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Topology I
Script
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Lecture 1 (October 15)

What is *Topology*? Very roughly: the study of *spaces* up to *continuous deformation*. There are various ways to make this precise, and we will see some later. To have an immediate (famous) picture in mind: a (usual) coffee cup, and a (usual) donut are the same up to deformation (imagine the cup made from not-yet-set clay, and you can form it into a ring without breaking or tearing). Intuitively, a donut and a ball are different, as are two- and three-dimensional space.
However, such negative results are much more subtle to make precise. Instead of describing a deformation, one needs to prove that none can possibly exist.
This is where *Algebraic Topology* enters the picture. Very coarsely, algebraic topology is concerned with attaching algebraic invariants (numbers, groups, rings, polynomials...) to spaces. The rationale is that algebraic invariants are more "rigid" and therefore much easier to distinguish. Spaces with different invariants cannot be the same, and so this perspective can give proofs that spaces are different. We will see this philosophy in play many times in the future.

Part I: Basics of Point-set topology. We know (e.g. from analysis courses) the following basic definitions:

Definition 0.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

(1) $\emptyset, X \in \mathcal{T}$.
(2) If I is some set, and $U_i \in \mathcal{T}$ for all i, then $\cup_{i \in I} U_i \in \mathcal{T}$.
(3) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.

We call the sets in $\mathcal{T}$ *open sets*. The pair $(X, \mathcal{T})$ is called a *topological space*.

Often we will abuse notation and simply say that X is a topological space, suppressing the mention of $\mathcal{T}$ from the notation.

Definition 0.2. A map $f : X \to Y$ between topological spaces is *continuous* if the preimage of every open set is open. A map is a *homeomorphism* if it is continuous, bijective, and its inverse is also continuous.

Some examples of topological spaces:

(1) $\mathbb{R}^n$ with the "usual topology": a set is open if it contains a positive radius Euclidean ball around any of its points.

(2) The *subspace topology*: If $A \subset X$ is any subset, and $\mathcal{T}$ a topology on X, then

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\}$$

is a topology on A, called the subspace topology.

(3) Using the previous two examples, any subset of $\mathbb{R}^n$ is naturally a topological space. It is good practice to think of "strange" subsets as well. Here is one such:

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \text{at least two } x_i \text{ are rational}\}$$

(the Nöbeling curve)

(4) The *quotient topology*: If X is a topological space, and $q : X \to Y$ a surjective map, then we can define a topology on Y be declaring $U \subset Y$ to be open exactly if $q^{-1}(U)$ is open (in X). This makes q automatically continuous.

(5) Using the previous two gives us more spaces. For example, we can start with the n–sphere

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

and take the quotient by the equivalence relation $x \sim -x$. The result is the so-called *real projective space* $\mathbb{R}P^n$.

(6) The *product topology*: If X, Y are two topological spaces, then we can put a topology on the Cartesian product $X \times Y$ in the following way: a set $O \subset X \times Y$ is open if for any $(x, y) \in O$ there are open sets $U \subset X, V \subset Y$ with $x \in U, y \in V$ and $U \times V \subset O$.

(7) Combining the previous things gives us yet more spaces, e.g. the *n–torus* $T^n = (S^1)^n = S^1 \times \cdots \times S^1$, the n–fold product of the circle with itself.

(8) Finally, there are topologies which seemingly have nothing to do with "geometry". E.g. we can define a topology on $\mathbb{Z}$ in the following way. Define for $a, b \in \mathbb{N}, a \neq 0$ the set:

$$S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$$

and say that $O \subset \mathbb{Z}$ is open, if for any $x \in O$ there is some $S(a, b) \subset O$ with $x \in S(a, b)$.

(This topology can be used to give a cute proof that there are infinitely many primes. It won't be important for the course, but search for "Fürstenberg's proof of the infinitude of primes" if you want to find it)

To illustrate why it is sometimes hard to distinguish spaces (and to show that continuous maps are really much more wild than one usually imagines) we will prove the following

Theorem 0.3 (Peano curves)**.** *For any n, there is a continous, surjective map*

$$c : [0, 1] \to [0, 1]^n$$

Proof. By induction, it suffices to show the case $n = 2$. To do so, we will use the *Cantor set*

$$C = \{\sum_{i=1}^{\infty} a_i \frac{1}{3^i} \mid a_i \in \{0, 2\}\}$$

It is often useful to think of a point $x \in X$ both as a real number and a sequence (a_i) with values in $\{0, 2\}$.
This will be done in three steps:

(1) There is a surjective continous map $C \to C \times C$
(2) There is a surjective map $C \to [0, 1]$
(3) Any continuous map $C \to [0, 1]^2$ extends to a continuous map defined on $[0, 1]$.

