Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: May 6, 2015; deadline for handing in the solutions:
May 13, 2015, 10 pm (22:00)

10. Let $\mathcal{A}$ be the C*-algebra of a quantum spin system on $\Gamma = \mathbb{Z}^d$, with $\mathcal{H}_x = \mathcal{H}$ for all $x \in \mathbb{Z}^d$. Let $\mathbb{Z}^d \ni z \mapsto \tau_z$ be the family of *-automorphisms of spatial translations. Prove that $\mathcal{A}$ is asymptotically abelian with respect to τ, viz.,

$$\lim_{|z|\to\infty} [\tau_z(a), b] = 0, \tag{12}$$

for all $a, b \in \mathcal{A}$.

11. Let $\mathcal{A}$ be a C*-algebra with a unit and let $\{\tau_t\}_{t\in\mathbb{R}}$ be a weakly continuous one-parameter group of *-automorphisms of $\mathcal{A}$, which by definition means

- for all $t \in \mathbb{R}$, τ_t is a *-automorphisms of $\mathcal{A}$
- $\tau_0 = \mathrm{id}$ and $\tau_s \circ \tau_t = \tau_{s+t}$ holds for all $s, t \in \mathbb{R}$
- for any state ω and $x \in \mathcal{A}$: $\lim_{t\to 0} \omega(\tau_t(x)) = \omega(x)$.

(i) Let ν be a τ_t-invariant state, $\nu \circ \tau_t = \nu$ for all $t \in \mathbb{R}$. Prove that there exists a densely defined self-adjoint operator H on the GNS Hilbert space $\mathcal{H}$ such that

$$\pi(\tau_t(x)) = \exp(\mathrm{i}tH)\pi(x)\exp(-\mathrm{i}tH), \quad \text{and} \quad H\Omega = 0 \tag{13}$$

Hint: Stone's theorem.

(ii) Show that there always exists a τ_t-invariant state
Hint: You can safely assume that $\mathcal{E}(\mathcal{A}) \neq \emptyset$. There is a natural operation on any state ω that yields a candidate invariant state.

12. Consider the C*-algebra $\mathcal{A}$ of a one-dimensional infinite chain of spins-1/2. Here, $\Gamma = \mathbb{Z}$ and the local algebras $\mathcal{A}_\Lambda = \bigotimes_{n\in\Lambda} \mathcal{A}_n$, with the on-site Hilbert spaces being $\mathcal{H}_n = \mathbb{C}^2$ for all $n \in \Gamma$ and $\mathcal{A}_n = M_{2\times 2}(\mathbb{C})$. Note that $\mathcal{A}_n$ is generated by the identity and the Pauli matrices $\sigma_n^x, \sigma_n^y, \sigma_n^z$, and each $A \in \mathcal{A}_n$ is identified with the corresponding element $\ldots \otimes 1_{n-1} \otimes A \otimes 1_{n+1} \otimes \ldots$ of $\mathcal{A}$. The goal of this exercise is to show that $\mathcal{A}$ admits two inequivalent representations $(\mathcal{H}_\pm, \pi_\pm)$.

Let

$$S_+ := \{s = (s_n)_{n\in\mathbb{Z}} : s_n \in \{-1, +1\} \text{ and } s_n \neq 1 \text{ for at most finitely many } n\}$$
$$S_- := \{s = (s_n)_{n\in\mathbb{Z}} : s_n \in \{-1, +1\} \text{ and } s_n \neq -1 \text{ for at most finitely many } n\}$$
$$\mathcal{H}_\pm = l^2(S_\pm) = \{f : S_\pm \to \mathbb{C} : \sum_{s\in S_\pm} |f(s)|^2 < \infty\} \tag{14}$$

Note that since $S_\pm$ are countable, then $l^2(S_\pm)$ is separable with canonical orthonormal basis $\{e_s\}_{s\in S_\pm}$ given by fixed spin configurations

$$e_s(t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

For any $n \in \mathbb{Z}$, let furthermore $\Theta_n : \mathcal{S}_\pm \to \mathcal{S}_\pm$

$$(\Theta_n(s))_m = \begin{cases} -s_m & \text{if } n = m \\ s_m & \text{otherwise} \end{cases} \tag{16}$$

Finally, let $\pi_\pm : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pm)$ be defined by

$$(\pi_\pm(1_n)(f))(s) := f(s) \tag{17}$$
$$(\pi_\pm(\sigma_n^x)(f))(s) := f(\Theta_n(s)) \tag{18}$$
$$(\pi_\pm(\sigma_n^y)(f))(s) := \mathrm{i}s_n f(\Theta_n(s)) \tag{19}$$
$$(\pi_\pm(\sigma_n^z)(f))(s) := s_n f(s) \tag{20}$$

for all $f \in \mathcal{H}_\pm, s \in S_\pm$.

(i) Prove that $\pi_\pm$ define representations of $\mathcal{A}$ in $\mathcal{H}_\pm$

(ii) Show that $\pi_\pm$ are irreducible representations
Hint: Recall that a representation is irreducible if and only if any vector is cyclic; for any $f \in \mathcal{H}_\pm$, any basis vector can be approximated arbitrarily well by $\pi_\pm(x_{i_N}) \cdots \pi_\pm(x_{i_1}) f$, where $x_j \in \mathcal{A}_{\{j\}}$ and of the form $(1_j \pm \sigma_j^z)/2$ or σ_j^x.

(iii) For each $N \in \mathbb{N}$, consider the local average magnetisation operator $M_N := \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma_n^z \in \mathcal{A}$. Prove that

$$\pi_\pm(M_N) \to \pm 1 \quad \text{weakly, in the operator sense} \tag{21}$$

i.e. for any $\phi_\pm, \psi_\pm \in \mathcal{H}_\pm$, $\lim_{N\to\infty} \langle \phi_\pm, \pi_\pm(M_N)\psi_\pm \rangle_{\mathcal{H}_\pm} = \langle \phi_\pm, \psi_\pm \rangle_{\mathcal{H}_\pm}$

(iv) Conclude that $\pi_\pm$ are inequivalent representations

(v) Argue that $\mathcal{A}$ admits in fact infinitely many inequivalent representations