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Topology

Problem Set 11

1. (20 points) Let X be a path-connected space. Recall that the cone CX over X is defined as $X \times I/(X \times \{1\})$. We identify X with the subspace of CX at height 0.
   (a) Show that CX is contractible.
   (b) Show that $H_1(CX, X) = 0$ and that the boundary map $H_n(CX, X) \to H_{n-1}(X)$ is an isomorphism for all $n \geq 2$.
   (c) There is an homeomorphism $C\Delta^{n-1} \cong \Delta^n$ defined by
   $$[(t_0, \dots, t_{n-1}), t] \mapsto (t, (1-t)t_0, \dots, (1-t)t_{n-1}).$$
   Thus we obtain a map $\phi: C_{n-1}(X) \to C_n(CX)$ by associating to $\sigma: \Delta^{n-1} \to X$ the composition $\Delta^n \cong C\Delta^{n-1} \to CX$ with the last map being the obvious map induced by σ. Show that $\sigma \mapsto (-1)^n \phi(\sigma)$ defines a chain map $C_{n-1}(X) \to C_n(CX, X)$ which induces an isomorphism $H_{n-1}(X) \to H_n(CX, X)$ inverse (up to sign) to the boundary map from above.

   We also define the *suspension of* X to be $SX := X \times [-1, 1]/\sim$ where the equivalence relation is generated by $(x, 1) \sim (y, 1)$ and $(x, -1) \sim (y, -1)$ for all $x, y \in X$. Note that CX is naturally a subspace of SX. We also denote by C^-X the subspace of SX of points with real coordinate ≤ 0.
   (d) We define $\psi: C_{n-1}(X) \to C_n(C^-X)$ analogous to ϕ (invert the real coordinate). Show that $\sigma \mapsto (-1)^n(\phi(\sigma) - \psi(\sigma))$ defines a chain map $C_{n-1}(X) \to C_n(SX)$ and that the resulting diagram
   $$\begin{array}{ccc} H_n(SX) & \longrightarrow & H_n(SX, C^-X) \\ \uparrow & & \uparrow \\ H_{n-1}(X) & \longrightarrow & H_n(CX, X) \end{array}$$
   commutes. Conclude that the left vertical map is an isomorphism.
   (e) Deduce that the map $SX \to SX$, $(x, t) \mapsto (x, -t)$ induces multiplication by (-1) on $H_n(SX)$ for $n \geq 1$.

2. (10 Points) Show that $S(S^n)$ is homeomorphic to S^{n+1}. Deduce that reflection along a hyperplane in $\mathbb{R}^{n+1}$ induces multiplication by -1 on $H_n(S^n)$. Conclude that $-\mathrm{id}_{S^n}$ is homotopic to id_{S^n} if and only if n is odd.

3. (10 Points) Compute $H_n(\Sigma_2)$ for all n.

   Hint: You might find Exercises 2 and 5*b* from sheet 10 helpful at some point. Also recall that in the latter we (should have) computed $H_1(T^2) = \mathbb{Z}^2$, $H_2(T^2) = \mathbb{Z}$, and $H_n(T^2) = 0$ for $n \geq 3$.

Please hand in your solutions on January 14 at the end of the lecture.