With these we are done: composing the map from (1) with the map from (2) in both coordinates we get $C \to [0, 1]^2$ continuous and surjective. By (3) this then extends to $[0, 1] \to [0, 1]^2$ continuous (and obviously still surjective).
We use the following facts about the Cantor set:

- If $(a_i), (b_i)$ are two sequences with values in $\{0, 2\}$, and $a_i = b_i$ for $i \leq n$, then
$$\left| \sum_{i=1}^{\infty} a_i \frac{1}{3^i} - \sum_{i=1}^{\infty} b_i \frac{1}{3^i} \right| \leq \frac{1}{3^{n-1}}$$
- If $(a_i), (b_i)$ are two sequences with values in $\{0, 2\}$, $a_i = b_i$ for $i \leq n - 1$, and $a_n \neq b_n$, then
$$\left| \sum_{i=1}^{\infty} a_i \frac{1}{3^i} - \sum_{i=1}^{\infty} b_i \frac{1}{3^i} \right| \geq \frac{1}{3^n}$$

Intuitively, these say that closeness in the Cantor set *exactly* corresponds to having long common initial segments of the defining sequence.
Now we are ready to prove (1)–(3). For (1), we define the map f in terms of sequences as

$$(a_1, a_2, \ldots) \mapsto (a_1, a_3, \ldots) \times (a_2, a_4, \ldots)$$

This is clearly a bijection between C and $C \times C$. To see that it is continuous, simply observe that if $(a_i), (b_i)$ have the same first $2n + 2$ terms, then both coordinates of $f((a_i)), f((b_i))$ have the same first n terms. By the two observations above, that shows continuity.
For (2), we simply define the map

$$\sum_{i=1}^{\infty} a_i \frac{1}{3^i} \mapsto \sum_{i=1}^{\infty} a_i/2 \frac{1}{2^i}$$

Arguing as above, it is easy to see that this is continuous. It is surjective since any number has a binary representative.
Finally, we prove (3). To this end, suppose we have a map $f : C \to [0, 1]^2$. Given any number $x \in [0, 1] - C$, there is a unique maximal open interval

I_x containing x which is disjoint from C. We call these *complementary intervals*. Let $I_x = (a_x, b_x)$. We then define $\widetilde{f}$ on I_x by

$$\widetilde{f}(x) = \frac{x - b_x}{a_x - b_x} f(a_x) + \frac{x - a_x}{b_x - a_x} f(b_x)$$

It is clear that this extends f. Showing continuity is a little bit tedious, but not hard. Namely, suppose that $x_n \to x$ is any convergent sequence in $[0, 1]$. If $x_n \in I_x$ for some x and all large n, then $\widetilde{f}(x_n) \to x$, as $\widetilde{f}$ is linear on I_x. Similarly, if $x_n \in C$ for all large n then $\widetilde{f}(x_n) = f(x_n) \to f(x)$ by continuity of f.

So, suppose that $x_n \notin C$ for all n, and not eventually contained in some I_y. For each i so that there is a y with $x_i \in I_y = (a, b)$, let z_i be the endpoint of the interval I_y so that $|f(z_i) - f(x)|$ is larger. By this choice, and the fact that $\widetilde{f}$ is linear on each interval I_z, it suffices to show that $f(z_i) \to f(x)$.

If every complementary interval contains only finitely many x_i, then the z_i converge to x as well (as the lengths of the intervals in which x_i, z_i are contained then converge to 0), and therefore $f(z_i) \to f(x)$.

