Normal forms and linearity in nonflat domains

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Scott, Plotkin, and sequentiality

- Dana Scott and Juri Ershov [late 60's–70's]: Scott–Ershov domains with Scott-continuous functions provide an appropriate framework for higher-type computability and semantics of programming languages.
- Gordon Plotkin [Plotkin 1977] and PCF: There are inherently nonsequential functionals in Scott's model:

$$\mathrm{pcond}(q, x, y) = \begin{cases} x & \text{if } q = \mathtt{tt}, \\ y & \text{if } q = \mathtt{ff}, \\ x \cap y & \text{if } q = \bot. \end{cases}$$

Berry, Zhang, and linearity

- Gerard Berry [Berry1978]: If a functional is sequential, it has to be stable (that is, preserve consistent infima).
- Guo-Qiang Zhang [Zhang 1989–1992]: In order to represent stable domains by information systems, we have to require linearity (in Munich, "atomicity"): if a consistent set entails a token of information, it must do so with a single witness.

Linear logic and higher-type computability

- Stability and linearity are quite relevant to classical [Girard *et al.* 1989] and intuitionistic linear logic [Bucciarelli *et al.* 2009-10].
- Helmut Schwichtenberg and the Munich group [Schwichtenberg, Huber, B., Ranzi 2006–] working towards a formal theory of computable functionals (TCF+), have been using *nonflat* base types (and have obtained density, preservation of values, adequacy, definability. . .), sometimes with linear systems, most of the time with nonlinear ones.

Flatness vs. nonflatness

- Why nonflat? (a) Trivially good reasons: injectivity of constructors and nonoverlapping of their ranges. (b) Deeper good reasons: more degrees of freedom in the model allow for stronger results—see [Escardò 1993] and Davide's talk.
- Why *not* nonflat? (a) Trivially good reasons: combinatorial chaos. (b) Deeper good reasons: flat base types are still refined enough to support relevant research; *flat* base types are linear but *nonflat* aren't.
- *But*: function spaces preserve linearity!

Coherent information systems

- Information system $A = (\mathrm{Tok}, \mathsf{Con}, \vdash)$

$$\begin{aligned}
&\{a\} \in \mathsf{Con}, \\
&U \subseteq V \wedge V \in \mathsf{Con} \to U \in \mathsf{Con}, \\
&U \in \mathsf{Con} \wedge a \in U \to U \vdash a, \\
&U \vdash V \wedge V \vdash c \to U \vdash c, \\
&U \in \mathsf{Con} \wedge U \vdash b \to U \cup \{b\} \in \mathsf{Con}.
\end{aligned}$$

- Coherence property

$$\forall_{a,a' \in U} \{a, a'\} \in \mathsf{Con} \to U \in \mathsf{Con}. \tag{1}$$

Write $a \asymp b$ for $\{a, b\} \in \mathsf{Con}$, and even $U \asymp V$ for $U \cup V \in \mathsf{Con}$.

Coherent function spaces

Function space $A \to B$

$$\begin{aligned}
\langle U, b\rangle \in \mathrm{Tok} &:= U \in \mathsf{Con}_A \wedge b \in \mathrm{Tok}_B, \\
\langle U, b\rangle \asymp \langle U', b'\rangle &:= U \asymp_A U' \to b \asymp_B b', \\
W \vdash \langle U, b\rangle &:= WU \vdash_B b,
\end{aligned}$$

where

$$b \in WU := \underset{U' \in \mathsf{Con}_A}{\exists} \left(\langle U', b\rangle \in W \wedge U \vdash_A U'\right).$$

Fact

The function space of two coherent information systems is itself a coherent information system.

Linear information systems

Linearity property

$$U \vdash b \to \exists_{a \in U} \{a\} \vdash b \qquad (2)$$

Fact

The function space of two linear information systems is itself a linear information system.

Objects as ideals

Ideal $x \in \mathsf{Ide}$

$$\forall_{U \subseteq^f x} \left(U \in \mathsf{Con} \wedge \forall_{b \in \mathrm{Tok}} (U \vdash b \to b \in x)\right)$$

Coherent domains (with countable bases) are algebraic bounded complete cpo's, where every set of compacts has a least upper bound exactly when each of its *pairs* has a least upper bound.

