

Another consequence of (3.4) is the following aperiodicity:

(5.5) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and $\mu(\{z\}) > 0$. Then there exists $n_0 \in \mathbf{N}$ such that*

$$\mathbf{P}(X_n^z = z) > 0 \quad \text{for} \quad n \geq n_0 \,.$$

Proof. As in the proof of (5.2) choose $m \in \mathbf{N}$ satisfying

$$\vartheta := \inf_{0\leq x\leq z} \mathbf{P}(X_m^x = z) > 0 \,.$$

With $\mu_1 := \mathcal{L}(X_1^z)$ it follows that

$$\begin{aligned} \mathbf{P}(X_{m+1}^z = z) &\geq \int_{x\leq z} \mathbf{P}(X_m^z = z)\, \mu_1(dx) \\ &\geq \vartheta\, \mathbf{P}(X_1^z \leq z) \;>\; 0 \,, \end{aligned}$$

because $\mathbf{P}(X_1^z \leq z) = 0$ by iteration would lead to $X_n^z > z$ for all $n \in \mathbf{N}$ almost surely, contradicting (5.1). From $\mathbf{P}(X_n^z = z) > 0$ for $n = m, m+1$ it follows, combining k periods of length m and l periods of length $m+1$, that

$$\mathbf{P}(X_n^z = z) > 0 \quad \text{for} \quad n = km + l(m+1) \,,$$

hence $n_0 = (m-1)m$ satisfies the assertion. □

This result implies in particular that z is a fixed point of some h in the semigroup generated by the support N of ν. While this property is clearly not sufficient for $\mu(\{z\}) > 0$, the following criterion holds:

(5.6) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ. Then for $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$ the following assertions are equivalent:*

(a) $$\mu(\{z\}) > 0 \,,$$

(b) $$\mathbf{P}(X_n^x = z \text{ for } 0 \leq x \leq t) > 0 \quad \text{for} \quad \text{some } n \in \mathbf{N} \,.$$

Proof. 1. If (a) is satisfied, combination of (5.1) and (5.3) provides $n \in \mathbf{N}$ such that

$$\mathbf{P}(X_n^0 = z = X_n^t) > 0 \,,$$

which by monotonicity agrees with the probability in (b).

2. If (b) is satisfied, then the estimate

$$\begin{aligned}\mu(\{z\}) &= \int_E \mathbf{P}(X_n^x = z)\, \mu(dx) \\ &\geq \mathbf{P}(X_n^x = z \text{ for } 0 \leq x \leq t)\, \mu([0,t])\end{aligned}$$

yields $\mu(\{z\}) > 0$, because $\mu([0,t]) > 0$ by (4.4). □

It is a consequence of this equivalence that $\overline{x} \in E$ implies $\mu(\{\overline{x}\}) > 0$. Indeed, in this case ν is recurrent and by definition there exists $n \in \mathbf{N}$ such that $\mathbf{P}(X_n^0 = \overline{x}) > 0$, hence due to $X_n^0 \leq X_n^x \leq \overline{x}$ for all $x \in E$ condition (b) is satisfied.

For the boundary values the criterion (5.6) simplifies:

(5.7) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and assume $\underline{x} < \overline{x}$. Then*

(a) $\mu(\{\underline{x}\}) > 0$ if and only if $\nu(h(x) = \underline{x}) > 0$ for some $x > \underline{x}$,

(b) $\mu(\{\overline{x}\}) > 0$ if and only if $\nu(h(x) = \overline{x}) > 0$ for some $x < \overline{x}$.

Proof. (a) The function

$$g(x) := \nu(h(x) = \underline{x}) \quad \text{for} \quad x \in E$$

decreases for $x \geq \underline{x}$, because $h(\underline{x}) \underset{\nu}{\geq} \underline{x}$ by (1.8), and satisfies $g(\underline{x}) < 1$, because otherwise $\underline{x} = \overline{x}$ by (1.9). Moreover, by (4.4)

$$\mu(\{\underline{x}\}) = \int_{x \geq \underline{x}} g(x)\, \mu(dx)\,.$$

Thus $\mu(\{\underline{x}\}) > 0$ implies $g(x) > 0$ for some $x > \underline{x}$, while $\mu(\{\underline{x}\}) = 0$ implies $g(x) = 0$ for $\mu-$almost all $x > \underline{x}$, hence for all $x > \underline{x}$ by (4.4).

(b) Under each condition $\overline{x}$ belongs to E, because $\nu(h(x) = \overline{x}) > 0$ for some $x < \overline{x}$ implies $\overline{x} \in E$ by (1.1). Therefore – with obvious modifications – the proof of (a) carries over to (b). □

A typical example for (a) is the queuing process, where in case of recurrence always $\underline{x} = 0$ and $\mu(\{\underline{x}\}) > 0$.

6. Ratio ergodic theorems

To derive ergodic theorems that are not restricted to continuous functions of the process $(X_n)_{n \geq 0}$, some preparations are necessary:

(6.1) Lemma *If $\nu \in \mathcal{N}[E]$ is recurrent, then for $\underline{x} < t \in E$ and arbitrary initial law*

$$\sum_{0 \leq m < n} \mathbf{P}(X_m \leq t) \Big/ \sum_{0 \leq m < n} \mathbf{P}(X_m^0 \leq t) \;\to\; 1\,.$$

Proof. 1. With the notations $g := 1_{[0,t]}$ and $f := 1 - g$ the assertion holds in the case $X_0 = s$, because

$$\sum_{0 \le m < n} \mathbf{P}(X_m^s \le t) \Big/ \sum_{0 \le m < n} \mathbf{P}(X_m^0 \le t)$$

$$= 1 - \Big(\sum_{0 \le m < n} (\mathbf{E}(f(X_m^s)) - \mathbf{E}(f(X_m^0))) \Big/ \sum_{0 \le m < n} \mathbf{P}(X_m^0 \le t) \Big) ,$$

where the last quotient tends to 0 by (3.1) and (2.4a).

2. For arbitrary $\mu_0 := \mathcal{L}(X_0)$ and $s \in E$ the estimate

$$\mathbf{E}(g(X_m^0)) \ge \mathbf{E}(g(X_m)) \ge \mathbf{E}(g(X_m^s))\, \mu_0([0,s])$$

holds, because the function g decreases. Summation over $0 \le m < n$ and division yields for the ratios r_n under consideration the bounds

$$\limsup_{n \to \infty} r_n \le 1 , \tag{1}$$

$$\liminf_{n \to \infty} r_n \ge \mu_0([0,s]) , \tag{2}$$

where (1) uses part 1 of the proof. The assertion now follows for $s \uparrow \overline{x}$ (or $s = \overline{x}$). $\square$

The next step relies on results for the regenerative case:

(6.2) Lemma *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and $\mu(\{z\}) > 0$. Then for $\underline{x} < t \in E$ and arbitrary initial law*

$$\sum_{0 \le m < n} \mathbf{P}(X_m = z) \Big/ \sum_{0 \le m < n} \mathbf{P}(X_m \le t) \ \to\ \mu(\{z\})/\mu([0,t]) .$$

Proof. 1. It suffices to consider the case $X_0 = z$. Indeed:
(1) In the nominator X_m may be replaced by X_m^z in view of (6.1).
(2) To extend this to the denominator, introduce the hitting time T_z of z by $(X_n)_{n \ge 0}$ and define

$$p_k := \mathbf{P}(X_k^z = z) \quad \text{for} \quad k \ge 0 .$$

Conditioning with respect to $\{X_0 \neq z\}$ shows that $\mathbf{P}(X_0 = z) = 0$ may be assumed. Then decomposition according to the first stay in z yields

$$\begin{aligned}
& \sum_{0 \le m < n} \mathbf{P}(X_m = z) \Big/ \sum_{0 \le m < n} \mathbf{P}(X_m^z = z) \\
= \ & \sum_{0 \le m < n} \Big(\sum_{0 \le k < m} \mathbf{P}(T_z = m - k)\, p_k \Big) \Big/ \sum_{0 \le k < n} p_k \\
= \ & \sum_{0 \le k < n} p_k\, \mathbf{P}(T_z < n - k) \Big/ \sum_{0 \le k < n} p_k \\
\to \ & \lim_{n \to \infty} \mathbf{P}(T_z < n) \ = \ 1 ,
\end{aligned}$$

because $\sum_{k \ge 0} p_k = \infty$ and $\mathbf{P}(T_z < \infty) = 1$ by (5.1).

2. To verify the assertion in the case $X_0 = z$, define

$$q_k := \mathbf{P}(X_k^z \le t \text{ and } T_z > k) \quad \text{for} \quad k \ge 0\,.$$

Then decomposition according to the last stay in z yields

$$\begin{aligned} & \sum_{0\le m<n} \mathbf{P}(X_m^z \le t) \Big/ \sum_{0\le m<n} \mathbf{P}(X_m^z = z) \\ = \ & \sum_{0\le m<n} \sum_{0\le k\le m} p_k\, q_{m-k} \Big/ \sum_{0\le k<n} p_k \\ = \ & \sum_{0\le k<n} p_k\, \mathbf{E}^z\Big(\sum_{0\le l<T_z\wedge(n-k)} 1_{[0,t]}(X_l)\Big) \Big/ \sum_{0\le k<n} p_k \\ \to \ & \lim_{n\to\infty} \mathbf{E}^z\Big(\sum_{0\le l<T_z\wedge(n-k)} 1_{[0,t]}(X_l)\Big) \\ = \ & \mathbf{E}^z\Big(\sum_{0\le l<T_z} 1_{[0,t]}(X_l)\Big)\,, \end{aligned}$$

again due to $\sum_{k\ge 0} p_k = \infty$, and the assertion follows by (5.2). $\square$

Now a first information on the fluctuation of a recurrent process $(X_n)_{n\ge 0}$ by means of its invariant measure can be obtained. In its general form it concerns one sequence $(H_n)_{n\in\mathbf{N}}$, but two possibly different initial variables:

(6.3) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and denote by $(X'_n)_{n\ge 0}$ a copy of $(X_n)_{n\ge 0}$ with X'_0 replacing X_0. Then for functions $f, g \in \mathcal{R}(E)$ with compact support*

$$\sum_{0\le m<n} \mathbf{E}(f(X_m)) \Big/ \sum_{0\le m<n} \mathbf{E}(g(X'_m)) \;\to\; \mu f/\mu g\,,$$

provided $\mu f \ne 0$ or $\mu g \ne 0$.