In the other case, suppose that there is some complementary interval I containing infinitely many x_i. In that case, x needs to be a boundary point of that interval. Now, we can break the sequence x_i into two subsequences: one of which is completely inside I (for which, as above, convergence is clear), and another which is completely outside. The latter cannot visit a complementary interval infinitely often (as x would then also have to be a boundary point of that interval), and therefore convergence is also clear. □

I want to emphasise two things about this result:

(1) It is intuitively very surprising. One might expect that continuous maps from $[0, 1]$ ("curves") should not be able to fill all of space, as a dimension 1 space should be smaller than a higher dimensional space.
(2) It shows that "seeing" dimension using topological tools is much harder than in algebra (compare invariance of dimension in linear algebra), or even for differentiable maps (a diffeomorphism would have invertible differential).

Lecture 2 (October 16)

We continue to recall some basic notions in point-set topology (which you probably know), but with the goal of distinguishing spaces. Our first triple of spaces that we want to tell apart are

$$X = [0, 1], Y = [0, 1] \cup [2, 3], Z = [0, 1] \cup [2, 3] \cup [4, 5]$$

What tells X and Y apart? We use

Definition 0.4. A space X is *disconnected* if there are nonempty open sets $U, V \subset X$ so that

$$X = U \cup Y$$

$$U \cap Y = \emptyset.$$

Otherwise, X is said to be connected.

We recall from analysis

Theorem 0.5. *The spaces* $(0,1), [0,1], [0,1)$ *(with the usual topology) are connected.*

On the other hand, the space Y is clearly disconnected (find the open sets!), and thus X and Y are not homeomorphic.
How to tell Y and Z apart? Both are disconnected, but Z has "one more connected piece". One way to make this precise is with the notion of path components.

Definition 0.6. A *path* in X is a continous map $\gamma : [0,1] \to X$. We define the *concatenation* of paths γ_1, γ_2 with $\gamma_1(1) = \gamma_2(0)$ as

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } t \leq 1/2 \\ \gamma_2(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

We define the inverse of a path γ as

$$\overline{\gamma}(t) = \gamma(1-t).$$

We will see later why the name "inverse" is justified. For now, recall

Definition 0.7. A space X is *path-connected* if for any two points $x, y \in X$ there is a path γ with $\gamma(0) = x, \gamma(1) = y$.

Lemma 0.8. *If X is path-connected, then it is connected.*

Proof. Suppose not, so there is a decomposition $X = U \cup V$ into disjoint open sets. Take $x \in U, y \in V$, and a path γ between them. Then $\gamma^{-1}(U), \gamma^{-1}(V)$ are nonempty, disjoint open sets, whose union is $[0,1]$. This contradicts connectivity of $[0,1]$. □

We will see later that the converse to the lemma is false. In fact, exactly the same proof of the previous lemma also shows the following useful statement.

Lemma 0.9. *Suppose that X is connected and that $f : X \to Y$ is continous. Then $f(X) \subset Y$ is connected.*

Definition 0.10. Suppose that X is a space. Define an equivalence relation $\sim$ on X by declaring $x \sim y$ if there is a path γ with $\gamma(0) = x, \gamma(1) = y$. We then put

$$\pi_0(X) = X/\sim .$$

The set $\pi_0(X)$ is called the set of path-components of X.

In addition to associating a set π_0 to every space, we will also associate maps of the sets π_0 to every continuous map in a reasonable. This idea is very important (it is called *functoriality*) and we will see it in action many times during this course. Here, it is very simple:

Lemma 0.11. *Suppose $f : X \to Y$ is a continous map. Then we have a map*

$$f_* : \pi_0(X) \to \pi_0(Y)$$

by associating to the equivalence class of a point p the equivalence class of the point $f(p)$. This has the following properties:

a) $\mathrm{id}_* = \mathrm{id}$

b) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then

$$(gf)_* = g_* f_*$$

Proof. Well-definedness follows from the observation that $f \circ \gamma$ is a path connecting $f(\gamma(0))$ to $f(\gamma(1))$. Property a) is obvious, and property b) is clear from the definition. □

The key point is now that as a *formal* consequence of the properties a), b) of the lemma we get that homeomorphic spaces have the same π_0 (up to set bijection). This allows us to distinguish the spaces Y, Z from the beginning of the class, as $\pi_0(Y)$ has two elements, and $\pi_0(Z)$ has three elements (fill in the details of why this is – it should be very quick).

