Dudeney and Frame-Stewart Numbers*

Andreas M. Hinz

hinz@math.lmu.de

2015–07–13

Stimulated by the paper [15] and a conversation with Uroš Milutinović, I was led to reconsider certain notions in connection with the Frame-Stewart numbers as defined in [14, Definition 5.1 and p. 186].

Throughout we will use the variables q, ν, $x \in \mathbb{N}_0$ and $h \in \mathbb{N}$.

0 Some Two-Dimensional Arrays

For every *dimension* q let the *hypertetrahedral* sequence Δ_q be defined by

$$\Delta_{q,\nu} = \binom{q+\nu-1}{q}.$$

Viewed as a two-dimensional array, this is built up just like Pascal's Arithmetical triangle for $\binom{q+\nu}{q}$, but with the 1s in the 0th column replaced by 0s.[1] For instance,

$$\begin{array}{rcll}
\Delta_{0,\nu} & = & (\nu \in \mathbb{N}) & \text{(characteristic function of } \mathbb{N}), \\
\Delta_{1,\nu} & = & \nu & \text{(natural numbers)}, \\
\Delta_{2,\nu} & = & \Delta_\nu & \text{(triangular numbers)}, \\
\Delta_{3,\nu} & = & T_\nu & \text{(tetrahedral numbers)}.
\end{array}$$

Then $\overline{\Delta_{q+1}} = \Delta_q$, $\Sigma(\Delta_q) = \Delta_{q+1}$.

[1] Pascal labelled rows and columns starting from 1.

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10
2	0	1	3	6	10	15	21	28	36	45	55
3	0	1	4	10	20	35	56	84	120	165	220
4	0	1	5	15	35	70	126	210	330	495	715
5	0	1	6	21	56	126	252	462	792	1287	2002
6	0	1	7	28	84	210	462	924	1716	3003	5005
7	0	1	8	36	120	330	792	1716	3432	6435	11440
8	0	1	9	45	165	495	1287	3003	6435	12870	24310
9	0	1	10	55	220	715	2002	5005	11440	24310	48620

Table 1: The hypertetrahedral array $\Delta_{q,\nu}$

Next we define for each q the sequence P_q by

$$P_{q,\nu} = (-1)^q \left(1 + \sum_{k=1}^{q} (-1)^k \Delta_{k,\nu} \right) .$$

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	-1	0	1	2	3	4	5	6	7	8	9
2	1	1	2	4	7	11	16	22	29	37	46
3	-1	0	2	6	13	24	40	62	91	128	174
4	1	1	3	9	22	46	86	148	239	367	541
5	-1	0	3	12	34	80	166	314	553	920	1461
6	1	1	4	16	50	130	296	610	1163	2083	3544
7	-1	0	4	20	70	200	496	1106	2269	4352	7896
8	1	1	5	25	95	295	791	1897	4166	8518	16414
9	-1	0	5	30	125	420	1211	3108	7274	15792	32206

Table 2: The P-array $P_{q,\nu}$

We immediately see that $P_{q,0} = (-1)^q,\ P_{0,\nu} = 1$ and from

$$P_{h,\nu} + P_{h-1,\nu} = \Delta_{h,\nu} \tag{0}$$

that

$$P_{1,\nu} = \nu - 1,\ P_{2,\nu} = \Delta_{\nu-1} + 1,\ P_{3,\nu} = T_{\nu-1} + \nu - 1\,.$$

More avanced properties are

$$P_{q+1,\nu+1} = P_{q+1,\nu} + P_{q,\nu+1} \tag{1}$$

and (cf. [14, p. 280])

$$2P_{q,\nu+1} = P_{q,\nu} + \Delta_{q,\nu+1}\,. \tag{2}$$

Formula (1) shows that the construction of the array $P_{q,\nu}$ is as the previous one for $\Delta_{q,\nu}$, but now the 0th column contains the alternating sign sequence $(-1)^q$.