Proof. 1. By comparing both the denominator and the nominator with the corresponding sum for $1_{[0,t]}$ the situation can be simplified to the case $g = 1_{[0,t]}$, where $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$. Moreover, $\mu(\{t\}) = 0$ may be supposed unless $t = \overline{x}$. Finally, the assumption $X'_0 = X_0$ means no real restriction in view of (6.1).

2. Now consider the measures ϱ_n defined in (4.3). If $\varrho_{n_k} \underset{v}{\to} \varrho \in \mathcal{M}(E)$ is any convergent subsequence, (4.3b) and the uniqueness of the invariant measure imply $\varrho = \gamma\,\mu$. Since g is $\mu-$almost continuous, the constant γ satisfies

$$\gamma\,\mu([0,t]) = \lim_{k\to\infty} \varrho_{n_k}([0,t]) = 1\,,$$

hence is independent of the subsequence, and thus by (4.3b)

$$\varrho_n f \to \varrho f = \mu f/\mu g \quad \text{for all} \quad f \in \mathcal{K}(E)\,.$$

3. For $a, b \in E$ with $a \le b$, approximating by $\mathcal{K}(E)$ from below and above, part 2 of the proof yields

$$\liminf_{n\to\infty} \varrho_n(]a,b[) \ge \varrho(]a,b[)\,, \tag{1}$$

(2) $$\limsup_{n\to\infty} \varrho_n([a,b]) \le \varrho([a,b])\,.$$

In particular, (2) implies that the convergence $\varrho_n(\{z\}) \to \varrho(\{z\})$, established in (6.2) for the case $\mu(\{z\}) > 0$, extends to the case $\mu(\{z\}) = 0$. Together with (1) this leads to

$$\begin{aligned}\liminf_{n\to\infty} \varrho_n([a,b]) &= \liminf_{n\to\infty} (\varrho_n(]a,b[) + \varrho_n(\{a\}) + \varrho_n(\{b\}))\\ &= \liminf_{n\to\infty} \varrho_n(]a,b[) + \varrho(\{a\}) + \varrho(\{b\})\\ &\ge \varrho(]a,b[) + \varrho(\{a\}) + \varrho(\{b\})\,,\end{aligned}$$

hence to

(3) $$\liminf_{n\to\infty} \varrho_n([a,b]) \ge \varrho([a,b])\,.$$

4. Combined, equations (2) and (3) yield

$$\varrho_n f \to \varrho f = \mu f/\mu g \quad \text{for all} \quad f = 1_{[a,b]} \in \mathcal{R}(E)\,.$$

Therefore the assertion holds for all step functions f with compact support. Thus, approximating a function $f \in \mathcal{R}(E)$ with compact support by such functions from below and above, the proof is completed. □

As a first application of this ergodic theorem consider the exchange process $(X_n)_{n\ge0}$ studied in Section 2. If it is recurrent (which can be tested by (2.4)) and $t > \underline{x}$ (which can be determined by (1.8)), then the sequence of ratios

$$\mathbf{P}(X_n^0 \le s)\,/\,\mathbf{P}(X_n^0 \le t) = \prod_{0\le m<n} F(s+m)/F(t+m)$$

converges for every $s \in E$ to a limit $G(s) \le \infty$, because the right-hand side decreases for $s \le t$ and increases for $s \ge t$. Then even more

$$\sum_{0\le m<n} \mathbf{P}(X_n^0 \le s) \Big/ \sum_{0\le m<n} \mathbf{P}(X_n^0 \le t) \to G(s)\,,$$

hence by (6.3) the function G is finite and the invariant measure μ is given by

$$\mu([0,s]) = \prod_{n\ge0} F(s+n)/F(t+n) \quad \text{for all} \quad s \in E\,.$$

Applied in particular to the recurrent case (1) with $F(t) = \dfrac{t}{t+1}$, it turns out that $\underline{x} = 0$ and μ is simply the Lebesgue measure on $\mathbf{R}_+$.

In stating the pointwise analogue of the mean version (6.3) more care has to be taken of (X_0, X_0'):

(6.4) Theorem *Let the assumptions of* (6.3) *be satisfied and in addition* (X_0, X_0') *and* $(H_n)_{n\in\mathbf{N}}$ *be independent. Then for functions* $f, g \in \mathcal{R}(E)$ *with compact support*

$$\sum_{0\le m<n} f(X_m) \Big/ \sum_{0\le m<n} g(X_m') \to \mu f/\mu g \quad a.s.\,,$$

provided $\mu f \neq 0$ *or* $\mu g \neq 0$.

Proof. By applying Fubini it follows that X_0 and X_0' may be assumed to be constant. Moreover, as in the proof of (6.3), the situation can be simplified to the case $g = 1_{[0,t]}$, where $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$. Since in a self-explanatory notation

$$\frac{S(f,x)}{S(g,y)} = \frac{S(f,x)}{S(g,x)} \frac{S(g,x)}{S(g,0)} \frac{S(g,0)}{S(g,y)},$$

only the following two assertions have to be verified:

$$\sum_{0 \leq m < n} f(X_m^x) \Big/ \sum_{0 \leq m < n} g(X_m^x) \to \mu f / \mu g \quad \text{a.s.}, \tag{1}$$

$$\sum_{0 \leq m < n} g(X_m^x) \Big/ \sum_{0 \leq m < n} g(X_m^0) \to 1 \quad \text{a.s.}. \tag{2}$$

Concerning (1), this follows from (4.5) and (4.6) in the case of step functions f and by approximation from below and above in the case $f \in \mathcal{R}(E)$. Concerning (2), this in fact has been derived already in part 1 of the proof of (4.5). □

Applied to the exchange process preceding this theorem, with λ denoting the Lebesgue measure, it follows for subintervals I_k of $\mathbf{R}_+$ of positive and finite length that

$$\sum_{0 \leq m < n} 1_{I_1}(X_m) \Big/ \sum_{0 \leq m < n} 1_{I_2}(X_m) \to \lambda(I_1)/\lambda(I_2) \quad \text{a.s.},$$

regardless of the initial law.

The next result concerns the orbits of the process themselves. In view of (4.4) it provides a simultaneous generalization of (1.5) and (1.6):

(6.5) Theorem *Let* $\nu \in \mathcal{N}[E]$ *be recurrent with invariant measure* μ *and define the random set*

$$L(\omega) := \{x \in E : x \textit{ is limit point of } (X_n(\omega))_{n \geq 0}\}.$$

Then with probability 1

$$L(\omega) = \operatorname{supp} \mu,$$

regardless of the initial law.

Proof. 1. Let $I_k, k \in \mathbf{N}$, be a countable base for E consisting of bounded open subintervals of E. Then the inclusion $L(\omega) \supset \operatorname{supp} \mu$ holds if and only if

$$\sum_{n \geq 0} 1_{I_k}(X_n(\omega)) = \infty \quad \text{whenever} \quad \mu(I_k) > 0.$$

Applying (6.4) to $f = 1_{I_k}$ and $g = 1_{[0,t]}$, where $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$, shows that this holds indeed with probability 1.

2. To prove the inverse inclusion, denote by $L_t(\omega)$ the analogue of $L(\omega)$ for the process $({}^tX_n)_{n\geq 0}$. Moreover, let D consist of $\overline{x}$ in the case $\overline{x} \in E$ and of a countable subset of $\{t \in E : t > \underline{x}\}$ with $\sup D = \overline{x}$ otherwise. Then clearly

$$L(\omega) = \bigcup_{t\in D} L_t(\omega)\,,$$

hence it suffices to verify that with probability 1

$$L_t(\omega) \subset \operatorname{supp}\mu \quad \text{for} \quad t \in D\,.$$

To this end let X_0 first be distributed according to (the trivial extension of) the normalized restriction of μ to $[0,t]$. Then $({}^tX_n)_{n\geq 0}$ is stationary by (4.2a) and thus

$$(*) \qquad \mathbf{P}({}^tX_n \notin I_k \text{ eventually}) = 1 \quad \text{whenever} \quad I_k \cap \operatorname{supp}\mu = \emptyset\,,$$

as desired. Finally, an application of (3.3b) to $f = 1_{I_k}$ shows that the distribution of X_0 in fact is irrelevant for $(*)$. $\square$

Together, (2.4) and (6.5) imply that the two familiar criteria for recurrence / transience from discrete Markov chain theory carry over to the present setting in the following form:

(1) If ν is recurrent, then for $x \in \operatorname{supp}\mu$ always

$$\mathbf{P}^{\,x}(X_n \in G \text{ infinitely often}) = 1\,,$$

hence

$$\mathbf{E}^x(|\{n \geq 0 : X_n \in G\}|) = \infty\,,$$

provided G is an open neighborhood of x.

(2) If ν is transient, then for $x \in E$ always

$$\mathbf{E}^x(|\{n \geq 0 : X_n \in K\}|) < \infty\,,$$

hence

$$\mathbf{P}^{\,x}(X_n \in K \text{ infinitely often}) = 0\,,$$

provided K is a compact subset of E.