Fact

Let $(\mathrm{Tok}, \mathsf{Con}, \vdash)$ *be a coherent information system. Then* $(\mathsf{Ide}, \subseteq, \varnothing)$ *is a coherent domain with compacts given by* $\{\overline{U} \mid U \in \mathsf{Con}\}$. *Conversely, every coherent domain can be represented by a coherent information system.*

Approximable mappings

Approximable mapping $r \subseteq \mathsf{Con}_A \times \mathsf{Con}_B$

$$\begin{aligned}
&\langle \varnothing, \varnothing \rangle \in r, \\
&\langle U, V_1 \rangle, \langle U, V_2 \rangle \in r \to \langle U, V_1 \cup V_2 \rangle \in r, \\
&U \vdash_\rho U' \wedge \langle U', V' \rangle \in r \wedge V' \vdash_\sigma V \to \langle U, V \rangle \in r.
\end{aligned}$$

Fact

There is a bijective correspondence between the approximable mappings from ρ to σ and the ideals of the function space $\rho \to \sigma$; domains (with Scott continuous functions) and information systems (with approximable mappings) are categorically equivalent [Scott 1982]. Moreover, the equivalence is preserved if we restrict ourselves to the coherent case [B 2013].

Types and partiality

- Base types ι, given by constructors

$$\begin{aligned}\mathbb{B} &= \{\mathtt{tt}, \mathtt{ff}\}\,,\\ \mathbb{N} &= \{\mathtt{0}, \mathtt{S0}, \mathtt{SS0}, \dots\}\,,\\ \mathbb{D} &= \{\mathtt{0}, \mathtt{1}, \mathtt{S0}, \dots, \mathtt{B01}, \dots, \mathtt{BS0B01}, \dots\}\,,\end{aligned}$$

  and higher types $\rho \to \sigma$.

- Partiality at base types ι is not a distinguished *token* but a distinguished *nullary constructor* $*_\iota$: the base types are already *nonflat*:

$$\begin{aligned}\mathbb{B} &= \{*, \mathtt{tt}, \mathtt{ff}\}\,,\\ \mathbb{N} &= \{*, \mathtt{0}, \mathtt{S}*, \mathtt{S0}, \mathtt{SS}*, \mathtt{SS0}, \dots\}\,,\\ \mathbb{D} &= \{*, \mathtt{0}, \mathtt{1}, \mathtt{S}*, \mathtt{S0}, \dots, \mathtt{B}*\mathtt{1}, \dots, \mathtt{BS}*\mathtt{B01}, \dots\}\,.\end{aligned}$$

The information system induced by $\mathbb{D}$:

$$\begin{aligned}
&*, \mathtt{0}, \mathtt{1} \in \mathrm{Tok}, \\
&a \in \mathrm{Tok} \to \mathtt{S}a \in \mathrm{Tok}, \\
&a, b \in \mathrm{Tok} \to \mathtt{B}ab \in \mathrm{Tok},
\end{aligned}$$

$$\begin{aligned}
&a \asymp * \wedge * \asymp a, \\
&a \asymp a' \to \mathtt{S}a \asymp \mathtt{S}a', \\
&a \asymp a' \wedge b \asymp b' \to \mathtt{B}ab \asymp \mathtt{B}a'b',
\end{aligned}$$

$$\begin{aligned}
&U \vdash *, \\
&U \vdash a \to \mathtt{S}U \vdash \mathtt{S}a, \text{ for } U \neq \varnothing, \\
&U \vdash a \wedge V \vdash b \to \mathtt{B}UV \vdash \mathtt{B}ab, \text{ for } U, V \neq \varnothing, \\
&U \vdash b \to U \cup \{*\} \vdash b,
\end{aligned}$$

where $\mathtt{B}UV := \{\mathtt{B}ab \mid a \in U, b \in V\}$.

Inconveniences

Fact

Let ι be an algebra given by constructors. The triple $(\mathrm{Tok}_\iota, \mathsf{Con}_\iota, \vdash_\iota)$ is a coherent information system.

The definition above is rather unduly involved—actually, I had to push some details concerning entailment under the rug, to keep the slides relatively light. Moreover, we have two main sources of inconvenience.