Proof of (1).

$$\begin{aligned}
P_{q+1,\nu+1} - P_{q+1,\nu} &= (-1)^{q+1}\left(\sum_{k=1}^{q+1}(-1)^k\Delta_{k,\nu+1} - \sum_{k=1}^{q+1}(-1)^k\Delta_{k,\nu}\right)\\
&= (-1)^{q+1}\sum_{k=1}^{q+1}(-1)^k\Delta_{k-1,\nu+1}\\
&= (-1)^q\left(1 + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu+1}\right)\\
&= P_{q,\nu+1}\,. \qquad \square
\end{aligned}$$

Proof of (2).

$$\begin{aligned}
(-1)^q \cdot 2P_{q,\nu+1} &= 2 + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu+1} + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu+1}\\
&= 2 + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu+1} + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu} + \sum_{k=1}^{q}(-1)^k\Delta_{k-1,\nu+1}\\
&= 1 + \sum_{k=1}^{q}(-1)^k\Delta_{k,\nu} + (-1)^q\Delta_{q,\nu+1}\\
&= (-1)^q\left(P_{q,\nu} + \Delta_{q,\nu+1}\right)\,. \qquad \square
\end{aligned}$$

We also have

$$\sum_{k=0}^{q}(-2)^k\binom{q+\nu}{q-k} = P_{q,\nu} = 2^{-\nu}\left((-1)^q + \sum_{k=0}^{\nu-1} 2^k \Delta_{q,k+1}\right) .$$

This is so since the left and right terms are $(-1)^q$ for $\nu = 0$ and fulfil the recurrence relation in (2). For the left identity (anticipated by Brousseau [8, p. 176]) see [14, p. 187], the right one also follows from (2) and [14, Lemma 2.18].

1 Dudeney's Array

In [2, p. 367f], Henry Ernest Dudeney posed the problem to find a shortest solution for the Tower of Hanoi with four pegs. He does not mention the Tower of Hanoi, though, but dresses his question into the fantastic story of *The Reve's puzzle*, in fact the first of his collection of *Canterbury puzzles.* Here the pegs are replaced by stools and the discs by "eight cheeses of graduating sizes". The text was then verbally adopted for the book [5, p. 1f] (with the inscrutable exception of "treat to" being replaced by "give"). The challenge is to solve the problem "in the fewest possible moves, first with 8, then with 10, and afterwards with 21 cheeses".

The solution given for $n = 8$ in [2, p. 480] (which has *not* been taken over for [5]) is an example of what has later been called *Frame's algorithm* (see [7, p. 216f]; cf. also [14, p. 175–177]).[2] Dudeney mentions both possible partitions of the topmost 7 cheeses, namely $4\,|\,3$ and $5\,|2$, leading to the conclusion that "[t]he least number of moves in which the cheeses can be so removed is thirty and three." However, no attempt has been made to *prove* minimality! For the remaining cases $n = 10$ and $n = 21$ again the claim of minimality of 49 and 321 moves, respectively, is made.[3] It is also noted that the partitions are unique, respectively.

The Reeve's puzzle [sic!] was taken up again by Dudeney as puzzle no. 447 in [3], citing several passages from [2]. Strangely enough, it is claimed that 6 cheeses "may all

[2]The respective partial move numbers contain a little clerical error.

[3]The minimality of the first two values, i.e. $\mathrm{d}_4(0^8, 1^8)$ and $\mathrm{d}_4(0^{10}, 1^{10})$, where d_p denotes the distance in graphs H_p^n, was shown in [11] and of the third one, $\mathrm{d}_4(0^{21}, 1^{21})$, in [13].

be removed . . . in sixty-three moves."; this is in fact the value if only three stools are used, i.e. $\mathrm{d}_3(0^6, 1^6) = 63$, for four stools $\mathrm{d}_4(0^6, 1^6) = 17$ moves are sufficient (and necessary). The "half-guinea prize"[4] was then offered for finding the "fewest moves in which *thirty-six* cheeses may be so removed", i.e. $\mathrm{d}_4(0^{36}, 1^{36})$.

Dudeney was quite surprised that he received the correct solution[5] within the time limit by one competitor only. This may have been the reason why he explains his own solution in great detail. Assuming that the largest cheese is moved only once,[6] he divides the solution into the three steps to form two piles with all cheeses except the largest, to move the latter and finally to reunite the others on top of it. He also observes that the first of these steps is the crucial one, the last one being just its reversal. He then presents "the curious point of the thing", namely triangular numbers, for which he claims that the formation of the two piles is unique, whereas there are two different ways otherwise. He characterizes triangular numbers as those n for which $8n + 1$ is a square, i.e. $n = \frac{\nu(\nu+1)}{2}$; the partition of the $n - 1$ smaller discs is then given by $\frac{\nu(\nu-1)}{2} \,|\, \nu - 1$. He finds this fact "[r]ather peculiar". For $n = 36$, i.e. $\nu = 8$, one now has to pile up the $\Delta_7 = 28$ smallest cheeses to one auxiliary stool in 769 moves, then the next 7 cheeses to the other auxiliary stool in 127 moves, to accomplish the first step in altogether 896, so that the whole solution takes $2 \cdot 896 + 1 = 1793$ moves.

