Mathematical Statistical Physics, 2015
Homework Problems, LMU

Issued: July 1, 2015; deadline for handing in the solutions:
July 7, 2015, 10 pm (22:00)

31. Consider the Heisenberg Hamiltonian on a hypercube $\Lambda = [-L, \cdots, L]^d \cap \mathbb{Z}^d$,

$$H_\Lambda^{(h,u)} := -2 \sum_{\{x,y\} \in E_\Lambda} \left(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 \right) - h \sum_{x \in \Lambda} S_x^3, \qquad u \in [-1,1]. \tag{77}$$

where E_Λ is the set of bonds of Λ and the last term corresponds to the interaction with a constant external magnetic field $h > 0$ in the '3'-direction. Let $M_\Lambda := |\Lambda|^{-1} \sum_{x \in \Lambda} S_x^3$ be the average magnetisation observable. Define the residual and spontaneous magnetisations as

$$m_{\text{res}} := \lim_{h \to 0^+} \liminf_{L \to \infty} \omega_{\beta,\Lambda}^{(h,u)} \left(M_\Lambda \right) \tag{78}$$

$$m_{\text{sp}} = \liminf_{L \to \infty} \omega_{\beta,\Lambda}^{(0,u)} \left(|M_\Lambda| \right) \tag{79}$$

where $\omega_{\beta,\Lambda}^{(h,u)}$ is the finite volume Gibbs state of $H_\Lambda^{(h,u)}$.

(i) Prove that there are constants $C_1, C_2 > 0$, uniform in Λ, such that

$$C_1 \omega_{\beta,\Lambda}^{(0,u)} \left(|M_\Lambda| \right)^2 \leq \omega_{\beta,\Lambda}^{(0,u)} \left((M_\Lambda)^2 \right) \leq C_2 \omega_{\beta,\Lambda}^{(0,u)} (|M_\Lambda|) \tag{80}$$

(ii) Prove that

$$m_{\text{res}} \geq m_{\text{sp}}. \tag{81}$$

32. Let $A, B \in \mathcal{A}_\Lambda$ and we shall drop the index Λ. We consider Duhamel's two-point function

$$(A, B)_\beta := Z(\beta)^{-1} \int_0^1 \mathrm{Tr}\left(\mathrm{e}^{-s\beta H} A \mathrm{e}^{-(1-s)\beta H} B\right) ds. \tag{82}$$

(i) *Basic properties.*
Prove that $(A, B)_\beta = (B, A)_\beta$.
Is the thermal two-point function $\omega_\beta(AB)$ also symmetric?
Prove Schwarz's inequality,

$$|(A, B)_\beta|^2 \leq (A^*, A)_\beta (B^*, B)_\beta \tag{83}$$

(ii) *Relation to thermal expectations and the KMS condition.*
Compute the thermal expectation value $\omega_\beta(A)$ using Duhamel's two-point function.
Conversely, let $\tau_t(A) = \exp(\mathrm{i}tH) A \exp(-\mathrm{i}tH)$, and let τ_z be its analytic continuation in the strip $\mathrm{Im}(z) \leq 1$. Show that

$$(A, B)_\beta = \int_0^1 \omega_\beta(B \tau_{\mathrm{i}s\beta}(A)) ds \tag{84}$$

(iii) *Bogoliubov's inequality.*
Use a convexity argument for the function

$$h_\beta(s) := \mathrm{Tr}\left(\mathrm{e}^{-s\beta H} A^* \mathrm{e}^{-(1-s)\beta H} A\right) \tag{85}$$

to show that

$$(A^*, A)_\beta \leq \frac{1}{2} \omega_\beta(\{A^*, A\}) \tag{86}$$

Further, prove that

$$\omega_\beta([A, B]) = ([A, \beta H], B)_\beta \tag{87}$$

and conclude that

$$|\omega_\beta([A, B])|^2 \leq \frac{\beta}{2} \omega_\beta([A^*, [H, A]]) \omega_\beta(\{B^*, B\}) \tag{88}$$