- The systems $\mathbb{B}$ and $\mathbb{N}$ are linear but $\mathbb{D}$ is not: $\{\mathtt{B0*}, \mathtt{B*1}\} \vdash \mathtt{B01}$ but $\{\mathtt{B0*}\} \nvdash \mathtt{B01}$ and $\{\mathtt{B*1}\} \nvdash \mathtt{B01}$.
- At base types antisymmetry holds for tokens, but neither for neighborhoods (e.g., $\{\mathtt{B0*}, \mathtt{B*1}\} \sim \{\mathtt{B01}\}$ and $\{\mathtt{S0}, \mathtt{S*}\} \sim \{\mathtt{S0}\}$) nor, consequently, at higher types.

Neighborhood mappings

Let ρ, σ be types. A mapping $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ is *compatible*, *monotone*, and *consistent* if

$$U_1 \sim_\rho U_2 \to f(U_1) \sim_\sigma f(U_2),$$
$$U_1 \vdash_\rho U_2 \to f(U_1) \vdash_\sigma f(U_2),$$
$$U_1 \asymp_\rho U_2 \to f(U_1) \asymp_\sigma f(U_2),$$

respectively.

Lemma

Let $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ be a neighborhood mapping.

1. *It is monotone if and only if it is compatible with equientailment and $f(U_1 \cup U_2) \vdash_\sigma f(U_1) \cup f(U_2)$ for every $U_1, U_2 \in \mathrm{Con}_\rho$ with $U_1 \asymp_\rho U_2$.*
2. *If it is monotone, then it is also consistent.*

Idealization

The *idealization* $\hat{f}$ of a neighborhood mapping $f : \mathrm{Con}_\rho \to \mathrm{Con}_\sigma$ is the token set

$$\hat{f} := \{\langle U, b\rangle \in \mathrm{Tok}_{\rho\to\sigma} \mid \underset{U_1,\dots,U_m}{\exists} (U \vdash_\rho \bigcup_j U_j \wedge \bigcup_j f(U_j) \vdash_\sigma b)\}$$

Theorem

Let ρ, σ be types, and f be a neighborhood mapping at type $\rho \to \sigma$. Then $\hat{f}$ is an ideal if and only if f is consistent.

Not all ideals are induced by neighborhood mappings: e.g., at type $\mathbb{N} \to \mathbb{N}$ take $\{\langle 0, \mathrm{S}^n * \rangle \mid n = 0, 1, \dots\}$. Neighborhood mappings are those approximable maps r for which $r(U)$ is covered by a finite collection $V_1, \dots, V_m \in \mathrm{Con}_\sigma$ for every $U \in \mathrm{Con}_\rho$.

Normal form mappings

Let ρ be a type. A neighborhood-mapping $f : \mathsf{Con}_\rho \to \mathsf{Con}_\rho$ is a *normal form mapping (at type ρ)* if it preserves information and identifies equivalent neighborhoods, that is,

$$\begin{aligned} &f(U) \sim_\rho U, \\ &U_1 \sim_\rho U_2 \to f(U_1) = f(U_2). \end{aligned}$$

Every normal form mapping is monotone (so by Lemma 6 also compatible and consistent).

Normal forms at base types: closures and suprema

- *Deductive closure*. Define

$$\overline{U} := \{b \in \mathrm{Tok} \mid U \vdash b\}\,.$$

  The mapping $U \mapsto \overline{U}$ is a normal form mapping at base types.

- *Supremum*. For $a, b \in \mathrm{Tok}_{\mathbb{D}}$, define $\sup(a, b)$ by

$$\begin{aligned} &\sup(a, *) = \sup(*, a) = a,\\ &\sup(\mathtt{S}a, \mathtt{S}a') = \mathtt{S}\sup(a, a'),\\ &\sup(\mathtt{B}ab, \mathtt{B}a'b') = \mathtt{B}\sup(a, a')\sup(b, b'). \end{aligned}$$

  For a neighborhood $U \in \mathrm{Con}_{\mathbb{D}}$ define $\sup(U) \in \mathrm{Tok}$ by

$$\begin{aligned} &\sup(\varnothing) := *,\\ &\sup(\{a_1, \dots, a_m\}) := \sup(\cdots \sup(a_1, a_2) \cdots, a_m). \end{aligned}$$

  The neighborhood mapping $U \mapsto \{\sup(U)\}$ is a normal form mapping at base types.