The most remarkable feature of Dudeney's exposition is a small table of three rows and eight columns. The first row contains the powers of two 2^ν for $\nu \in [8]$, the second row are these numbers reduced by 1, i.e. the *Mersenne numbers* M_ν, and for the third row c Dudeney gives the formative rule $1 = c_1$, $2c_\nu + M_{\nu+1} = c_{\nu+1}$. He then deduces that $c_8 = 1793$ is the number of moves for the 8th triangular number 36. This, in fact, anticipates *Stewart's algorithm* (see [7, p. 217–219]; cf. also [14, p. 166]), provided that the number of cheeses is a triangular number.

The sparse reaction to his puzzle no. 447 might also have caused Dudeney's "intention

[4] This was the weekly prize awarded by the journal. A half-guinea was 10s6d or 10/6 as it can be seen on The Hatter's hat in Lewis Carroll's "Alice's Adventures in Wonderland".

[5] again up to minimality which is still open!

[6] This assumption has been justified for the first time in [11, p. 119]; cf. [14, Proposition 5.8].

to advance the subject one stage further." This he did in puzzle no. 494 in [4] by proposing "the case where there is one stool more and one cheese less", i.e. five stools and 35 cheeses. Two weeks later he revealed the solution he had in mind, namely 351 moves, but postponed the explanation to the next issue of the journal. This time ten readers had submitted the desired number.

Dudeney's justification starts with a table for $\Delta_{h,\nu}$, with $h \in [3]$ representing the number of auxiliary stools and $\nu \in [7]$, but he stresses that this table could be extended "to any required length". It shows the figurate numbers of cheeses he considers for three ($h = 1$), four ($h = 2$), and five ($h = 3$) stools, respectively. For the construction of this table Dudeney says that the sequence in the second row is made up from the partial sums of the first and similarly for the third row of tetrahedral numbers, which he calls "pyramidal". The first two rows of a second table repeat the last two rows of the table in [3], i.e. M_ν and c_ν, shortened to $\nu \in [7]$. Again the third row of this new table is obtained from the second as the latter is from the first. The claim is now that the entries in this table yield the optimal number of moves for the respective number of stools and cheeses from the first table. Since 35 is the fifth number in the third row of the first table, i.e. the fifth tetrahedral number, the previously announced solution 351 can be found at the fifth position of the third row in the second table.

We generalize these ideas and define the array a by (cf. Table 3)

$$a_{q,0} = 0,\ a_{0,\nu} = \Delta_{0,\nu},\ a_{q+1,\nu+1} = 2a_{q+1,\nu} + a_{q,\nu+1}\,. \tag{3}$$

Dudeney's claim is now that given the number $p = h + 2$ of stools and a figurate number $n = \Delta_{h,\nu}$ of cheeses, we can deduce the corresponding value for ν from Table 1 and with this ν read the number of moves $a_{h,\nu}$ for his algorithm from Table 3. Dudeney also observes that $\Delta_{h,\nu+1}$ cheeses on $h + 2$ stools have to be split successively into piles of $\Delta_{h,\nu}$, $\Delta_{h-1,\nu}$, . . . , $\Delta_{1,\nu}$, and $\Delta_{0,\nu} = 1$ discs from small to the largest. So it can be said that Dudeney has provided an argument for the statement

$$\mathrm{d}_{h+2}\left(0^{\Delta_{h,\nu}}, 1^{\Delta_{h,\nu}}\right) \leq a_{h,\nu}\,. \tag{4}$$

In the last column of [4], Dudeney deals with the problem of how to proceed if the

$q \setminus \nu$	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	0	1	3	7	15	31	63	127	255
2	0	1	5	17	49	129	321	769	1793
3	0	1	7	31	111	351	1023	2815	7423
4	0	1	9	49	209	769	2561	7937	23297
5	0	1	11	71	351	1471	5503	18943	61183
6	0	1	13	97	545	2561	10625	40193	141569
7	0	1	15	127	799	4159	18943	78079	297727

Table 3: Dudeney's array $a_{q,\nu}$

number of cheeses is *not* tetrahedral in the case of five stools. He presents a "little subsidiary table" of three rows and six columns. The first row contains $\Delta_{3,\nu}$ for $\nu \in [6]$, the second has the powers of two $2^{\nu-1}$, and for the third row s he gives the rule $0 = s_1$, $\Delta_{3,\nu} \cdot 2^{\nu-1} + s_\nu = s_{\nu+1}$. He then presents, with two examples, his recipe to get the (presumed) optimal number of moves $\widetilde{d_5}(0^n, 1^n)$ for n cheeses, which amounts to

$$\mu := \min\{\nu \in \mathbb{N} \mid \Delta_{3,\nu} > n\},\ \widetilde{d_5}(0^n, 1^n) = n2^{\mu-1} - s_\mu\,. \tag{5}$$

Although Dudeney mentions that for non-tetrahedral cheese numbers there "will always be more than one way in which the cheeses may be piled", he does *not* say how.