The final result of this section is related to (5.5) and (5.6):

(6.6) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ. Then for each open subset G of E satisfying $\mu(G) > 0$ and every $t \in E$ there exists $n_0 \in \mathbf{N}$ such that*

$$\mathbf{P}(X_n^x \in G \textit{ for } 0 \leq x \leq t) > 0 \quad \textit{for} \quad n \geq n_0\,.$$

Proof. Since G may be assumed to be a bounded interval, $f = 1_G$ in view of (6.3) satisfies all conditions in (3.4). Accordingly there exists $m \in \mathbf{N}$ such that

$$\vartheta := \mathbf{P}(X_m^x \in G \text{ for } 0 \leq x \leq t) > 0\,,$$

where the assumption $t > \underline{x}$ or $t = \overline{x}$ means no loss of generality. By applying Fubini it follows that

$$\begin{aligned} \mathbf{P}(X_{l+m}^x \in G \text{ for } 0 \leq x \leq t) &= \int_{\mathcal{H}[E]} \mathbf{P}(X_m^{h(x)} \in G \text{ for } 0 \leq x \leq t)\, \nu^l(dh) \\ &\geq \vartheta\, \nu^l(h(t) \leq t) \quad \text{ for } \ l \in \mathbf{N}\,, \end{aligned}$$

because $x \leq t$ and $h(t) \leq t$ imply $h(x) \leq t$. Now $\nu^l(h(t) \leq t) > 0$ follows from (2.5) and (1.8) in the case $t > \underline{x}$ and is trivial in the case $t = \overline{x}$. Therefore $n_0 = m$ satisfies the assertion. □

7. Properties of the attractor

While in the transient case $\overline{x}$ attracts the process $(X_n)_{n\geq 0}$, in the recurrent case (6.5) suggests the following terminology:

(7.1) Definition *If $\nu \in \mathcal{N}[E]$ is recurrent with invariant measure μ, the set $M := \operatorname{supp} \mu$ is called the "attractor" of ν.*

As a first information (4.4) yields

$$\inf M = \underline{x} \quad \text{ and } \quad \sup M = \overline{x}\,,$$

where $\underline{x} \in M$ in any case, while $\overline{x} \in M$ only in the case $\overline{x} \in E$.

Similarly to (1.1), there is an implicit characterization:

(7.2) Theorem *If $\nu \in \mathcal{N}[E]$ is recurrent, its attractor is the smallest nonempty closed subset F of E satisfying the condition*

(a) $$\nu(h[F] \subset F) = 1\,,$$

or, equivalently, the condition

(b) $$h[F] \subset F \quad \textit{for all} \quad h \in N\,.$$

Proof. 1. Since the set of mappings $h \in \mathcal{H}[E]$ with $h[F] \subset F$ is closed, both conditions are clearly equivalent. Moreover, any nonempty closed set $F \subset E$ satisfying (a) satisfies the corresponding condition with respect to ν^n as well. For any $x \in F$ this implies $\mathbf{P}(X_n^x \in F) = 1$ for all $n \geq 0$, and thus the process $(X_n^x)_{n\geq 0}$ with probability 1 has all its limit points in F. Therefore the inclusion $M \subset F$ is a consequence of (6.5).

2. It remains to verify that (b) is satisfied for $F = M$. To this end consider $x = h_0(x_0)$ with $x_0 \in M$ and $h_0 \in N$, and let $G \subset E$ be any open neigborhood of x. Then, due to the continuity of the mapping $(x, h) \to h(x)$, there are open sets $G_0 \subset E$ and $\mathcal{H}_0 \subset \mathcal{H}[E]$ such that

$$(x_0, h_0) \in G_0 \times \mathcal{H}_0 \quad \text{ and } \quad h(x) \in G \quad \text{for} \quad x \in G_0, h \in \mathcal{H}_0\,.$$

By the invariance of μ this yields

$$\begin{aligned}\mu(G) &= (\mu\otimes\nu)\{(x,h) : h(x)\in G\}\\ &\geq (\mu\otimes\nu)\,(G_0\times\mathcal{H}_0) \;>\; 0\,.\end{aligned}$$

Therefore x has to be contained in the support of μ. $\square$

It is a consequence of (b) that the attractor of a recurrent distribution ν depends on it only through its support N.

In the following two propositions $\overline{B}$ denotes the closure of a subset B of E. Then M and N are related by an equation that is basic in the context of self-similar sets (see e.g. [12, 14]):

(7.3) Proposition *If $\nu\in\mathcal{N}[E]$ is recurrent, its attractor M satisfies*

$$M = \overline{\bigcup_{h\in N} h[M]}\,.$$

Proof. If F denotes the set on the right-hand side, the inclusion $F\subset M$ follows from . Conversely, the continuity of $h\in\mathcal{H}[E]$ yields

$$h[F]\subset \overline{h[\bigcup_{h'\in N} h'[M]]}\,,$$

where for $h'\in N$, again by (7.2), $h'[M]\subset M$. Therefore

$$h[F]\subset \overline{h[M]}\subset F \quad \text{for all} \quad h\in N\,,$$

and the inclusion $M\subset F$ follows, once more from (7.2). $\square$

Whenever both M and N are compact, due to the continuity of the mapping $(x,h)\to h(x)$, this result holds without taking the closure. To see that, without assuming M to be compact, this may fail even if N is finite, consider the autoregressive process

$$X_n = \frac{1}{3}X_{n-1}+V_n \quad \text{for} \quad n\in\mathbf{N}\,,$$

where the i.i.d. variables $V_n, n\in\mathbf{N}$, attain the values 0 and $\frac{2}{3}$ with probability $\frac{1}{2}$. With $E=[0,1[$ as state space the corresponding distribution ν is supported by the two mappings $h_1 : x\to x/3$ and $h_2 : x\to (x+2)/3$. It is clearly irreducible and by (2.2) recurrent. In view of $\sup M = 1$ by (7.2) also $\frac{1}{3}=\sup h_1[M]\in M$, while on the other hand $\frac{1}{3}\notin h_1[E]\cup h_2[E]$.

Next, M will be described by means of the semigroup N^* generated by N:

(7.4) Proposition *If $\nu\in\mathcal{N}[E]$ is recurrent, its attractor M satisfies*

$$M=\overline{\{h(x) : h\in N^*\}} \quad \textit{for every } x\in M\,.$$

Proof. If F denotes the set on the right-hand side, the inclusion $F \subset M$ follows from (7.2). Conversely, as in the proof of (7.3),

$$\begin{aligned} h[F] &\subset \overline{\{h \circ h'(x) : h' \in N^*\}} \\ &\subset \overline{\{h''(x) : h'' \in N^*\}} \\ &= F \quad \text{for all} \quad h \in N\,, \end{aligned}$$

and the inclusion $M \subset F$ follows again from (7.2). □

The assumption $x \in M$ is clearly essential for the inclusion $F \subset M$, while the proof shows that the inclusion $M \subset F$ holds for any $x \in E$.

The most explicit characterization of M uses the closure N^{**} of N^*:

(7.5) Theorem *If $\nu \in \mathcal{N}[E]$ is recurrent, its attractor M satisfies*

$$x \in M \quad \textit{if and only if} \quad j(x) \in N^{**}\,,$$

where j is the canonical injection of E into $\mathcal{H}[E]$.

Proof. 1. To prove the condition to be necessary, consider $h_0 = j(x_0)$ with $x_0 \in M$. By definition of the topology in $\mathcal{H}[E]$ it has to be shown that

$$\{h \in N^* : h(x) \in G_0 \text{ for } 0 \le x \le t\} \neq \emptyset$$

for arbitrary $t \in E$ and open sets $G_0 \subset E$ containing x_0. Since $x_0 \in M$ implies $\mu(G_0) > 0$ for these sets, (6.6) applies and provides $n \in \mathbf{N}$ with

$$\mathbf{P}(X_n^x \in G_0 \text{ for } 0 \le x \le t) > 0\,.$$

Thus there exist $h_1, \ldots, h_n \in N$ such that

$$h_n \circ \ldots \circ h_1(x) \in G_0 \quad \text{for} \quad 0 \le x \le t\,,$$

as had to be shown.

2. To prove the converse, let $h_n \in N^*$ converge to $h_0 = j(x_0)$ and apply (7.4) to $x = \underline{x} \in M$, leading to

$$x_0 = h_0(\underline{x}) = \lim_{n\to\infty} h_n(\underline{x}) \in M\,. \quad \square$$

It is a consequence of this result, that the fixed points of the mappings $h \in N^*$ are dense in M. Indeed, since this is trivial for $\underline{x} = \overline{x}$, assume

$$M \cap]s,t[\neq \emptyset \quad \text{with} \quad \underline{x} < s < t < \overline{x}\,.$$

Then any $h \in N^*$ with $s < h(x) < t$ for $0 \le x \le t$ satisfies $s < h(s) \le h(t) < t$ and thus by its continuity $h(x) = x$ for some $x \in]s,t[$.

On the other hand, fixed points of mappings in N^* need not belong to M. This is seen, for instance, replacing ν by $\nu' = (\nu + \nu_0)/2$, where ν_0 is concentrated on the identity map; for it is obvious that, passing from ν to ν', neither recurrence nor invariant measure are concerned. It is, however, true that a fixed point x of a mapping $h \in N^*$ belongs to M, if it is minimal under the condition $x \geq \underline{x}$. Indeed, in this case the increasing sequence $(h^n(\underline{x}))_{n\geq 0}$ converges to x and (7.4) applies.

The rest of this section concerns conditions under which the attractor is an interval. The first one is quite natural:

(7.6) Proposition *If $\nu \in \mathcal{N}[E]$ is recurrent, its attractor M is connected whenever N is connected.*

Proof. Fix some $y \in M$ and assume $\underline{x} < \underline{y} < \overline{y} < \overline{x}$. Then, according to (4.4) and (6.6), there exists $n \in \mathbf{N}$ such that

$$\mathbf{P}(X_n^y < \underline{y}) > 0 \quad \text{and} \quad \mathbf{P}(X_n^y > \overline{y}) > 0\,.$$

Thus the set

$$B := \{h_n \circ \ldots \circ h_1(y) : h_1, \ldots h_n \in N\}$$

is a subset of M by (7.2) and contains elements $x < \underline{y}$ and $x > \overline{y}$. Moreover it is connected, because in $\mathcal{H}[E]$ composition and evaluation are continuous. Therefore $[\underline{y}, \overline{y}] \subset M$, as had to be shown. □

The connectedness of N is by no means necessary for that of M as is seen, for instance, by the autoregressive process following (3.3), where the invariant measure is the uniform distribution on [0,1[.

