Prof. Dr. Sebastian Hensel
Leopold Zoller

Topology

Problem Set 5

1. (15 points) We call a graph Γ *finite* if the vertex and edge sets are finite. We call a graph Γ *connected* if for any two distinct $v_1, v_2 \in V(\Gamma)$ there are $e_1, \dots, e_n \in E(\Gamma)$ so that $i(e_1) = v_1, t(e_n) = v_2$, and $t(e_j) = i(e_{j+1})$ for all $j = 1, \dots, n-1$. We call a graph T a *tree*, if it is connected, and furthermore for any edge $e \in E(T)$ the graph obtained by removing $e, \overline{e}$ from $E(T)$ is not connected.
   (a) Show that a graph is connected (in the sense above) if and only if its topological realisation is path-connected.
   (b) Show that the realisation X_Γ of a finite tree deformation retracts (strongly) to a point, i.e. there exists a point $p \in X_\Gamma$ and a homotopy H from id_{X_Γ} to the constant map to p such that $H(p,t) = p$ for any $t \in [0,1]$ (Hint: Show first that there needs to be a vertex which is the endpoint of a unique edge).
   (c) Show that any finite, connected graph is homotopy equivalent to a wedge

   $$S^1 \vee \cdots \vee S^1$$

   of finitely many circles (Hint: Show first that any graph has a subgraph containing all vertices which is a tree).

2. (15 points)
   (a) For $0 < k \in \mathbb{N}$ construct a connected covering space of S^1 such that every fiber of the covering map consists of k elements.
   (b) Construct a connected covering space of $S^2 \vee S^1$ such that every fiber of the covering map is infinite.
   (c) Construct a connected covering space of $S^2 \cup \{(x,0,0) \mid x \in [-1,1]\} \subset \mathbb{R}^3$ such that every fiber of the covering map is infinite.

3. (10 points) Show that it is possible, using only a piece of thread and two nails, to hang a picture on a wall in such a way that it will drop if only one of the nails is removed. In mathematical terms: for distinct points $x, p, q \in \mathbb{R}^2$, show the existence of a nontrivial element of $\pi_1(\mathbb{R}^2 \backslash \{p,q\}, x)$ which maps to a trivial element of $\pi_1(\mathbb{R}^2 \backslash \{p\}, x)$ and $\pi_1(\mathbb{R}^2 \backslash \{q\}, x)$ under the maps induced by the inclusions $\mathbb{R}^2 \backslash \{p,q\} \to \mathbb{R}^2 \backslash \{p\}$ and $\mathbb{R}^2 \backslash \{p,q\} \to \mathbb{R}^2 \backslash \{q\}$.

Please hand in your solutions on November 19 at the end of the lecture.