This question, also for the case of non-triangular cheese numbers with four stools, is left to "the reader to work out for himself" in the solutions section of [5, p. 131f], which repeats the first two tables from the previous discussion, but not the subsidiary table and its interpretation. Maybe the author had meanwhile realized that $s_\mu = a_{4,\mu-1}$, as we will see below, such that all information is already contained in the two tables for hypertetrahedral and Dudeney numbers. Even more remarkable is the fact that in his final solution Dudeney is *not* addressing the question of optimality at all, but writes that his "8 cheeses can be removed in 33 moves; 10 cheeses in 49 moves; and 21 cheeses in 321 moves."

We will now analyze properties of the Dudeney array. For instance,

$$a_{h,\nu} = P_{h-1,\nu} \cdot 2^\nu + (-1)^h \,; \tag{6}$$

in particular (cf. (0)):

$$a_{q+1,\nu} + a_{q,\nu} = 2^\nu \Delta_{q,\nu} \,. \tag{7}$$

and

$$a_{1,\nu} = 2^\nu - 1,\ a_{2,\nu} = (\nu - 1)2^\nu + 1,\ a_{3,\nu} = (\Delta_{\nu-1} + 1)2^\nu - 1,\ a_{4,\nu} = (T_{\nu-1} + \nu - 1)2^\nu + 1\,.$$

Note that a_2 is Dudeney's original sequence c; it will be called the *Dudeney sequence.*

Proof of (6). Let the right-hand term in (6) be called $\widetilde{a}_{h,\nu}$. Then $\widetilde{a}_{h,0} = 0$, $\widetilde{a}_{1,\nu} = 2^\nu - 1$, and with the aid of (1) we get

$$\begin{aligned} \widetilde{a}_{h+1,\nu+1} &= P_{h,\nu+1} \cdot 2^{\nu+1} + (-1)^{h+1} \\ &= P_{h,\nu} \cdot 2^{\nu+1} + 2(-1)^{h+1} + P_{h-1,\nu+1} \cdot 2^{\nu+1} + (-1)^h \\ &= 2\widetilde{a}_{h+1,\nu} + \widetilde{a}_{h,\nu+1} \,. \end{aligned}$$

So $\widetilde{a}$ fulfills the recurrence (3) starting at $q = h = 1$. □

Together with (2) we get

$$\begin{aligned} a_{h,\nu+1} - a_{h,\nu} &= P_{h-1,\nu+1} \cdot 2^{\nu+1} - P_{h-1,\nu} \cdot 2^\nu \\ &= 2^\nu (2P_{h-1,\nu+1} - P_{h-1,\nu}) \\ &= 2^\nu \Delta_{h-1,\nu+1} \,. \end{aligned} \tag{8}$$

With the aid of (8) (put $h = 4$ and $\nu = \mu - 1$) we can now perform the induction step in the proof of the above mentioned identity $s_\mu = a_{4,\mu-1}$ for Dudeney's subsidiary sequence s.

Before we turn to the relation between the array a and the *Frame-Stewart numbers*, we note an interesting alternative representation (cf. [14, Corollary 5.6], [15, Lemme 2.1] for $h = 2$ and [12, Theorem 4.6] for the general case). We will make use of the *hypertetrahedral root* $\nabla_{h,k}$ of $k \in \mathbb{N}_0$, namely

$$\nabla_{h,k} = \max\left\{\mu \in \mathbb{N}_0 \mid \Delta_{h,\mu} \le k\right\} \,;$$

for instance, $\nabla_{1,k} = k$ (*trivial root*) and $\nabla_{2,k} = \left\lfloor \frac{\sqrt{8k+1}-1}{2} \right\rfloor$ (*triangular root*). (There is no simple closed expression for the *tetrahedral root* ∇_3 though; it behaves asymptotically like $\sqrt[3]{6k}$. In general, $\nabla_{h,k} \sim \sqrt[h]{h!k}$.) From $\Delta_{h,\mu} \leq \Delta_{h+1,\mu}$ we get $\nabla_{h+1,k} \leq \nabla_{h,k}$. The sequence ∇_h is (not strictly) increasing, and we have