We end with the standard example showing that connectivity does not imply path-connectivity. The space is usually called "topologists sine curve" and is built in two steps. First, we consider the following graph:

$$S = \{(\sin \frac{1}{x}, x) \in \mathbb{R}^2 \mid x \in [0, \infty)\}.$$

For later, observe that it is clearly path-connected, and therefore connected. Next, we define

$$S^+ = S \cup \{(y, 0) \in \mathbb{R}^2 \mid y \in [-1, 1].\}$$

We claim that S^+ is connected. This is actually a consequence of a useful general lemma. To state it, we need yet another definition.

Definition 0.12. Let X be a topological space, and $Y \subset X$ be a subset. The *closure* of Y, usually denoted by $\overline{Y}$, is defined to be the smallest (with respect to inclusion) closed set containing Y.

Explicitly, one way to describe the closure is

$$\overline{Y} = \bigcap_{Y \subset C, C \text{ closed}} C.$$

There is another useful description of the closure.

Definition 0.13. A *limit point* of a subset Y of a topological space X is a point $x \in X$ with the following property: if $U \subset X$ is any open set with $x \in U$, then $U \cap Y \neq \emptyset$.

Observe that this agrees with the notion of limit point we know from analysis (for the usual topology on $\mathbb{R}^n$).

Lemma 0.14. *A closed set $Y \subset X$ contains all of its limit points.*

Proof. We need to show that no point $x \notin Y$ is a limit point of Y. But that is clear: $X \setminus Y$ is an open set which is disjoint from Y and contains any such point. □

We can now give the promised description of the closure:

Lemma 0.15. *The closure of Y is the union of Y and its limit points.*

Proof. One inclusion is easy: if $C \supset Y$ is a closed set, and $x \in X$ is a limit point of Y, then it clearly also is a limit point of C. Thus, by the previous lemma, it is a point of C. Since this is true for any such C, it follows that the closure contains Y and all of its limit points.
Next, we want to show that the union Z of Y and its limit points is itself closed. That would show the lemma. To see this, take $x \in X \setminus Z$ (i.e. x is neither an element, nor a limit point of X). By definition, this means that there is an open set $U \subset X$, $x \in U$, $U \cap X = \emptyset$. In particular, any other $x' \in U$ is also not a limit point of X, and not contained in X. Thus, $U \subset X \setminus Z$. Since this was true for any $x \notin Z$, we see that Z is indeed closed. □

Back to the topologists sine curve. Using the description of the closure in terms of limit points, it is easy to see that S^+ is the closure of S. Connectivity of S^+ now follows from this useful lemma:

Lemma 0.16. *Suppose that $Y \subset X$ is connected. Then the closure $\overline{Y}$ is also connected.*

Proof. Suppose now, i.e. write

$$\overline{Y} = U \cup V$$

for nonempty disjoint open sets in $\overline{Y}$ (with the subspace topology). Now, Y is assumed to be connected, and therefore up to possibly swapping labels, we have $U \cap Y = Y$, $V \cap Y = \emptyset$.
Next, observe that since U is closed in $\overline{Y}$ (it is the complement of the open V), and therefore by the definition of the subspace topology, there is a closed set $C \subset X$ so that

$$U = C \cap \overline{Y}.$$

In particular, $C \supset Y$ (as U contains Y), and so $C \supset \overline{Y}$. But then $U = \overline{Y}$, showing $V = \emptyset$. □

Finally, it is not hard to see that S^+ is not path-connected: it is impossible to join a point in S to a point outside S with a path. The details were discussed in the problem session.

Lecture 3 (October 22)

We will continue with our review of basic terms in order to distinguish some examples of spaces. This time, we will look at

$$X = \mathbb{Z}, Y = \mathbb{Q}, Z = \text{Cantor}.$$

All of them have infinitely many path-components, but their local structure is very different.

Definition 0.17. A point $x \in X$ in a topological space is *isolated*, if $\{x\}$ is open.

This is particularly intuitive for the subspace topology: a point $z \in Z \subset X$ is isolated, if there is an open set of X intersecting Z just in z. We observe easily that every point of $\mathbb{Z}$ is isolated, but no point of $\mathbb{Q}$ (or the Cantor set) is. This is a useful property, so it has a name:

Definition 0.18. A space X is *perfect* if no point is isolated.

How can we distinguish the rationals from the Cantor set?

