Prof. Dr. Sebastian Hensel
Leopold Zoller

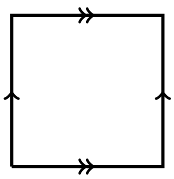
Topology

Problem Set 1

1. (10 points) Let $f\colon X \to Y$ be a continuous map between topological spaces.
   (a) If X is compact, then $\operatorname{im} f \subset Y$ is compact when endowed with the subspace topology.
   (b) If $K \subset Y$ is compact and Y is Hausdorff, then K is closed
   (c) If X is compact, Y is Hausdorff, and $f\colon X \to Y$ is bijective, then f is already a homeomorphism.
   (d) Give an example of a bijective continuous map which is not a homeomorphism.

2. (10 points) Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $\pi\colon X \to X/_\sim$ be the canonical projection where the space $X/_\sim$ is endowed with the quotient topology. Prove the following universal property: a map $f\colon X/_\sim \to Y$ is continuous if and only if the composition $f \circ \pi$ is continuous.

3. (10 points) Let $I = [0,1] \subset \mathbb{R}$ be the unit interval and consider the space $I^2 = I \times I$ with the product topology. Let $\sim$ be the equivalence relation on I^2 which is generated by the relations $(0,x) \sim (1,x)$ and $(x,0) \sim (x,1)$ for every $x \in I$ (it is the smallest equivalence relation containing the given relations). So the quotient space $I^2/_\sim$ is a square with opposite sides identified as indicated by the arrows in the following picture.



   Show that $I^2/_\sim$ is homeomorphic to the two dimensional *torus* T^2 which is defined as $S^1 \times S^1$ with the product topology, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the *unit circle*.

4. (10 points) Show that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic.

Please hand in your solutions on Tuesday at the end of the lecture.