$$\forall\, x \in [\Delta_{h-1,\nu+1}]_0 :\ \nabla_{h,\Delta_{h,\nu}+x} = \nu\,; \tag{9}$$

in particular, $\nabla_{h,\Delta_{h,\nu}} = \nu$, but $\mu - \Delta_{h,\nabla_{h,\mu}} \in [\Delta_{h-1,\nabla_{h,\mu}+1}]_0$. It is interesting to note that we have what Bousch calls a *Galois correspondence* in [15, p. 897]:

$$\Delta_{h,\mu} \leq \nu \Leftrightarrow \mu \leq \nabla_{h,\nu} \ (\text{or } \nu < \Delta_{h,\mu} \Leftrightarrow \nabla_{h,\nu} < \mu)\,,$$

i.e. Δ_h and ∇_h engender a *Galois connection* in $\mathbb{N}_0$.

The statement on a now reads as follows.

Proposition 1.

$$a_{h,\nu} = \sum_{k=0}^{\Delta_{h,\nu}-1} 2^{\nabla_{h,k}}\,.$$

The *proof* is by induction on ν. The case $\nu = 0$ is trivial. Making use of the induction assumption, (9), (6), and (2), we get:

$$\begin{aligned}
\sum_{k=0}^{\Delta_{h,\nu+1}-1} 2^{\nabla_{h,k}} &= \sum_{k=0}^{\Delta_{h,\nu}-1} 2^{\nabla_{h,k}} + \sum_{k=\Delta_{h,\nu}}^{\Delta_{h,\nu+1}-1} 2^{\nabla_{h,k}} \\
&= a_{h,\nu} + \Delta_{h-1,\nu+1} \cdot 2^{\nu} \\
&= P_{h-1,\nu} \cdot 2^{\nu} + (-1)^h + \Delta_{h-1,\nu+1} \cdot 2^{\nu} \\
&= (P_{h-1,\nu} + \Delta_{h-1,\nu+1}) 2^{\nu} + (-1)^h \\
&= P_{h-1,\nu+1} \cdot 2^{\nu+1} + (-1)^h \\
&= a_{h,\nu+1}\,. \qquad \square
\end{aligned}$$

2 The Frame-Stewart Numbers

So far everything could be formulated and proved without reference to the Frame-Stewart numbers [6, 7] or to the Tower of Hanoi [1].

Definition 1. $\forall n \in \mathbb{N}_0 : \ FS_3^n = 2^n - 1 = M_n \,,$

$$\forall p \in \mathbb{N} \setminus [3] : \ FS_p^0 = 0,\ \forall n \in \mathbb{N} : \ FS_p^n = \min\{2FS_p^m + FS_{p-1}^{n-m} \mid m \in [n]_0\}\,,$$

$$\forall n \in \mathbb{N}_0 : \ \overline{FS}_p^n = \tfrac{1}{2}(FS_p^{n+1} - 1)\,.$$

The following theorem, which establishes the interrelation between FS and a, is *not* easy to prove. If we interpret h as the number of spare pegs in the ToH, i.e. those which are neither the starting nor the goal peg, the case $h = 1$ is classical. For $h = 2$ it can be found as [10, Theorem 1] and, more comprehensive, as [14, Theorem 5.4]; cf. also [15, (1.2)][7]. The general case is [14, Theorem 5.16] and goes back to [9, Theorem 3] and [12, Theorem 3.1].

Theorem 1. $\forall x \leq \Delta_{h-1,\nu+1} : \ FS_{h+2}^{\Delta_{h,\nu}+x} = a_{h,\nu} + x \cdot 2^\nu\,.$

From (7) we get, if we put $n = \Delta_{h,\nabla_{h,n}} + x$:

Corollary 1. $\forall n \in \mathbb{N}_0 : \ FS_{h+2}^n = n \cdot 2^{\nabla_{h,n}} - a_{h+1,\nabla_{h,n}}\,.$

For $h = 3$, i.e. $p = 5$, this corresponds to Dudeney's formula (5). The first few Frame-Stewart numbers are presented in Table 4.

$h \setminus n$	0	1	2	3	4	5	6	7	8	9
1	0	1	3	7	15	31	63	127	255	511
2	0	1	3	5	9	13	17	25	33	41
3	0	1	3	5	7	11	15	19	23	27
4	0	1	3	5	7	9	13	17	21	25

Table 4: Frame-Stewart numbers FS_{h+2}^n

Another immediate consequence of Theorem 1, together with Proposition 1 and (9), is:

Corollary 2. $\forall p \in \mathbb{N} \setminus [2]\ \forall n \in \mathbb{N}_0 : \ FS_p^n = \sum_{k=0}^{n-1} 2^{\nabla_{p-2,k}},\ \overline{FS}_p^n = \frac{1}{2}\sum_{k=1}^{n} 2^{\nabla_{p-2,k}}\,.$

[7] On page 896 the author says that it can be shown by simple calculations; on page 897, however, he admits that it is less easy to verify!