To exhibit a nontrivial example with disconnected attractor, choose $E = [0, 1[$ and let N consist of the two mappings defined by

$$h_1(x) = \frac{x}{3} \vee (x - \frac{2}{5}) \quad \text{and} \quad h_2(x) = (x + \frac{2}{5}) \wedge \frac{x+2}{3}\,,$$

satisfying the symmetry condition $h_2(1-x) = 1 - h_1(x)$. The corresponding distribution ν is recurrent by (2.2) with $\underline{x} = 0$ by (1.8), and a simple sketch shows that the open set $G =]\frac{1}{5}, \frac{2}{5}[\cup]\frac{3}{5}, \frac{4}{5}[$ satisfies $h_i[E \setminus G] \subset E \setminus G$ for $i = 1, 2$. By (7.2) this implies $M \subset E \setminus G$, hence M is disconnected, even though the mappings h_i are nonexpansive and the images $h_1[E]$ and $h_2[E]$ cover E.

The final result of this section is related to (2.2) and of interest in Section 9:

(7.7) Proposition *If $\nu \in \mathcal{N}[E]$ is recurrent, its attractor M is connected whenever the following two conditions are satisfied:*

(a) $$\nu(\sup h \in E) = 0\,,$$

(b) $$M \supset \,]\,t, \overline{x}\,[\quad \textit{for some } t \in E\,.$$

Proof. 1. The crucial point is the implication

$$(*) \qquad [s,t] \subset M \quad \text{if } s < t \text{ satisfies } \nu(h(t) < s) > 0\,.$$

To derive it, consider a nonempty open subset G of $]s,t[$. Then condition (a) and the assumption $\nu(h(t) < s) > 0$ yield

$$\nu(\mathcal{H}_s) > 0 \quad \text{for} \quad \mathcal{H}_s := \{h \in \mathcal{H}[E] : h(t) < s < t < \sup h\}\,.$$

For $h \in \mathcal{H}_s$, due to the continuity, $h^{-1}[G]$ is a nonempty open subset of $]t, \overline{x}\,[$, hence the invariant measure satisfies $\mu(h^{-1}[G]) > 0$. Therefore

$$\begin{aligned} \mu(G) &= \int_{\mathcal{H}[E]} \mu(h(x) \in G)\,\nu(dh) \\ &\geq \int_{\mathcal{H}_s} \mu(h^{-1}[G])\,\nu(dh) \;>\; 0\,, \end{aligned}$$

as had to be shown.

2. Now let t in condition (b) be chosen minimal. Then by $(*)$

$$\nu(h(t) < s) = 0 \quad \text{for} \quad s < t\,,$$

hence $s \uparrow t$ yields $h(t) \underset{\nu}{\geq} t$, which by (1.8) implies $t \leq \underline{x}$. □

To see condition (a) to be essential for this result, consider the independent system (E, μ), where the invariant measure coincides with μ.

8. Properties of the invariant measure

Since an explicit determination of the invariant measure in general is out of reach, at least a qualitative description is desirable. Some basic facts concerning the case $E = [0,1]$ can be found in [7]. A rather general result can be derived from the mean ergodic theorem:

(8.1) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and exclude the case $E = \{0\}$. Then μ is nonatomic whenever*

$$(*) \qquad \nu(h(x_1) = z = h(x_2)) = 0 \quad \textit{for} \quad x_i, z \in E \textit{ with } x_1 < x_2\,.$$

Proof. 1. If D is a countable dense subset of E, application of $(*)$ to all pairs $x_1 < x_2$ with $x_i \in D$ yields

$$|h^{-1}[\{z\}]| \leq 1 \quad \text{for} \quad \nu\text{-almost all } h \in \mathcal{H}[E]\,.$$

Consequently, the kernel P transforms nonatomic measures $\varrho \in \mathcal{M}_1(E)$ into measures of the same type, because

$$\varrho P(\{z\}) = \int_{\mathcal{H}[E]} \varrho(h(x) = z)\,\nu(dh)\,,$$

where the integrand vanishes for $\nu-$almost all $h \in \mathcal{H}[E]$.

2. Starting with any nonatomic initial law μ_0 on $E \neq \{0\}$, it follows from part 1 of the proof that $\mu_n := \mathcal{L}(X_n)$ is nonatomic for all $n \in \mathbf{N}$. An application of (6.3) to $f = 1_{\{z\}}$ and an admissible function g with $\mu g \neq 0$ under $X_0 = X_0'$ yields

$$\mu(\{z\})/\mu g = \lim_{n\to\infty} \sum_{0\le m<n} \mu_m(\{z\}) \Big/ \sum_{0\le m<n} \mu_m g$$

and thus $\mu(\{z\}) = 0$ for all $z \in E$. □

Since condition $(*)$ is apparently satisfied whenever the underlying mappings h are injective, the regenerative case may rightly be considered as an exception. This holds even more, as $(*)$ is not a necessary condition. To see this, consider the example following (7.6), where the invariant measure μ is nonatomic by (8.1). A slight modification of the mappings h_i on the set G due to $\mu(G) = 0$ does not affect the equation defining μ.

The next result is of interest primarily for affine recursions, the distinguished measure being the Lebesgue measure. In its simplest form the argument can be traced back to [15]. Since Lebesgue null sets are not invariant under conjugation, the proper version is as follows:

(8.2) Proposition *Let $\nu \in \mathcal{N}[E]$ be recurrent with invariant measure μ and let $\varrho \in \mathcal{M}[E]$ satisfy ($h(\varrho)$ denoting the image of ϱ under h)*

$$\nu(\, h(\varrho) \text{ absolutely continuous with respect to } \varrho\,) = 1\,.$$

Then μ is either absolutely continuous or singular with respect to ϱ.

Proof. 1. First, measurability of the set

$$\mathcal{H}_0 := \{h \in \mathcal{H}[E] : h(\varrho) \ll \varrho\}$$

has to be settled, where the symbol "$\ll$" stands for absolute continuity and ϱ may be assumed to be a finite measure. Using the $\varepsilon,\delta-$criterion for $h(\varrho) \ll \varrho$ and approximating $B \in \mathcal{B}(E)$ from outside by open sets and $h^{-1}[B]$ from inside by compact sets, it is no problem to verify the representation

$$\mathcal{H}[E] \setminus \mathcal{H}_0 = \bigcup_{k\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} \bigcup \{h \in \mathcal{H}[E] : h[K] \subset G\}\,,$$

where the inner union is extended over all pairs of compact sets K with $\varrho(K) > 1/k$ and open sets G with $\varrho(G) < 1/l$, hence is an open subset of $\mathcal{H}[E]$. Therefore $\mathcal{H}_0$ itself is of type $\mathrm{F}_{\sigma\delta}$.

2. Let now μ be decomposed into its absolutely continuous part μ_c and its singular part μ_s with respect to ϱ. Then the equation

$$\mu_c P(B) = \int_{\mathcal{H}_0} \mu_c(h^{-1}[B])\,\nu(dh) \quad \text{for} \quad B \in \mathcal{B}(E)$$

implies $\mu_c P \ll \varrho$, hence the equation

$$\mu_c P + \mu_s P = \mu P = \mu = \mu_c + \mu_s$$

implies $\mu_c P \ll \mu_c$. By the uniqueness of the invariant measure, therefore, $\mu_c = \gamma_c \mu$ with some constant γ_c, hence $\mu_s = \gamma_s \mu$ with some constant γ_s. Now $\mu_c \wedge \mu_s = 0$ implies $\gamma_c \wedge \gamma_s = 0$ and proves the assertion. □

In general, there is no further information available about the alternative in this proposition. An obvious exception is the case, where the common distribution of the variables $H_n(x), n \in \mathbf{N}$, is absolutely continuous with respect to ϱ for all $x \in E$, because in this case the equation

$$\mu(B) = \int_E \mathbf{P}(H_n(x) \in B)\, \mu(dx) \quad \text{for} \quad B \in \mathcal{B}(E)$$

implies $\mu \ll \varrho$.

The rest of this section is devoted to stability results. This requires some preparation:

(8.3) Lemma *If $\nu_k, \nu \in \mathcal{N}[E]$ and $\mu_k, \mu \in \mathcal{M}(E)$ satisfy*

(a) $\qquad \mu_k$ *is excessive for* ν_k ,

(b) $\qquad \nu_k \underset{w}{\to} \nu \quad$ *and* $\quad \mu_k \underset{v}{\to} \mu$,

then μ is excessive for ν.

Proof. Let $g \in \mathcal{K}(E)$ satisfy $0 \le g \le 1$ and define $\varrho_k, \varrho \in \mathcal{M}(E)$ by

$$d\varrho_k / d\mu_k = g = d\varrho / d\mu \,.$$

Since $f \in \mathcal{C}(E)$ implies $fg \in \mathcal{K}(E)$, the vague convergence in (b) yields $\varrho_k \underset{w}{\to} \varrho$. Together with the weak convergence in (b) this leads to

$$\varrho_k \otimes \nu_k \underset{w}{\to} \varrho \otimes \nu \,,$$

because the multiplication of measures is continuous in the weak topology. Now for $0 \le f \in \mathcal{K}(E)$ the mapping $(x,h) \to f(h(x))$ is continuous, hence

$$\begin{aligned} \int_E \int_{\mathcal{H}[E]} f(h(x)) g(x)\, \mu(dx)\, \nu(dh) &= \lim_{k\to\infty} \int_E \int_{\mathcal{H}[E]} f(h(x)) g(x)\, \mu_k(dx)\, \nu_k(dh) \\ &\le \liminf_{k\to\infty} \int_E \int_{\mathcal{H}[E]} f(h(x))\, \mu_k(dx)\, \nu_k(dh) \\ &\le \liminf_{k\to\infty} \int_E f\, d\mu_k \\ &= \int_E f\, d\mu \,, \end{aligned}$$

where the second inequality uses (a). By letting g increase to 1, therefore

$$\mu P f \le \mu f \quad \text{for} \quad 0 \le f \in \mathcal{K}(E) \,,$$

as had to be shown. □

If $\nu_k, \nu \in \mathcal{N}[E]$ satisfy $\nu_k \underset{w}{\to} \nu$, all that can be said about their restrictions in the sense of (1.1) is the inequality

$$\liminf_{k\to\infty} \overline{x}_k \geq \overline{x}$$

for the respective upper limits. For simplicity, therefore, in the sequel all occurring distributions will be assumed to be irreducible with respect to the same state space. Then the following stability criterion holds:

(8.4) Theorem *Let $\nu_k, \nu \in \mathcal{N}[E]$ be recurrent with invariant measures μ_k, μ and denote the respective lower limits by $\underline{x}_k, \underline{x}$. Then the convergence $\nu_k \underset{w}{\to} \nu$ implies the existence of norming constants γ_k with $\gamma_k \mu_k \underset{v}{\to} \mu$ if and only if*

$$(*) \qquad \limsup_{k\to\infty} \underline{x}_k \leq \underline{x}\,.$$

Proof. 1. In verifying the condition $(*)$ to be necessary $\underline{x} < \overline{x}$ may be assumed, because otherwise $\underline{x}_k \leq \underline{x}$ for all $k \in \mathbf{N}$. For any $t \in E$ with $t > \underline{x}$ the convergence $\gamma_k \mu_k \underset{v}{\to} \mu$ yields

$$\liminf_{k\to\infty} \gamma_k \mu_k([0,t[) \geq \mu([0,t[) > 0\,,$$

hence $\underline{x}_k \leq t$ for almost all $k \in \mathbf{N}$ by (4.4), as had to be shown.