Definition 0.19. A space X is *compact* if for any collection $U_i, i \in I$ of open sets so that $X = \cup_{i \in I} U_i$, there is a finite number of indices $i_1, \ldots, i_n$ so that

$$X = U_{i_1} \cup \cdots \cup U_{i_n}.$$

In $\mathbb{R}^n$ with the usual topology, compactness is easy to test:

Theorem 0.20 (Heine-Borel)**.** *A subset $X \subset \mathbb{R}^n$ (with the subspace topology) is compact if and only if it is closed and bounded (with respect to the usual topology and any norm).*

In more general spaces, compactness is much more delicate. On the problem set, you will prove the following very useful lemma:

Lemma 0.21. *A closed subset of a compact space is compact.*

The converse of this is not true. To see why, we build a space by gluing two unit intervals, except at 0. To make this precise, first define

$$Y = \{(y, i) \mid y \in [0, 1], i \in \{0, 1\}\},$$

and then define the quotient

$$X = Y / \sim$$

where $(y, i) \sim (y, 1 - i)$ for all $y \neq 0$[1]
Explicitly, we can identify X (as a set) with $\{0_1, = 0_2\} \cup (0, 1]$. Open neighbourhoods of points $x \in (0, 1) \subset X$ look like usual intervals $(o - \epsilon, o + \epsilon)$. Neighbourhoods of 0_i look like $\{0_i\} \cup (0, \epsilon)$. Make sure you understand why – this is how the quotient topology works. In particular, note that any neighbouhood of 0_1 and any neighbourhood of 0_2 intersect.
Now, (again by how the quotient topology works), the map $[0, 1] \to X$ induced by

$$x \mapsto (x, 0)$$

[1] *I will often describe an equivalence relation by the "interesting" relations. To be formally correct, one should always say "$\sim$ is the equivalence relation generated by the following..."*

is continuous, and therefore has compact image (see problem set). Thus, $\{0_1\} \cup (0,1]$ is compact – but it is not closed, as its complement is $\{0_2\}$. What goes wrong here? The problem is the pair of points $\{0_1, 0_2\}$ which cannot be separated by open sets. Recall

Definition 0.22. A space X is *Hausdorff* if for any $x, y \in X$, $x \neq y$, there are open sets $U, V \subset X$ with $U \cap V = \emptyset$ and $x \in U, y \in V$.

Also on the problem set you will show:

Lemma 0.23. *If X is Hausdorff, and $C \subset X$ is compact, then C is closed.*

Before moving on, we will briefly discuss how these basic topological notions interact with our basic constuctions, in particular products and quotients.

- The quotient of a compact space is compact. This follows since images of compact spaces under continuous maps are compact.
- The quotient of a Hausdorff space is usually not Hausdorff. We have seen this above.
- The product of two compact spaces is compact. This is not hard, and if you have not seen it before you should try to prove it.

  Also true, and much harder is the fact that any product of compact spaces is compact. For this to be true, we need to carefully define the product topology on an infinite product. Namely, if $X_i, i \in I$ is any collection of topological spaces, define a *basic open set* to be a set of the form
  $$\prod U_i \subset \prod X_i,$$
  where $U_i \subset X_i$ is open for all i, and $U_i \neq X_i$ for only finitely many i. Then, call a set $O \subset \prod X_i$ open if for any $o \in O$ there is a basic open set B so that $o \in B \subset O$. Note that for the product of two spaces this is the same topology we defined above.

  Theorem 0.24 (Tychonoff)**.** *The product of compact spaces is compact.*

  The proof is a bit tricky. Since we do not use infinite products in this course, I won't be giving a proof. If you are interested, there is a readable account in the book "Topology and Geometry" by Bredon.
- Products of Hausdorff spaces are Hausdorff. This is easy, even in the general case: if $(x_i), (y_i)$ are distinct points of the product, then there is an index j with $x_j \neq y_j$. Using Hausdorff in X_j yields open sets $U_j, V_j \subset X_j$ which are disjoint and so that $x_j \in U_j, y_j \in V_j$. Put $U_k = V_k = X_k$ for all $k \neq j$. Then $U = \prod U_i, V = \prod V_i$ are the desired open sets.