Normal forms at base types: paths I

Path reduced neighborhood. Define the paths in $\mathbb{D}$, $\mathrm{Tok}^p_{\mathbb{D}}$, by

$$\begin{aligned}
&*, \mathtt{0}, \mathtt{1} \in \mathrm{Tok}^p_{\mathbb{D}}, \\
&a \in \mathrm{Tok}^p_{\mathbb{D}} \to \mathtt{S}a \in \mathrm{Tok}^p_{\mathbb{D}}, \\
&a, b \in \mathrm{Tok}^p_{\mathbb{D}} \to \mathtt{B}a*, \mathtt{B}*b \in \mathrm{Tok}^p_{\mathbb{D}}.
\end{aligned}$$

At a base type ι, let $p \in \mathrm{Tok}^p_\iota$, $a, b \in \mathrm{Tok}_\iota$, and $U \in \mathrm{Con}_\iota \setminus \varnothing$. The following hold.

- *path comparability*: $p \vdash_\iota a \wedge p \vdash_\iota b \to a \vdash_\iota b \vee b \vdash_\iota a$
- *downward closure*: $p \vdash_\iota a \to a \in \mathrm{Tok}^p_\iota$
- *path linearity*: $U \vdash_\iota p \to \exists_{a \in U} \{a\} \vdash_\iota p$

Normal forms at base types: paths II

A *path reduced neighborhood* is an inhabited neighborhood whose every token is maximal and a path.

Theorem (Path normal form)

There exists a normal form mapping $\mathrm{nf}^p : \mathsf{Con}_\iota \to \mathsf{Con}_\iota$*, such that* $\mathrm{nf}^p(U)$ *is path reduced for every* $U \in \mathsf{Con}_\iota$*.*

Moving on to higher types

Let $W = \{\langle U_1, b_1\rangle, \ldots, \langle U_m, b_m\rangle\} \in \mathsf{Con}_{\rho\to\sigma}$. Let

$$L(W) := \bigcup_{i=1}^{m} U_i = \{a \in U_i \mid i = 1, \ldots, m\},$$
$$R(W) := \{b_i \mid i = 1, \ldots, m\}.$$

These finite sets are not necessarily consistent! Also, write

$$\langle U, V\rangle := \{\langle U, b\rangle \mid b \in V\}.$$

Eigen-neighborhoods I

An *eigen-neighborhood* of W is a neighborhood $H = \langle U, V \rangle$, where $U \in \mathsf{Con}_{L(W)}$ (a subset of $L(W)$ which is consistent) and furthermore

$$U = \overline{U} \cap L(W) \wedge V = \overline{WU} \cap R(W).$$

Write $H \in \mathrm{Eig}_W$. The *eigenform* of W is given by the neighborhood mapping

$$\mathrm{eig}(W) := \bigcup_{U \in \mathsf{Con}_{L(W)}} \left\langle \overline{U} \cap L(W), \overline{WU} \cap R(W) \right\rangle,$$

that is, it is the union $\bigcup \mathrm{Eig}_W$ of its eigen-neighborhoods. (At base types we use the convention $\mathrm{eig}(U) := U$.)

Eigen-neighborhoods II

Lemma (Eigenform)

Let ρ and σ be types, and $W, W_1, W_2 \in \mathsf{Con}_{\rho\to\sigma}$.

1. *The eigenform mapping is information preserving, that is,* $W \sim_{\rho\to\sigma} \mathrm{eig}(W)$*, and idempotent, that is* $\mathrm{eig}(\mathrm{eig}(W)) = \mathrm{eig}(W)$*.*
2. *It is*

$$W_1 \vdash_{\rho\to\sigma} W_2 \leftrightarrow \underset{H_2 \in \mathrm{Eig}_{W_2}}{\forall} \; \underset{H_1 \in \mathrm{Eig}_{W_1}}{\exists} H_1 \vdash_{\rho\to\sigma} H_2,$$

$$W_1 \asymp_{\rho\to\sigma} W_2 \leftrightarrow \underset{H_1 \in \mathrm{Eig}_{W_1}}{\forall} \; \underset{H_2 \in \mathrm{Eig}_{W_2}}{\forall} H_1 \asymp_{\rho\to\sigma} H_2.$$

Note: The mapping eig is *not* a normal form mapping!

Eigen-maximal neighborhoods

Write Eig^0_W for the inhabited eigen-neighborhoods of W. Call $W \in \mathsf{Con}_{\rho\to\sigma}$ *eigen-maximal* if $W = \mathrm{eig}(W)$, and each $H \in \mathrm{Eig}_W$ is either empty or maximal, that is, if $H \in \mathrm{Eig}^0_W$, then for all $H' \in \mathrm{Eig}_W$ with $H' \vdash_{\rho\to\sigma} H$, it is $H' \sim_{\rho\to\sigma} H$.
An eigen-maximal neighborhood is "flat", in the sense that the inclusion diagram of its eigen-neighborhoods forms a flat tree.