2. To prove sufficiency, choose $t \in E$ with $t > \underline{x}$ or $t = \overline{x}$ and assume $\mu(\{t\}) = 0$ unless $t = \overline{x}$. Then (4.4) and the inequality $(*)$ yield $\mu([0,t]) > 0$ and $\mu_k([0,t]) > 0$ for almost all $k \in \mathbf{N}$. Therefore it means no loss of generality to assume

$$(1) \qquad \mu_k([0,t]) = \mu([0,t]) = 1 \quad \text{for all} \quad k \in \mathbf{N}\,.$$

From this it will be derived in part 3 of the proof that

$$(2) \qquad \sup_{k\in\mathbf{N}} \mu_k([0,s]) < \infty \quad \text{for all} \quad s \in E\,,$$

i.e. the measures $\mu_k, k \in \mathbf{N}$, are uniformly locally finite. It follows as in the proof of (4.3) that $\{\mu_k : k \in \mathbf{N}\}$ is a sequentially compact subset of $\mathcal{M}(E)$. If $(\mu'_k)_{k\in\mathbf{N}}$ is any convergent subsequence, its limit by (8.3) is excessive and thus by (4.7) is of the form $\gamma\mu$. Since $1_{[0,t]}$ is $\mu-$almost continuous, the constant γ satisfies by (1)

$$\gamma = \gamma\,\mu([0,t]) = \lim_{k\to\infty} \mu'_k([0,t]) = 1\,.$$

Thus the sequence $(\mu_k)_{k\in\mathbf{N}}$ itself satisfies $\mu_k \underset{v}{\to} \mu$.

3. Since (2) is implied by (1) in the case $t = \overline{x}$, in the sequel $t > \underline{x}$ may be assumed. Thus there exists $n \in \mathbf{N}$ with $\nu^n(h(s) < t) > 0$. Since $\nu_k \underset{w}{\to} \nu$ implies $\nu_k^n \underset{w}{\to} \nu^n$, therefore

$$\liminf_{k\to\infty} \nu_k^n(h(s) < t) \geq \nu^n(h(s) < t) > 0\,,$$

and thus it is no real restriction to assume

$$\vartheta := \inf_{k\in\mathbf{N}} \nu_k^n(h(s) < t) > 0\,.$$

Since μ_k is invariant for ν_k^n as well, this yields

$$\begin{aligned}\mu_k([0,s]) &\le \vartheta^{-1}\,\mu_k([0,s])\,\nu_k^n(h(s)<t)\\ &\le \vartheta^{-1}\,(\mu_k\otimes\nu_k^n)\,(\{(x,h) : h(x)<t\})\\ &= \vartheta^{-1}\,\mu_k([0,t[)\\ &\le \vartheta^{-1} \quad \text{for all} \quad k\in\mathbf{N}\,,\end{aligned}$$

and the proof is completed. □

As an example violating the condition (∗) consider on $E = [0,1[$ the mappings defined by

$$h_1(x) = \frac{1}{2}\,(x+1)\,,\; h_2(x) = x \wedge \frac{1}{2} \quad \text{and} \quad h_2^k(x) = (1-\frac{1}{k})\,x \wedge \frac{1}{2}\,.$$

If ν_k and ν assign mass $\frac{1}{2}$ to h_1, h_2^k and h_1, h_2, respectively, then clearly $\nu_k \underset{w}{\to} \nu$. Moreover, ν_k and ν are recurrent by (2.2), their lower limits, however, are $\underline{x}_k = 0$ and $\underline{x} = \frac{1}{2}$ by (1.8).

For an application of (8.4) let ν_0 be concentrated on the constant mapping $h=0$ and approximate a recurrent distribution ν by

$$\nu_k := (1-\frac{1}{k})\,\nu + \frac{1}{k}\,\nu_0 \quad \text{for} \quad n\in\mathbf{N}\,.$$

For the same reasons as above these distributions are recurrent with $\underline{x}_k = 0$, hence the corresponding invariant measures, suitably normalized, satisfy $\mu_k \underset{v}{\to} \mu$. In this case $\mu_k(\{0\}) > 0$ by (5.6) or (5.7), and thus μ_k has a representation according to (5.2).

Finally, the special case $\overline{x} \in E$ has to be mentioned, where the invariant measures can be normalized to $\mu_k, \mu \in \mathcal{M}_1(E)$. Since in this case vague and weak convergence coincide, the constants γ_k are needless and thus $\mu_k \underset{w}{\to} \mu$. That this convergence in the case $\overline{x} \notin E$ and $\mu_k, \mu \in \mathcal{M}_1(E)$ may fail, will be seen in the next section.

9. Positive recurrence and null recurrence

The following classification is adopted from the discrete situation:

(9.1) Definition *If $\nu\in\mathcal{N}[E]$ is recurrent with invariant measure μ, the distribution ν (or the kernel P or the process $(X_n)_{n\ge 0}$) is called*

(a) *"positive recurrent" if $\mu(E) < \infty$,*

(b) *"null recurrent" if $\mu(E) = \infty$.*

As the alternative recurrent / transient, considered in (2.5), this classification is invariant under a passage from ν to ν^n, because this does not affect the invariant measure.

The following criterion, where again ν is identified with the corresponding kernel P, includes the transient case:

(9.2) Proposition *The irreducible distribution $\nu \in \mathcal{N}[E]$ is positive recurrent if and only if there exists a stationary distribution $\mu \in \mathcal{M}_1(E)$ for ν.*

Proof. It suffices to deduce recurrence of ν, if μ is stationary for ν. Otherwise, with μ as initial law, (2.4b) would yield

$$\sum_{n \geq 0} \mu([0,t]) = \sum_{n \geq 0} \mathbf{P}(X_n \leq t) < \infty \quad \text{for all} \quad t < \overline{x}\,,$$

hence $\mu([0,t]) = 0$, implying $\overline{x} \in E$ and contradicting transience by (2.2). □

Next, the result of (2.2) can be strengthened:

(9.3) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible and satisfies*

$$\nu(\sup_{x \in E} h(x) \in E) > 0\ ,$$

then ν is positive recurrent.

Proof. Let μ be the invariant measure for ν that is ensured by (2.2) and choose $t \in E$ satisfying

$$\vartheta := \nu(\sup_{x \in E} h(x) \leq t) > 0\,.$$

Then the invariance of μ implies

$$\mu([0,t]) = \int_E \nu(h(x) \leq t)\, \mu(dx) \geq \vartheta\, \mu(E)\,.$$

Since the left-hand side is finite, the assertion follows. □

That the sufficient condition in this proposition fails to be necessary is seen by the queuing process. It is well-known to possess a stationary distribution if and only if the associated random walk diverges to $-\infty$.

At this point it can be clarified what measures $\mu \in \mathcal{M}(E)$ arise as invariant measure of some recurrent distribution $\nu \in \mathcal{N}[E]$. The independent system (E,μ) shows that the only condition to be satisfied in the case of a measure $\mu \in \mathcal{M}_1(E)$ is given by

$$\mu(\{x \in E : x \geq t\}) > 0 \quad \text{for all} \quad t \in E\,.$$

In the case of an infinite measure μ, however, this condition fails to be sufficient, because for a null recurrent distribution $\nu-$almost all $h \in \mathcal{H}[E]$ are unbounded in E by (9.3) and thus (7.7) applies.

In the sequel – suggested by ideas from queuing theory (see e.g. [21, 22]) – the "dual process" $(H_1 \circ \ldots \circ H_n(0))_{n\in\mathbf{N}}$ will be investigated. Though it fails, in general, to be a Markov process, it is of particular interest for distinguishing positive and null recurrence. To this end an improper random variable taking its values in the (possibly) enlarged state space $\overline{E} = E \cup \{\overline{x}\}$ has to be introduced:

(9.4) Definition *For irreducible $\nu \in \mathcal{N}[E]$ the random variable*

$$Y := \sup_{n\in\mathbf{N}} H_1 \circ \ldots \circ H_n(0)$$

is called the "dual limit" of ν.

Often, Y can be given in an explicit form, for instance

(1) if $(X_n)_{n\geq 0}$ is the queuing process, then

$$Y = \sup_{n\geq 0} (U_1 + \ldots + U_n)\,,$$

(2) if $(X_n)_{n\geq 0}$ is an exchange process with $U_n = 1$, then

$$Y = \sup_{n\in\mathbf{N}} (V_n - (n-1))\,,$$

(3) if $(X_n)_{n\geq 0}$ is an affine recursion as in the introduction, then

$$Y = \sum_{n\in\mathbf{N}} U_1 \ldots U_{n-1} V_n\,.$$

By means of the results in Section 3 the terminology in (9.4) can be justified:

(9.5) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible, the sequence*

$$Y_n^x := H_1 \circ \ldots \circ H_n(x) \quad \textit{for} \quad n \in \mathbf{N}$$

satisfies

(a) $$Y_n^x \uparrow Y \quad \textit{a.s.} \quad \textit{for} \quad x = 0\,,$$

(b) $$Y_n^x \to Y \quad \textit{a.s.} \quad \textit{for every } x \in E\,.$$

Proof. (a) This is immediate from the monotonicity of $h \in \mathcal{H}[E]$.