In his attempt to prove the *Frame-Stewart Conjecture*, i.e. $\mathrm{d}_p(0^n, 1^n) = FS_p^n$, for the case $p = 4$, i.e. $h = 2$, or in other words to solve The Reve's Puzzle, Bousch defines the function, or rather integer sequence, Φ as the right-hand side of the identity in Corollary 2, i.e.

$$\forall\, n \in \mathbb{N}_0 : \ \Phi(n) = \sum_{k=0}^{n-1} 2^{\nabla_{2,k}} .$$

But instead of making use of the corollary to show that [15, (1.2)]

$$\forall\, n \in \mathbb{N} : \ \Phi(n) = \min \left\{ 2\Phi(m) + 2^{n-m} - 1 \mid m \in [n]_0 \right\} ,$$

he uses only the weaker statement [15, (2.1)]

$$\forall\, m, n \in \mathbb{N}_0 : \ \Phi(m+n) \leq 2\Phi(m) + 2^n - 1 . \tag{10}$$

We can give a direct argument for (10), which is the case $q = 0$, if we define more generally

$$\forall\, q \in \mathbb{N}_0 \ \forall\, n \in \mathbb{N}_0 : \ \Phi_q(n) = \sum_{k=0}^{n-1} 2^{\nabla_{q+1,k}} ,$$

of

Corollary 3. $\forall\, q, m, n \in \mathbb{N}_0 : \ \Phi_{q+1}(m+n) \leq 2\Phi_{q+1}(m) + \Phi_q(n)$.

Proof for $q = 0$.[8] We show by double induction on m and n that

$$\forall\, m, n \in \mathbb{N}_0 : \ \Phi(m+n) \leq 2\Phi(m) + M_n .$$

For $m = 0$ we have $\forall\, n \in \mathbb{N}_0 : \ \Phi(n) \leq M_n$, since $\nabla_k := \nabla_{2,k} \leq k$. Now let $m \in \mathbb{N}_0$ and assume that

$$\forall\, n \in \mathbb{N}_0 : \ \Phi(m+n) \leq 2\Phi(m) + M_n . \tag{11}$$

Then clearly $\Phi(m+1, 0) \leq 2\Phi(m+1) + M_0$ and we assume that for $n \in \mathbb{N}_0$,

$$\Phi(m+1+n) \leq 2\Phi(m+1) + M_n . \tag{12}$$

[8] following Pascal Stucky

Now, making use of (11),

$$\begin{aligned}
\Phi(m+1+n+1) &= \Phi(m+n+2) \\
&\le 2\Phi(m) + M_{n+2} \\
&= 2\Phi(m) + 2^{n+1} + M_{n+1} \\
&= 2\left(\Phi(m) + 2^{n}\right) + M_{n+1} \\
&\le 2\left(\Phi(m) + 2^{\nabla_m}\right) + M_{n+1} \;=\; 2\Phi(m+1) + M_{n+1}\,,
\end{aligned}$$

where the last inequality is only true if $n \le \nabla_m$. If this is not the case, then $n > \nabla_m \Rightarrow \nabla_{\Delta_n} > \nabla_m \Rightarrow m < \Delta_n = \Delta_{n+1} - n - 1 \Rightarrow \Delta_{\nabla_{m+n+1}} \le m+n+1 < \Delta_{n+1} \Rightarrow \nabla_{m+n+1} < n+1 \Rightarrow \nabla_{m+n+1} \le n$, so that with (12):

$$\begin{aligned}
\Phi(m+1+n+1) &= \Phi(m+1+n) + 2^{\nabla_{m+1+n}} \\
&\le 2\Phi(m+1) + M_n + 2^{\nabla_{m+1+n}} \\
&\le 2\Phi(m+1) + M_n + 2^{n} \;=\; 2\Phi(m+1) + M_{n+1}\,. \qquad \square
\end{aligned}$$

A direct proof for the general case of Corollary 3, i.e. independent of Theorem 1, is still lacking.

