Mathematical Quantum Mechanics, 2014/15
Homework Problems, LMU

Issued: January 14, 2015; deadline for handing in the solutions: Wednesday, January 21, 2015, 4 pm

35. Consider the Hamilton operator h_Z in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ defined by $h_Z\psi = -\psi''$ on the domain

$$\mathcal{D}(h_Z) = \{\psi \in \mathcal{H} \,|\, \psi \in \mathrm{AC}(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}), \lim_{\epsilon \downarrow 0}(\psi'(-\epsilon) - \psi'(\epsilon)) = Z\psi(0), h_Z\psi \in \mathcal{H}\} \quad (40)$$

with $Z \in \mathbb{R}$.

(i) Show that h_Z is associated to the quadratic form

$$q_Z(\psi, \phi) = \int \overline{\psi'(x)}\phi'(x)\,dx - Z\overline{\psi}(0)\phi(0) \quad (41)$$

on $\mathcal{D}(q_Z) = H^1(\mathbb{R})$. Prove that h_Z is bounded from below and self-adjoint.

(ii) Find the spectrum $\sigma(h_Z)$, and, if $Z > 0$, compute the eigenvalues and eigenfunctions of h_Z.

36. With the help of the operator h_Z defined in Problem 35, we construct the (essentially self-adjoint) two-particle operator

$$H^{(0)}_{Z_1,Z_2} = h_{Z_1} \otimes 1 + 1 \otimes h_{Z_2} \quad (42)$$

on the domain $\mathcal{D}(H^{(0)}_{Z_1,Z_2}) = \mathcal{D}(H_{Z_1}) \otimes \mathcal{D}(H_{Z_2}) \subset L^2(\mathbb{R}^2)$. For $Z_i > 0$, $i = 1, 2$, determine the eigenvalues and eigenfunctions and the spectrum $\sigma(\overline{H}^{(0)}_{Z_1,Z_2})$ of $\overline{H}^{(0)}_{Z_1,Z_2}$.

37. Assuming a repulsive Coulomb interaction $W_\gamma(x_1, x_2) = \gamma |x_1 - x_2|^{-1}$ with coupling parameter $\gamma > 0$ between both particles, the operator $H^{(0)}_{Z_1,Z_2}$ of problem 36 is modified into an interacting Hamiltonian by setting

$$H^{(\gamma)}_{Z_1,Z_2} = \overline{H}^{(0)}_{Z_1,Z_2} + W_\gamma = h_{Z_1} \otimes 1 + 1 \otimes h_{Z_2} + W_\gamma \tag{43}$$

where W_γ is defined on its maximal domain $\mathcal{D}(W_\gamma) = \{\psi \in L^2(\mathbb{R}^2) \,|\, W_\gamma \psi \in L^2(\mathbb{R}^2)\}$.

(i) Construct a self-adjoint operator $\tilde{H}^{(\gamma)}_{Z_1,Z_2}$ that extends the operator $H^{(\gamma)}_{Z_1,Z_2}$ defined on $\mathcal{D}(\overline{H}^{(0)}_{Z_1,Z_2}) \cap \mathcal{D}(W_\gamma)$.

(ii) For $Z_i > 0$, $i = 1, 2$, prove an HVZ theorem for $\tilde{H}^{(\gamma)}_{Z_1,Z_2}$ analogously as it has been done in the lecture for the two-electron atom.

(iii) For given $Z_i > 0$, $i = 1, 2$, find a $\gamma_0 > 0$ that guarantees the existence of bound states of $\tilde{H}^{(\gamma)}_{Z_1,Z_2}$ for all $0 \leq \gamma < \gamma_0$.