Lemma

Let ρ, σ be types. There exists a neighborhood mapping emax *such that for every $W \in \mathsf{Con}_{\rho\to\sigma}$ the neighborhood $\mathrm{emax}(W)$ is eigen-maximal and $W \sim_{\rho\to\sigma} \mathrm{emax}(W)$.*

Note: The mapping emax is (still) *not* a normal form mapping!

Eigen-products of neighborhood mappings

Write Fin_ρ for all (not necessarily consistent) finite token sets at type ρ. If f : $\mathsf{Con}_\rho \to \mathsf{Con}_\rho$ and g : $\mathsf{Con}_\sigma \to \mathsf{Con}_\sigma$, define their *eigenproduct* $\langle f, g \rangle$: $\mathsf{Con}_{\rho\to\sigma} \to \mathrm{Fin}_{\rho\to\sigma}$ by

$$\langle f, g \rangle (W) := \bigcup_{H \in \mathrm{Eig}\,^0_W} \langle f(L(H)), g(R(H)) \rangle .$$

Lemma

Let f and g be normal form mappings at types ρ and σ respectively. Then their eigenproduct is a normal form mapping at type $\rho \to \sigma$, when restricted to eigen-maximal neighborhoods.

Normal forms at higher types

As a corollary we obtain the following.

Theorem (Inductive normal forms)

Let f and g be normal form mappings at types ρ and σ respectively. Then the mapping $\langle f, g\rangle \circ \mathrm{emax}$ is a normal form mapping at type $\rho \to \sigma$.

Implicit linearity

Call a type *implicitly linear* when every neighborhood has an equivalent one which is linear.
All base types are implicitly linear, since there are normal forms for every neighborhood which are linear, like the closure and the supremum.

Theorem

Let ρ be an arbitrary type. There exists a neighborhood mapping $\mathrm{at}_\rho : \mathsf{Con}_\rho \to \mathsf{Con}_\rho$, such that $\mathrm{at}_\rho(U)$ is linear and equivalent to U for all $U \in \mathsf{Con}_\rho$.

Witness.

$\mathrm{at}_{\rho\to\sigma}(W) := \langle \mathrm{id}, \mathrm{at}_\sigma \rangle (W)$. □

Explicit linearity I

Fact 2 (i.e., the preservation of linearity by exponentiation) bluntly suggests the following simple strategy: build your base type information systems in a linear manner and you're done.
The only challenge is to avoid missing some ideals while restricting to linear base types.

Explicit linearity II

Write $\rho \cong \sigma$ if the ideals of ρ and the ideals of σ are in a bijective correspondence.

Theorem

Let ι be a finitary base type. There exists a linear-coherent information system ι', such that $\iota' \cong \iota$.

Explicit linearity III

Proofsketch.

Given a finitary base type ι, define the *path subsystem of* ι, ι^p, by letting

$$\begin{aligned} \mathrm{Tok}_{\iota^p} &:= \mathrm{Tok}_\iota^p, \\ \mathsf{Con}_{\iota^p} &:= \mathsf{Con}_\iota \cap \mathcal{P}_f(\mathrm{Tok}_{\iota^p}), \\ \vdash_{\iota^p} &:= \vdash_\iota \cap (\mathsf{Con}_{\iota^p} \times \mathrm{Tok}_{\iota^p}). \end{aligned}$$

The triple ι^p is a coherent information system and it is $\iota^p \cong \iota$. To see that it is linear, let $U \in \mathsf{Con}_{\iota^p}$ and $q \in \mathrm{Tok}_{\iota^p}$ be such that $U \vdash_{\iota^p} q$. By path linearity there is a $p \in U$ with $\{p\} \vdash_\iota q$. But p is itself a path, so $\{p\} \vdash_{\iota^p} q$. $\square$

Outlook

- Can linearity help us prove definability also for base types with superunary constructors? [Huber, B, Schwichtenberg 2010]
- How exactly does "linearity" manifest in a formal topological setting a la Padua?
- Is there something to be gained by pursuing nonflat models of linear logic?