Another immediate consequence of Corollary 2, Theorem 1 and (6) is

Lemma 1. $\forall\, x \in [\Delta_{h-1,\nu+1} + 1]_0$:

$$\Phi_{h-1}(\Delta_{h,\nu} + x) = a_{h,\nu} + x \cdot 2^{\nu} = (P_{h-1,\nu} + x)2^{\nu} + (-1)^{h}\,.$$

We are now left with proving that

$$\forall\, t \in [h]^n :\; \mathrm{d}_{h+2}(0^n, t) \ge \overline{\Phi}_{h-1}(n)\,,$$

where $\overline{\Phi}_q(n) = \dfrac{1}{2}\displaystyle\sum_{k=1}^{n} 2^{\nabla_{q+1,k}}$.

3 The Reve's Puzzle

We will now concentrate on the case of $p = 4$ pegs, i.e. $h = 2$. For brevety we put $\mathrm{d} := \mathrm{d}_4$. We already saw that $\Delta_\nu = \Delta_{2,\nu}$ and and $\Phi = \Phi_1$; we will also write $\overline{\Phi}$ for $\overline{\Phi}_1$.

The numbering in this section is according to [15]. Our goal is to prove that

$$\forall\, t \in (Q \setminus B)^n : \ \mathrm{d}(0^n, t) \geq \overline{\Phi}(n) \,. \tag{13}$$

Lemma 1 translates to

Lemma 2.1. $\forall\, x \in [\nu+2]_0 : \Phi(\Delta_\nu + x) = (\nu - 1 + x)2^\nu + 1\,.$

Let $2_0^{\mathbb{N}_0} := \left\{ N \in 2^{\mathbb{N}_0} \mid |N| < \infty \right\}$ and define for $E \in 2_0^{\mathbb{N}_0}$:

$$\forall\, \mu \in \mathbb{N}_0 : \ \Psi_\mu(E) = \sum_{k \in E} 2^{\min\{\nabla_k, \mu\}} - c_\mu \,,$$

where c is again Dudeney's sequence, and

$$\Psi(E) = \max_{\mu \in \mathbb{N}_0} \Psi_\mu(E) \ (\geq \Psi_0(E) = |E|) \,.$$

Lemma 2.2. $\Psi[n] := \Psi\left([n]_0\right) = \overline{\Phi}(n)\,.$

Proof. W.l.o.g. $n \in \mathbb{N}$, i.e. $n = \Delta_\nu + x, \ x \in [\nu+1]_0$. Then

$$\begin{aligned}
\Psi_{\mu+1}[n] - \Psi_\mu[n] \ &= \ \sum_{k=0}^{n-1} \left(2^{\min\{\nabla_k, \mu+1\}} - 2^{\min\{\nabla_k, \mu\}}\right) - (\mu+1)2^\mu \\
&= \ 2^\mu \left(|[n]_0 \setminus [\Delta_{\mu+1}]_0| - (\mu+1)\right) \\
&= \ 2^\mu \left((n - \Delta_{\mu+1})(n > \Delta_{\mu+1}) - (\mu+1)\right) > 0 \\
&\Leftrightarrow \ (n - \Delta_{\mu+1})(n > \Delta_{\mu+1}) > \mu + 1 \\
&\Leftrightarrow \ n \geq \Delta_{\mu+1} + \mu + 2 = \Delta_{\mu+2} \\
&\Leftrightarrow \ \nu = \nabla_n \geq \mu + 2 \ \Leftrightarrow \ \mu < \nu - 1\,.
\end{aligned}$$

Therefore, $\Psi_\mu[n]$ is maximal at $\mu = \nu - 1$, so that

$$\begin{aligned}
\Psi[n] = \Psi_{\nu-1}([n]_0) \ &= \ \sum_{k=0}^{n-1} 2^{\min\{\nabla_k, \nu-1\}} - c_{\nu-1} \\
&= \ \sum_{k=0}^{\Delta_\nu - 1} 2^{\nabla_k} + \sum_{k=\Delta_\nu}^{n-1} 2^{\nu-1} - c_{\nu-1} \\
&= \ \Phi(\Delta_\nu) + (n - \Delta_\nu)2^{\nu-1} - c_{\nu-1} \\
&= \ (\nu + x)2^{\nu-1} = \overline{\Phi}(n)\,,
\end{aligned}$$

the latter two equalities coming from Lemma 2.1. ∎

It follows with (10) that ([15, (2.2)])

$$\forall\, m,n \in \mathbb{N}_0 :\ \Psi[m+n] \leq 2\Psi[m] + 2^{n-1}\,. \tag{14}$$

The subsequent Lemmas 2.3 to 2.8 on properties of Ψ are needed for the proof of Theorem 2.9.