(b) Since the distributions of Y_n^x and X_n^x agree, the fundamental inequality (3.1) implies as well

$$\sum_{n\geq 0} \mathbf{E}(f(Y_n^x) - f(Y_n^0)) < \infty$$

for each bounded increasing function $f : E \to \mathbf{R}_+$. In complete analogy to the derivation of (3.3b) this leads to

$$1_I(Y_n^x) - 1_I(Y_n^0) \to 0 \quad \text{a.s.}$$

for every subinterval I of E. If I runs through a countable base of E, therefore, (b) is a consequence of (a). □

The uniform convergence on compact sets carries over from Section 3, i.e.

$$\sup_{0 \le x \le t} |f(Y_n^x) - f(Y_n^0)| \to 0 \quad \text{a.s.} \quad \text{for} \quad f \in \mathcal{R}(E) \quad \text{and} \quad t \in E\,.$$

If $\overline{x} \in E$, moreover, (a) can be complemented by $Y_n^x \downarrow Y$ a.s. for $x = \overline{x}$.

There is a zero-one law for the (possibly) improper random variable Y:

(9.6) Proposition *If $\nu \in \mathcal{N}[E]$ is irreducible, its dual limit satisfies*

$$\mathbf{P}(Y \in E) = 0 \ \text{or} \ 1\,.$$

Proof. 1. Consider first the case that

$$\mathbf{P}(\sup_{x \in E} H_n(x) \le t) > 0 \quad \text{for some } t \in E\,.$$

Then the random time

$$T := \inf\{n > 1 : \sup_{x \in E} H_n(x) \le t\}$$

is almost surely finite and satisfies

$$Y \le H_1 \circ \ldots \circ H_{T-1}(t) \in E \quad \text{a.s.}.$$

2. Otherwise H_1 is almost surely unbounded in E, hence

$$\{\sup_{n \in \mathbf{N}} H_1(H_2 \circ \ldots \circ H_n(0)) \in E\} = \{\sup_{n > 1} H_2 \circ \ldots \circ H_n(0) \in E\} \quad \text{a.s.}.$$

Continuing, it follows that the event $\{Y \in E\}$ is contained in the completed tail σ-field of $(H_n)_{n \in \mathbf{N}}$ and thus has probability 0 or 1. □

Now the dual limit can be shown to play the same role in singling out positive recurrence as the lower limit does for recurrence. The first result is related to a "principle" in [20]:

(9.7) Theorem *The irreducible distribution $\nu \in \mathcal{N}[E]$ is positive recurrent if and only if $\mathbf{P}(Y \in E) = 1$. In this case*

(a) $\mu := \mathcal{L}(Y)$ is the unique stationary distribution,

(b) $\mathcal{L}(X_n) \underset{w}{\to} \mu$ for arbitrary initial law.

Proof. 1. If ν is positive recurrent with stationary distribution μ, then

$$\begin{aligned} \mu([0,t]) &= (\liminf_{n\to\infty}) \int_E \mathbf{P}(X_n^x \le t)\, \mu(dx) \\ &\le \liminf_{n\to\infty} \int_E \mathbf{P}(X_n^0 \le t)\, \mu(dx) \\ &= \liminf_{n\to\infty} \mathbf{P}(Y_n^0 \le t) \\ &= \mathbf{P}(Y \le t) \\ &\le \mathbf{P}(Y \in E) \quad \text{for all} \quad t \in E \end{aligned}$$

by (9.5a). Thus $t \uparrow \overline{x}$ (or $t = \overline{x}$) implies $\mathbf{P}(Y \in E) = 1$.

2. If conversely $Y \in E$ almost surely, then

$$Y = H_1(Y') \text{ a.s.} \quad \text{with} \quad Y' := \sup_{n>1} H_2 \circ \ldots \circ H_n(0)\,,$$

where H_1 and Y' are independent and in addition Y and Y' have the same distribution. Thus $\mu = \mathcal{L}(Y)$ is a stationary distribution, hence ν is positive recurrent by (9.2), and (a) is established.

3. Finally, if $\mu_0 := \mathcal{L}(X_0)$ is arbitrary, then

$$\begin{aligned} \mathbf{E}(f(X_n)) &= \int_E \mathbf{E}(f(X_n^x))\, \mu_0(dx) \\ &= \int_E \mathbf{E}(f(Y_n^x))\, \mu_0(dx) \\ &\to \mathbf{E}(f(Y)) \quad \text{for all} \quad f \in \mathcal{C}(E) \end{aligned}$$

by (9.5b), and (b) is established. □

As a simple application consider the exchange process studied in Sections 2 and 6. The representation following (9.4) implies $\mathbf{P}(Y \in E) = 1$ in the case $\overline{x} < \infty$, because then the supremum is in fact a maximum. In the case $\overline{x} = \infty$ choose any t satisfying $F(t) > 0$ for the underlying distribution function. Then

$$\mathbf{P}(Y \le t) = \prod_{n\ge 0} F(t+n) > 0$$

if and only if the series $\sum_{n\ge 0} (1 - F(t+n))$ converges. Therefore the process $(X_n)_{n\ge 0}$ is positive recurrent if and only if the variables $V_n, n \in \mathbf{N}$, have a finite expectation (for extensions see [13]).

Instead of presenting further examples (as can be found in [6, 11, 20]), the fractal character of stationary distributions and their support will briefly be discussed. To this end choose $E = [0,1[$ and let the support of $\nu \in \mathcal{N}[E]$ consist of two injective, but not necessarily contractive, mappings $h_0, h_1 \in \mathcal{H}[E]$ satisfying

$$h_0(x) < x < h_1(x) \quad \text{for } 0 < x < 1 \quad \text{and} \quad \sup_{x\in E} h_0(x) < h_1(0)\,.$$

Then ν is positive recurrent by (9.3) with $\underline{x} = 0$ (and $\overline{x} = 1$). Now endow the space $J := \{0,1\}^{\mathbf{N}}$ with the product topology and define

$$\varphi : (j_n)_{n\in\mathbf{N}} \to \sup_{n\in\mathbf{N}} h_{j_1} \circ \ldots \circ h_{j_n}(0)\,.$$

It follows that φ is an embedding of J into [0,1] (see e.g. [12, 14]). Moreover, the stationary distribution μ is the image under φ of the product measure ϱ with factors $\nu(\{h_0\})\,\varepsilon_0 + \nu(\{h_1\})\,\varepsilon_1$. Since $\operatorname{supp}\varrho = J$, the support M of μ equals $\varphi[J] \cap E$ and is thus, as J, totally disconnected. Finally, since the measures ϱ belonging to different ν are pairwise orthogonal, this holds for the corresponding distributions μ as well.

Before establishing the counterpart of (9.7) the stability problem mentioned at the end of the preceding section will be settled. To this end choose $E = \mathbf{R}_+$ and let ν correspond to a positive recurrent exchange process as above, satisfying $\mathbf{P}(V_n = 0) > 0$ and thus $\underline{x} = 0$. Now consider the perturbed distributions

$$\nu_k := (1 - \frac{1}{k})\,\nu + \frac{1}{k}\,\nu_k' \quad \text{for} \quad k \in \mathbf{N}\,,$$

where ν_k' is concentrated on the constant mapping $h_k = k^2$. Then ν_k is again positive recurrent by (9.3), with lower limit 0 by (1.8). Thus (8.4) applies, i.e. the corresponding stationary distributions μ_k and μ satisfy $\gamma_k\mu_k \underset{v}{\to} \mu$ for suitable constants γ_k. If $\mathbf{P}_k$ refers to ν_k, however, $h(x) \geq x - 1$ for $\nu-$almost all $h \in \mathcal{H}[E]$ implies

$$\begin{aligned}\mathbf{P}_k(\sup_{n\in\mathbf{N}} H_1 \circ \ldots \circ H_n(0) \leq l) &\leq \mathbf{P}_k(H_m \neq h_k \text{ for } 1 \leq m \leq k^2 - l)\\ &= (1 - \frac{1}{k})^{k^2-l}\\ &\to 0 \quad \text{for all} \quad l \in \mathbf{N}\,,\end{aligned}$$

hence $\mu_k \underset{w}{\to} \varepsilon_{\overline{x}}$ on $\overline{E} = E \cup \{\overline{x}\}$ by (9.7).