Part II: Homotopy. We now begin with a discussion of one of the most imporant notions in (algebraic) topology. The basic definition is very simple:

Definition 0.25. Let X, Y be topological spaces, and $f, g : X \to Y$ be continuous maps. A *homotopy between* f, g is a continuous map

$$H : X \times [0,1] \to Y$$

so that $H(x,0) = f(x), H(x,1) = g(x)$ for all $x \in X$. If such a H exists, we say that f, g *are homotopic*. If $A \subset X$ is a subset, then the homotopy is called *relative to* A (often simply: rel A), if $H(a,t) = H(a,0)$ for all $a \in A$ and $t \in [0,1]$.

Intuitively: f, g are homotopic, if one can continously deform one into the other. Here are some basic examples:

- The constant 0 map and the identity of $\mathbb{R}^n$ are homotopic. In fact, we can take
$$H(x,t) = tx.$$
- A homotopy between maps of the one-point-space $\{\mathrm{pt}\}$ into X is the same thing as a path. This is a little bit silly, but it sometimes allows us to give quick proofs about paths from statements about homotopies.

Definition 0.26. A continuous map $f : X \to Y$ between topological spaces is called a *homotopy equivalence* if there is a map $g : Y \to X$ (called *homotopy inverse*) so that $f \circ g$ and $g \circ f$ are both homotopic to the identity (of Y, X respectively).

Intuitively, this means that "X can be deformed into Y" – but the deformations are allowed to squish parts of the space to points, as long as no information is lost up to homotopy. Examples make this clearer:

- $\mathbb{R}^n$ and the one-point space are homotopy equivalent. We have seen this above.
- $\mathbb{R}^n \setminus \{0\}$ and S^{n-1} are homotopy equivalent. One map is simply the inclusion, the other is $x \mapsto x/\|x\|$.
- A graph in the shape of θ and a graph in the shape of 8 are homotopy equivalent. We will soon see a way to prove this formally and quickly – but it is instructive to think about how one would build the maps and homotopies by hand.

Lecture 4 (October 23)

We continue with homotopies. First, we'll prove a useful lemma about homotopies between homotopies:

Lemma 0.27 (Reparametrisation lemma)**.** *Suppose X, Y are topological spaces, and*

$$F : X \times [0,1] \to Y$$

is a homotopy. Suppose that $\phi_1, \phi_2 : [0,1] \to [0,1]$ are two continuous maps with $\phi_1(t) = \phi_2(t)$ for $t = 0, 1$. Then the maps $G_1, G_2 : X \times [0,1] \to Y$ defined by $G_i(x,t) = F(x, \phi_i(t))$ are homotopic relative to $X \times \{0,1\}$.

Proof. Simply define $H : X \times [0,1] \times [0,1]$ by

$$H(x,t,s) = F(x, s\phi_2(t) + (1-s)\phi_1(t)).$$

This is clearly continuous, and it is equal to G_1, G_2 for $s = 0,1$. Since for $t = 0,1$ the two functions ϕ_1, ϕ_2 take the same value, it is a homotopy rel $X \times \{0,1\}$. □

Here is a typical application: similar to paths, given homotopies $F, G : X \times [0,1] \to Y$ we define the *concatenation*

$$F * G(x,t) = \begin{cases} F(x,2t) & \text{if } t \leq 1/2 \\ G(x,2t-1) & \text{if } t \geq 1/2 \end{cases}$$

We define the inverse as

$$\overline{F}(x,t) = F(x,1-t).$$

The lemma then immidiately implies the following: the concatenation $F * \overline{F}$ is homotopic, relative to $X \times \{0,1\}$ to the constant homotopy $C(x,t) = F(x,0)$ (for all x,t). In that sense, the inverse really deserves its name: it is an inverse up to homotopy.
Another standard application is the following: the concatenation $F * G$ is homotopic to

$$\begin{cases} F(x, f_1(t)) & \text{if } t \leq c \\ G(x, f_2(c)) & \text{if } t \geq c \end{cases}$$

if $f_1 : [0,c] \to [0,1], f_2 : [c,1] \to [0,1]$ are any surjective increasing continuous maps. In other words, it does not matter (up to homotopy) how exactly we define concatenation, as long as we traverse first F, then G in the correct direction. To prove the claim, apply the lemma to $F * G$, ϕ_1 the identity map, and ϕ_2 defined as

$$\phi_2(t) = \begin{cases} \frac{1}{2} f_1(t) & \text{if } t \leq c \\ \frac{1}{2}(f_2(t)+1) & \text{if } t \geq c \end{cases}$$