Lemma 2.3. $\Psi[n+2] \geq 2^{\nabla_n + 1}$.

Lemma 2.4. $\forall\, E \in 2_0^{\mathbb{N}_0} :\ |E| \leq \Psi[|E|] \leq \Psi(E) \leq 2^{|E|} - 1$.

Lemma 2.5. $\forall\, A, B \in 2_0^{\mathbb{N}_0} :\ \Psi(A) - \Psi(B) \leq \sum\limits_{\nu \in A\setminus B} 2^{\nabla_\nu}$.

Lemma 2.6. *Let $A \in 2_0^{\mathbb{N}_0}$ and $s \in \mathbb{N}_0$ with $|A \setminus [\Delta_s]_0| \leq s$. Then*

$$\forall\, a \in A :\ \Psi(A) - \Psi\left(A \setminus \{a\}\right) \leq 2^{s-1}\,.$$

Lemma 2.7. *Let $s \in \mathbb{N}$, $n \in \mathbb{N}_0 \setminus [\Delta_{s-1}]_0$, and $A \subset [n]_0$. Then*

$$\forall\, b \in \mathbb{N}_0^s :\ \Psi(A \cup b([s])) - \Psi(A) \leq \Psi[n+s] - \Psi[n]\,.$$

Lemma 2.8. $\forall\, A, B \in 2_0^{\mathbb{N}_0} :\ \Psi(A) + \Psi(B) \geq \frac{1}{4}\left(\overline{\Phi}(|A \cup B| + 2) - 3\right)$.

Finally, Bousch announces his main result

Theorem 2.9. *Let $s \in Q^n$, $t \in (Q \setminus B)^n$. Then*

$$\mathrm{d}(s,t) \geq \Psi\left(\{k \in [n]_0 \mid s_{k+1} = 0\}\right)\,;$$

this inequality is [15, (2.3)].

In view of (13) we can conclude:

Corollary 4.1. $\forall\, n \in \mathbb{N}_0 :\ \mathrm{d}_4(0^n, 1^n) = FS_4^n$.

References

[1] Claus de Siam, N., La Tour d'Hanoï, Jeu de calcul, Science et Nature 1(1884), 127–128.

[2] Dudeney, H. E., The Canterbury puzzles, The London Magazine 8(46)(May 1902), 367–371; 8(47)(June 1902), 480–482.

[3] Dudeney, H. E., Puzzles and prizes, No. 447. The Reeve's puzzle, The Weekly Dispatch (1902–05–25) 13; (1902–06–15) 13.

[4] Dudeney, H. E., Puzzles and prizes, No. 494. Five stools and cheeses, The Weekly Dispatch (1903–03–15) 13; (1903–03–29) 13; (1903–04–05) 13.

[5] Dudeney, H. E., The Canterbury puzzles and other curious problems, E. P. Dutton, New York NY, 1908 ($\stackrel{?}{=}$ W. Heinemann, London, 1907).

[6] Stewart, B. M., Advanced problem 3918, Amer. Math. Monthly 46(1939), 363.

[7] Frame, J. S., Stewart, B. M., Dunkel, O., Solution to Advanced problem 3918, Amer. Math. Monthly 48(1941), 216–219.

[8] Brousseau, A., Tower of Hanoi with more pegs, J. Recreat. Math. 8(1975/6), 169–176.

[9] Lu, X.-M., Towers of Hanoi Problem with Arbitrary $k \geq 3$ Pegs, Int. J. Comput. Math. 24(1988), 39–54.

[10] Hinz, A. M., An Iterative Algorithm for the Tower of Hanoi with Four Pegs, Computing 42(1989), 133–140.

[11] Bode, J.-P., Hinz, A. M., Results and open problems on the Tower of Hanoi, Congr. Numer. 139(1999), 113–122.

[12] Klavžar, S., Milutinović, U., Petr, C., On the Frame-Stewart algorithm for the multi-peg Tower of Hanoi problem, Discrete Appl. Math. 120(2002), 141–157.

[13] Korf, R. E., Best-First Frontier Search with Delayed Duplicate Detection, in: Nineteenth National Conference on Artificial Intelligence (AAAI-2004), The MIT Press, Cambridge MA, 2004, 650–657.

[14] Hinz, A. M., Klavžar, S., Milutinović, U., Petr, C., The Tower of Hanoi—Myths and Maths, Springer, Basel, 2013.

[15] Bousch, T., La quatrième tour de Hanoï, Bull. Belg. Math. Soc. Simon Stevin 21(2014), 895–912.