This improper convergence appears again in the following criterion:

(9.8) Theorem *The irreducible distribution $\nu \in \mathcal{N}[E]$ is null recurrent or transient if and only if $\mathbf{P}(Y \in E) = 0$. In this case for arbitrary initial law*

$$\mathcal{L}(X_n) \underset{w}{\to} \varepsilon_{\overline{x}} \quad \textit{on} \quad \overline{E} = E \cup \{\overline{x}\}\,.$$

Proof. The asserted equivalence is immediate from (9.6) and (9.7). The asserted convergence follows from

$$\begin{aligned}\limsup_{n\to\infty} \mathbf{P}(X_n \leq t) &\leq \limsup_{n\to\infty} \mathbf{P}(X_n^0 \leq t)\\ &= \limsup_{n\to\infty} \mathbf{P}(Y_n^0 \leq t)\\ &= \mathbf{P}(Y \leq t)\\ &= 0 \quad \text{for all} \quad t \in E\,. \quad \square\end{aligned}$$

The results concerning the classification can now be summarized as follows:

(1) if ν is positive recurrent, then

$$\liminf_{n\to\infty} X_n^0 \in E \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} Y_n^0 \in E \text{ a.s.},$$

(2) if ν is null recurrent, then

$$\liminf_{n\to\infty} X_n^0 \in E \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} Y_n^0 \notin E \text{ a.s.},$$

(3) if ν is transient, then

$$\liminf_{n\to\infty} X_n^0 \notin E \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} Y_n^0 \notin E \text{ a.s.}.$$

10. Further ergodic theorems

The convergence in (9.7) is unnecessarily restricted to functions $f \in \mathcal{C}(E)$:

(10.1) Theorem *Let $\nu \in \mathcal{N}[E]$ be positive recurrent with stationary distribution μ. Then the convergence*

$$\mu_n f \to \mu f \quad \textit{with} \quad \mu_n := \mathcal{L}(X_n)$$

holds in each of the following cases:

(a) $f \in \mathcal{R}(E)$,

(b) $f : E \to \mathbf{R}_+$ increasing and $\operatorname{supp} \mu_0$ *compact.*

Proof. (a) Since f is bounded, application of (3.3b) with initial variables $X_0 = x_0$ resp. $X_0 = x$ yields

$$\mu_n f - \mu f = \int_E \int_E \mathbf{E}(f(X_n^{x_0}) - f(X_n^x))\, \mu_0(dx_0)\, \mu(dx) \to 0\,.$$

(b) Application of (a) to $f \wedge k$ with $k \to \infty$ leads to

$$\liminf_{n\to\infty} \mu_n f \geq \mu f\,,$$

establishing the case $\mu f = \infty$. Otherwise, (9.5a) and (9.7) imply

$$(*) \qquad \mathbf{E}(f(X_n^0)) = \mathbf{E}(f(Y_n^0)) \leq \mu f < \infty \quad \text{for all} \quad n \geq 0\,.$$

If $t \in E$ satisfies $\operatorname{supp} \mu_0 \subset [0,t]$, therefore

$$\begin{aligned} \mu_n f &\leq \mathbf{E}(f(X_n^t)) \\ &\leq \mu f + (\mathbf{E}(f(X_n^t)) - \mathbf{E}(f(X_n^0)))\,. \end{aligned}$$

In view of $(*)$ the fundamental inequality (3.1) yields

$$\limsup_{n\to\infty} \mu_n f \leq \mu f\,. \quad \square$$

To see the regularity condition in case (a) to be essential, consider the autoregressive process following (3.3). It is positive recurrent with the uniform distribution on $[0,1[$ as stationary distribution. Here, the convergence $\mu_n f \to \mu f$ fails, for instance, if $X_0 = x_0 \in D$ and $f = 1_D$ with D denoting the set of dyadic numbers in E.

The compactness condition in case (b) is essential as well. Indeed, consider an exchange process with $E = \mathbf{R}_+$ and $\mathbf{E}(V_n) < \infty$, proved to be positive recurrent by (9.7). Assume in addition $\int_E x \, d\mu < \infty$ for its stationary distribution μ – a condition that can be checked to amount to $\prod_{n > \underline{x}} (F(n))^n > 0$ for the underlying distribution function. Then $\mathbf{E}(X_0) = \infty$ by $X_n \geq X_{n-1} - 1$ implies $\mathbf{E}(X_n) = \infty$ for all $n \in \mathbf{N}$.

Application of (10.1) yields in particular pointwise convergence of the associated distribution functions. Together with (2.4) this implies that the familiar classification from discrete Markov chain theory for $\underline{x} < t \in E$ carries over in the following form:

(1) ν positive recurrent $\Leftrightarrow \lim_{n \to \infty} \mathbf{P}(X_n \leq t) > 0$,

(2) ν null recurrent $\Leftrightarrow \lim_{n \to \infty} \mathbf{P}(X_n \leq t) = 0$ and $\sum_{n \geq 0} \mathbf{P}(X_n \leq t) = \infty$,

while in the transient case $\sum_{n \geq 0} \mathbf{P}(X_n \leq t) < \infty$ for all $t \in E$.

To prove a law of large numbers not restricted to functions $f \in \mathcal{C}(E)$, ergodicity will be established first. More generally the following holds:

(10.2) Theorem *Let $\nu \in \mathcal{N}[E]$ be positive recurrent with stationary distribution μ. Then the process $(X_n)_{n \geq 0}$ with $\mathcal{L}(X_0) = \mu$ is mixing.*

Proof. 1. Extending $H_n, n \in \mathbf{N}$, let $H_n, n \in \mathbf{Z}$, be independent variables with distribution ν. Then by (9.7)

$$X'_n := \sup_{m \leq n} H_n \circ \ldots \circ H_m(0) \in E \quad \text{a.s.}$$

and, moreover, $\mathcal{L}(X'_n) = \mu$ for $n \in \mathbf{Z}$. The continuity of $h \in \mathcal{H}[E]$ yields

$$X'_n = H_n(X'_{n-1}) \quad \text{a.s.} \quad \text{for} \quad n \in \mathbf{N}\,.$$

Since X'_0 is independent of $(H_n)_{n \in \mathbf{N}}$, the processes $(X_n)_{n \geq 0}$ and $(X'_n)_{n \geq 0}$ have the same distribution, and it suffices to prove the assertion for $(X'_n)_{n \geq 0}$.

2. Denote by σ resp. σ' the shift in $W := \prod_{n \in \mathbf{Z}} \mathcal{H}[E]$ resp. $W' := \prod_{n \geq 0} \overline{E}$ with $\overline{E} = E \cup \{\overline{x}\}$ and consider the (measurable) mapping

$$\tau : (h_n)_{n \in \mathbf{Z}} \to \Big(\sup_{m \leq n} h_n \circ \ldots \circ h_m(0) \Big)_{n \geq 0}$$

from W to W'. It is compatible with σ and σ', i.e. $\tau \circ \sigma = \sigma' \circ \tau$. Therefore the mixing property of σ with respect to the product measure $\bigotimes_{n\in\mathbf{Z}} \nu$ carries over to σ' with respect to its image by τ. Since this apparently is the distribution of $(X'_n)_{n\geq 0}$, the assertion follows. □

In contrast to the result of (5.4), in general the tail σ-field of $(X_n)_{n\geq 0}$, even under stationarity, need not be trivial. A counterexample is provided by any distribution $\nu \in \mathcal{N}[E]$, whose support consists of two injective mappings $h_0, h_1 \in \mathcal{H}[E]$ with disjoint images $h_i[E]$, as considered in Section 9. Since in this case X_{n-1} can be reconstructed from X_n with probability 1, the tail σ-field of $(X_n)_{n\geq 0}$ coincides with the full σ-field generated by the process up to sets of probability 0.

If the underlying mappings $h \in \mathcal{H}[E]$ satisfy a Lipschitz condition, laws of large numbers regarding functions $f \in \mathcal{C}(E)$ can be found in [3, 9]. In the order context more general results are available:

(10.3) Theorem *Let $\nu \in \mathcal{N}[E]$ be positive recurrent with stationary distribution μ. Then for arbitrary initial law the convergence*

$$\frac{1}{n} \sum_{0\leq m<n} f(X_m) \to \mu f \quad a.s.$$

holds in each of the following cases:

(a) *$f \in \mathcal{R}(E)$,*

(b) *$f : E \to \mathbf{R}_+$ increasing.*

Proof. (a) Let $(X'_n)_{n\geq 0}$ be a copy of $(X_n)_{n\geq 0}$ as in (6.3) and assume $\mathcal{L}(X'_0) = \mu$. Then the process $(X'_n)_{n\geq 0}$ is ergodic by (10.2), hence

$$\frac{1}{n} \sum_{0\leq m<n} f(X'_m) \to \mu f \quad \text{a.s.}.$$

This convergence carries over to $(X_n)_{n\geq 0}$, because $f(X_n) - f(X'_n) \to 0$ a.s. by (3.3b).

(b) It follows from (a) as in the proof of (10.1b) that it is no real restriction to assume $\mu f < \infty$. In this case it follows, again as in the proof of (10.1b), that $\sup_{n\geq 0} \mathbf{E}(f(X^0_n)) < \infty$. Therefore, (3.2b) can replace (3.3b) to continue the proof as in (a). □

Case (b) implies in particular the classical law of large numbers

$$\frac{1}{n} \sum_{0\leq m<n} X_m \to \int_E x\, \mu(dx) \quad \text{a.s.},$$

holding regardless of initial law and proper existence of the integral.

Case (a) cannot dispense with the regularity condition on f as follows by the same counterexample as considered for (10.1a). Moreover, even under continuity, boundedness of f cannot be replaced by integrability with respect to the stationary distribution. To exhibit an appropriate counterexample requires a recursive construction as in the context of (3.3). Choose $E = \mathbf{R}_+$ and let the support of ν consist of the two mappings defined by $h_1(x) = x + 1$ and $h_2(x) = x/(x+1)$. Then ν is positive recurrent by (9.3) with $\underline{x} = 0$ (and $\overline{x} = \infty$). For $X_0 = 0$ consider the random times

$$T_k := \inf\{n \geq 0 : X_n \geq k\} \quad \text{for} \quad k \geq 0\,.$$

Due to $X_n \leq X_{n+1} - 1$ they satisfy $T_0 < T_1 < \ldots$ almost surely. As in Section 3 choose now $n_k \in \mathbf{N}$ such that

$$\limsup_{k\to\infty} \mathbf{P}(T_k \leq n_k) = 1 \tag{1}$$

and finite subsets B_k of $[k, k+1[$ such that

$$\mathbf{P}(T_k \leq n_k, X_{T_k} \notin B_k) = 0 \quad \text{for} \quad k \geq 0\,. \tag{2}$$

Since the stationary distribution μ is nonatomic by (8.1), a continuous function $f : \mathbf{R}_+ \to \mathbf{R}_+$ can be found satisfying the conditions

$$f(x) \geq k\, n_k \quad \text{for} \quad x \in B_k\,, \tag{3}$$

$$\int_{[k,k+1[} f\, d\mu < 2^{-k} \quad \text{for} \quad k \in \mathbf{N}\,. \tag{4}$$

Then the sequence

$$Z_n := \frac{1}{n} \sum_{1 \leq m \leq n} f(X_m) \quad \text{for} \quad n \in \mathbf{N}$$

satisfies

$$\begin{aligned}
\mathbf{P}(\limsup_{n\to\infty} Z_n = \infty) \;&\geq\; \mathbf{P}(\limsup_{k\to\infty} \{Z_{T_k} \geq k\}) \\
&\geq\; \mathbf{P}(\limsup_{k\to\infty} \{T_k \leq n_k\}),
\end{aligned}$$

where the last inequality holds, because (2) and (3) yield

$$Z_{T_k} \geq \frac{1}{T_k} f(X_{T_k}) \geq k \quad \text{a.s.} \quad \text{for} \quad T_k \leq n_k\,.$$

In view of (1) this implies

$$\limsup_{n\to\infty} Z_n = \infty \quad \text{a.s.}\,,$$

while in view of (4) on the other hand $\mu f \leq 1$.

