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# Topology

## Problem Set 2

1. (10 points) Let $X$ be a set and $\mathcal{T} \subset P(X)$ a topology. A *subbasis* for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ such that every $U \in \mathcal{T}$ can be written as a union of finite intersections of sets from $\mathcal{B}$ (if one only needs to consider unions, then $\mathcal{B}$ is called a *basis*).

   (a) Show that for any subset $\mathcal{B} \subset P(X)$, there is a unique topology $\mathcal{T} \subset P(X)$ such that $\mathcal{B}$ is a subbasis for $\mathcal{T}$.

   A *neighbourhood basis* of a topology $\mathcal{T}$ at a point $x \in X$ is a subset $\mathcal{B}_x \subset \mathcal{T}$ consisting of open neighbourhoods of $x$ such that any $U \in \mathcal{T}$ which contains $x$ also contains a neighbourhood from $\mathcal{B}_x$. We say that $(X, \mathcal{T})$ is a *first-countable* space if every point has a countable neighbourhood basis.

   (b) Prove that if $x \in X$ has a countable neighbourhood basis, then there is also a countable neighbourhood basis $(U_i)_{i \in \mathbb{N}}$ satisfying $U_{i+1} \subset U_i$. Furthermore, show that if $(x_i)_{i \in \mathbb{N}}$ is a sequence such that $x_i \in U_i$ then $(x_i)$ converges to $x$.

   (c) Show that a compact first-countable space is sequentially compact.

2. (10 points) Let $(X, d)$ be a metric space. Show:

   (a) If $K_1, K_2 \subset X$ are compact, then
   $$\sup_{\substack{x \in K_1,\\ y \in K_2}} d(x,y) < \infty.$$

   (b) If $K \subset X$ is compact, $A \subset X$ is closed, and $K \cap A = \varnothing$, then
   $$\inf_{\substack{x \in K,\\ y \in A}} d(x,y) > 0.$$

   Hint: it can be helpful to observe that metric spaces are first-countable and argue via sequential compactness.

3. (20 points) Let $X$ and $Y$ be topological spaces and denote by $C(X,Y)$ the set of continuous maps from $X$ to $Y$. Then one can endow $C(X,Y)$ with the *compact open topology*, which is generated by the subbasis consisting of the sets of the form
   $$U(K,V) = \{f \in C(X,Y) \mid f(K) \subset V\}$$
   for any compact $K \subset X$ and any open $V \subset Y$. We show that this rather abstract topology has a nice interpretation for metric spaces:

   If $(Y, d)$ is a metric space, then one can consider the topology on $C(X,Y)$ with (sub)basis consisting of sets of the form
   $$B_\varepsilon^K(f) = \{g \in C(X,Y) \mid d(g(x), f(x)) < \varepsilon \text{ for all } x \in K\}$$
   where $K \subset X$ is any compact subset and $f \in C(X,Y)$ is any function.

(a) Show that in this case both topologies agree.
(b) In the case where $X$ is compact and $Y$ is a metric space, define a metric on $C(X, Y)$ which induces the compact open topology.

**Please hand in your solutions on October 29 at the end of the lecture.**