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Topology

Problem Set 9

1. (10 points) Let X be a reasonable space and $Y = X \coprod_f D^2$ the space that arises from X by attaching a disk via some map $f\colon S^1 \to X$, i.e. we identify $x \in S^1$ with $f(x)$. Now let $p \in S^1$ and consider f as a loop in X based at $f(p)$. Show that if $\pi_1(X, f(p)) = \langle S|R\rangle$ is a presentation and r is a word in the free group $F\langle S\rangle$ which represents the homotopy class of f, then $\pi_1(Y, f(p)) = \langle S|R \cup \{r\}\rangle$. Conclude that every finitely presented group (that is a group $\langle S|R\rangle$ with S and R finite) appears as the fundamental group of a space.

2. (10 points) Let k be a field and C_* a chain complex of k-vector spaces. Prove that if only finitely many of the C_i are nonzero and those are finite dimensional, then we have
$$\sum_{i\in\mathbb{Z}}(-1)^i \dim_k C_i = \sum_{i\in\mathbb{Z}}(-1)^i \dim_k H_i(C_*).$$
We call the above number the *Euler characteristic* of the complex.

3. (20 points) We fix an identification of the 1-simplex with the unit interval. Thus a path γ in a topological space X defines an element $c_\gamma \in C_1(X)$. Show:
   (a) c_γ is closed if γ is a loop.
   (b) if γ and η are paths with $\gamma(1) = \eta(0)$, then $c_{\gamma*\eta} - c_\gamma - c_\eta$ is exact.
   (c) for homotopic loops γ and η the chain $c_\gamma - c_\eta$ is exact.
   (d) Show that $[\gamma] \mapsto [c_\gamma]$ defines a homomorphism $\pi_1(X, x) \to H^1(X)$. Give an example where it is not injective. Is it always surjective?

Please hand in your solutions on December 17 at the end of the lecture.