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Topology

Problem Set 4

1. (10 points) Let $x_0 \in X$ and $y_0 \in Y$. Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, x_0).$$

2. (10 points) Let $A \subset X$ be a subspace and consider the following statements:
   - (i) The collapsing map $X \to X/A$ is a homotopy equivalence.
   - (ii) The identity id_X is homotopic to a map $X \to X$ which sends A to a single point.
   - (iii) The inclusion $X \to C_i$ into the mapping cone of the inclusion $i: A \to X$ is a homotopy equivalence.

   Show that $(i) \Rightarrow (ii) \Rightarrow (iii)$.

3. (20 points)
   - (a) Show that the sphere S^3 is homeomorphic to the gluing of two solid tori $S^1 \times D^2$ and $D^2 \times S^1$ along their boundaries, i.e. $(x, y) \sim (x', y')$ if $(x, y) = (x', y') \in S^1 \times S^1$.
     Hint: decompose $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ into the subsets with $|z|^2 \leq 1/2$ and $|z|^2 \geq 1/2$.
   - (b) Show that $S^3 \backslash (\{0\} \times S^1) \subset \mathbb{C}^2$ is homotopy equivalent to S^1.
   - (c) Consider $X = \mathbb{R}^3 \backslash K$, where K is the circle $\{(0, \cos(2\pi t), \sin(2\pi t)) \mid t \in [0, 1]\}$. Show that the loop $\gamma: [0, 1] \to X$ given by $\gamma(t) = (\sin(2\pi t), \cos(2\pi t) - 1, 0)$ is not contractible, i.e. it defines a nontrivial element of $\pi_1(X, 0)$.
     Hint: to prove this formally, it can be helpful to recall that the inverse of the stereographic projection $S^n \backslash \{(1, 0, \ldots, 0)\} \to \mathbb{R}^n$,

$$(x_0, \ldots, x_n) \mapsto (1 - x_0)^{-1} \cdot (x_1, \ldots, x_n)$$

     is given by

$$(x_1, \ldots, x_n) \mapsto \left(1 + \sum x_i^2\right)^{-1} \cdot \left(-1 + \sum x_i, 2x_1, \ldots, 2x_n\right).$$

Please hand in your solutions on November 12 at the end of the lecture.