11. Mean passage times

In accordance with the order structure of the state space the hitting times for regions above or below some level are of special interest. The first result in this direction follows from (2.3) in the transient case, but holds as well in the recurrent case:

(11.1) Theorem *Let $\nu \in \mathcal{N}[E]$ be irreducible and fix $t \in E$. Then the hitting time T^t of $\{x \in E : x \geq t\}$ by $(X_n)_{n \geq 0}$ satisfies:*

$$\mathbf{E}^0(\exp(uT^t)) < \infty \quad \textit{for some } u > 0\,,$$

hence in particular $\mathbf{E}^0(T^t) < \infty$.

Proof. Choose $l \in \mathbf{N}$ such that $\vartheta := \mathbf{P}(X_l^0 \geq t) > 0$. Then monotonicity and independence imply

$$\begin{aligned} \mathbf{P}^0(T^t > kl) &\leq \mathbf{P}(H_{(i+1)l} \circ \ldots \circ H_{il+1}(0) < t \text{ for } 0 \leq i < k) \\ &= (1-\vartheta)^k \quad \text{for all} \quad k \geq 0\,. \end{aligned}$$

Partial integration shows that each $u < -\frac{1}{l}\log(1-\vartheta)$ satisfies the assertion. □

The following counterpart of (11.1) separates the two kinds of recurrence:

(11.2) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent and fix $t \in E$ with $t > \underline{x}$. Then the hitting time T_t of $\{x \in E : x \leq t\}$ by $(X_n)_{n \geq 0}$ satisfies*

*(*a*) $\mathbf{E}^x(T_t) < \infty$ for all $x \in E$ whenever ν is positive recurrent,*

*(*b*) $\mathbf{E}^x(T_t) = \infty$ for $t \leq x \in E$ whenever ν is null recurrent.*

Proof. (a) Clearly, the process $(X_n)_{n \geq 0}$ may be assumed to be stationary. Moreover, the assumption $x > t$ means no loss of generality, because the assertion is trivial in the case $t = \overline{x}$. Since $\mathbf{P}(X_0 \leq t) > 0$ by (4.4), any $n \in \mathbf{N}$ with $\mathbf{P}(X_n^0 \geq x) > 0$ satisfies

$$(*) \qquad \mathbf{P}(X_0 \leq t, X_n \geq x) > 0\,.$$

If n is chosen minimal with respect to $(*)$, then

$$\mathbf{P}(X_0 \leq t, X_m \leq t, X_n \geq x) = 0 \quad \text{for} \quad 0 < m < n\,,$$

because otherwise by stationarity $n - m$ would satisfy $(*)$ as well. Thus $\mathbf{P}(A) > 0$ for the event

$$A := \{X_0 \leq t, X_1 > t, \ldots, X_{n-1} > t, X_n \geq x\} \subset \{T_t > n\}\,.$$

With the increasing function

$$g(y) := \mathbf{E}^y(T_t) \quad \text{for} \quad y \in E$$

the recurrence theorem by Kac and the Markov property imply

$$\begin{aligned}
\mathbf{P}(T_t < \infty) &= \int_{\{X_0 \le t\}} T_t \, d\mathbf{P} \\
&\ge \int_A T_t \, d\mathbf{P} \\
&= \int_A (n + g(X_n)) \, d\mathbf{P} \\
&\ge \mathbf{P}(A) \, (n + g(x)) \, .
\end{aligned}$$

Therefore $g(x) < \infty$, as had to be shown.

(b) If μ is the invariant measure and $t < y \in E$, then

$$\mu'(B) := (\mu([0,y])^{-1}\mu(B) \quad \text{for} \quad B \in \mathcal{B}([0,y])$$

by (4.2a) and (4.4) defines a stationary distribution with respect to ${}^yP$. Now let X_0 be distributed according to (the trivial extension of) μ' and let T'_t denote the hitting time of $[0,t]$ by $({}^yX_n)_{n \ge 0}$. Then it is obvious that $T'_t \le T_t$, and it follows, again from the recurrence theorem by Kac, that

$$\begin{aligned}
\mathbf{P}(T'_t < \infty) &= \int_{\{{}^yX_0 \le t\}} T'_t \, d\mathbf{P} \\
&= \int_{x_0 \le t} \mathbf{E}^{x_0}(T'_t) \, \mu'(dx_0) \\
&\le \int_{x_0 \le t} \mathbf{E}^{x_0}(T_t) \, \mu'(dx_0) \\
&\le \mu'([0,t]) \, \mathbf{E}^t(T_t) \, .
\end{aligned}$$

Since T'_t is almost surely finite, therefore

$$\mu([0,y]) \le \mu([0,t]) \, \mathbf{E}^t(T_t) \, .$$

Letting y increase to $\overline{x}$ in view of $\mu([0,t]) < \infty$ yields $\mathbf{E}^t(T_t) = \infty$, hence $\mathbf{E}^x(T_t) = \infty$ for all $x \in E$ with $x \ge t$. □

Since $\mathbf{P}^t(T_t < \infty) = 1$ clearly implies $\underline{x} \le t$, in the transient case the equation $\mathbf{E}^t(T_t) = \infty$ holds for all $t \in E$.

Next, the topological structure of the state space is taken into account:

(11.3) Proposition *Let $\nu \in \mathcal{N}[E]$ be positive recurrent with stationary distribution μ. Then for each open subset G of E satisfying $\mu(G) > 0$ and every $t \in E$ the stopping time*

$$T := \inf\{n \in \mathbf{N} : X_n^x \in G \textit{ for } 0 \le x \le t\}$$

has a finite expectation.

Proof. 1. By (6.6) there exists $n \in \mathbf{N}$ such that

$$\vartheta := \mathbf{P}(X_n^x \in G \text{ for } 0 \le x \le t) > 0 \, .$$

If T' denotes the analogous stopping time with respect to $\nu' = \nu^n$, then apparently $T \le nT'$, hence $n = 1$ may be assumed in view of (2.5).

2. Moreover, $t > \underline{x}$ or $t = \overline{x}$ means no loss of generality. Then

$$S_0 := 0 \quad \text{and} \quad S_{k+1} := \inf\{n > S_k : H_n \circ \ldots \circ H_{S_k+1}(t) \le t\}$$

are almost surely finite stopping times with respect to $(H_n)_{n \in \mathbf{N}}$, which by (11.2a) satisfy

$$\text{(1)} \qquad \mathbf{E}(S_k - S_{k-1}) = \mathbf{E}^t(T_t) < \infty \quad \text{for} \quad k \in \mathbf{N}\,.$$

Finally, the events

$$A_k := \{H_{S_k+1}(x) \in G \text{ for } 0 \le x \le t\}$$

by the assumption $n = 1$ satisfy

$$\text{(2)} \qquad \mathbf{P}(A_k) = \vartheta > 0 \quad \text{for} \quad k \ge 0\,.$$

3. By construction the variables $1_{A_0}, \ldots, 1_{A_{k-1}}, S_{k+1} - S_k$ are independent for each $k \ge 0$. Moreover, the estimate

$$T \le 1 + \sum_{k \ge 0} \prod_{i \le k} (1 - 1_{A_i})\,(S_{k+1} - S_k)$$

holds, because the right-hand side for fixed $\omega \in \Omega$ equals $S_k(\omega) + 1$, if k is the first index with $\omega \in A_k$, and is infinite, if there is no such index. If for each k the factor with $i = k$ is cancelled, the bound for T is increased and the summands are composed of independent factors. By (1) and (2) this yields

$$\mathbf{E}(T) \le 1 + \vartheta^{-1}\,\mathbf{E}^t(T_t) < \infty\,. \quad \Box$$

If in particular $\overline{x} \in E$, this result implies $\sup_{x \in E} \mathbf{E}^x(T_G) < \infty$, where T_G denotes the hitting time of G by $(X_n)_{n \ge 0}$. With this notation the familiar criterion for positive / null recurrence by mean passage times carries over from discrete Markov chain theory in the following form:

(11.4) Theorem *Let $\nu \in \mathcal{N}[E]$ be recurrent with attractor M and fix $x \in M$. Then ν is positive recurrent if and only if*

$$\mathbf{E}^x(T_G) < \infty \quad \textit{for all open subsets } G \textit{ of } E \textit{ containing } x\,.$$

Proof. According to (11.3) only the sufficiency of the condition has to be established. To this end assume ν to be null recurrent. Then $\overline{x} \notin E$ by (9.3), hence $x < \overline{x}$ and thus $\nu(h(x) > x) > 0$ by (1.1). This implies the existence of $t \in E$ with $t > x$ such that $\vartheta := \nu(h(x) > t) > 0$. With the notation $\mu_1 := \mathcal{L}(X_1^x)$ an application of (11.2b) yields

$$\begin{aligned} \mathbf{E}^x(T_t) &= (1-\vartheta) + \int_{y>t} (1+\mathbf{E}^y(T_t))\,\mu_1(dy) \\ &\geq \int_{y>t} \mathbf{E}^t(T_t)\,\mu_1(dy) \\ &= \vartheta\,\mathbf{E}^t(T_t) \;=\; \infty\,, \end{aligned}$$

i.e. the condition is violated for $G = [0, t[\ni x$. $\square$

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