Definition 0.28. Let X be a topological space, and $x \in X$. We define

$$\pi_1(X,p) = \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = p\}/\sim$$

where the equivalence relation is homotopy relative to $\{0,1\}$.
The constant path as identity element, concatenation of paths, and inverses of paths turn $\pi_1(X,p)$ into a group, called the *fundamental group.*

The fact that inverses of paths are inverses in $\pi_1(X,p)$ is exactly the observation about homotopies above (recall: homotopies of one-point-maps are paths!). Similarly, that the constant path really is the identity, and that concatenation is associative (up to homotopy). Work out these details for yourself; they are all straightforward applications of the reparametrisation lemma.
Next, just as for π_0 we have the following functoriality:

- If $f : X \to Y$ is a continuous map, and $p \in X$ is a point, then there is a group homomorphism
$$f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$$
defined by $f_*[\gamma] = [f \circ \gamma]$.
- We have $\mathrm{id}_* = \mathrm{id}$ for the identity map $\mathrm{id} : X \to X$, and
- $(fg)_* = f_* g_*$ if $f : Y \to Z, g : X \to Y$ are continous maps.

As for π_0, these properties guarantee that π_1 is a *topological invariant* – homeomorphic spaces have the same fundamental group (up to group isomorphism).
However, we have to be a little bit careful with basepoints! If $f : X \to Y$ is a homeomorphism, then we get an induced group isomorphism
$$f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$$
Is the choice of $p, f(p)$ important? Not for path-connected spaces:

Lemma 0.29 (Basepoint dependence of π_1)**.** *Suppose that X is a topological space and that $\rho : [0, 1] \to X$ is a path connecting $p = \gamma(0)$ to $q = \gamma(1)$. Then there is a group isomorphism*
$$\pi_1(X, p) \to \pi_1(X, q)$$
defined by
$$[\gamma] \mapsto [\overline{\rho} * \gamma * \rho]$$

Proof. First, we need to check that the map is well-defined. This means that $\overline{\rho} * \gamma * \rho, \overline{\rho} * \gamma' * \rho$ are homotopic if γ, γ' are homotopic, which is clear (check if you agree!). Next, we need to check that is a group homomorphism. This means that
$$\overline{\rho} * \gamma * \gamma' * \rho \quad \text{and} \quad \overline{\rho} * \gamma * \rho * \overline{\rho} * \gamma' * \rho$$
are homotopic rel $\{0, 1\}$. But this is just the reparametrisation lemma again! Similarly, we see that the assigment
$$[\gamma] \mapsto [\overline{\rho} * \rho * \gamma * \overline{\rho} * \rho]$$
is the identity map of $\pi_1(X, p)$, and therefore the map from the lemma has an inverse. □

Next week we will start developing tools which are able to compute π_1, and that will give us the ability to tell apart many more spaces.
But first, we want to briefly talk about one very useful tool to show that spaces are homotopy equivalent. This uses the following notion.

Definition 0.30. Suppose that $f : X \to Y$ is a continuous map. The *mapping cylinder* is the space
$$M_f = Y \cup X \times [0, 1]/ \sim$$
where the equivalence relation is generated by $f(x) \sim (x, 0)$ for all $x \in X$. The *mapping cone* is the space
$$C_f = M_f \ \sim$$

where now we also identify $(x,1) \sim (x',1)$ for all $x, x' \in X$.

This is a useful, and very general construction. One crucial special case is the process of *attaching a cell*. We will discuss this in detail later in the course, but will already give an indication now. Namely, suppose that X is a topological space. We denote by $D^n = \{p \in \mathbb{R}^n \mid \|p\| \leq 1\}$ the n–*dimensional cell*. The boundary $\partial D^n = S^{n-1}$ is the $(n-1)$-dimensional sphere. Suppose we are given a map $f : S^{n-1} \to X$. Then we define

$$Y = X \cup D^n / \sim$$

where $f(x) \sim x$ for all $x \in S^{n-1}$ and say that Y *is obtained from* X *by attaching a* n*–cell*. Often one writes $Y = X \cup_f D^n$.
What does this have to do with mapping cones? We have a homeomorphism

$$X \cup_f D^n \simeq C_f.$$

Namely, $S^{n-1} \times [0,1]/ \sim = D^n$, where $(s,1) \sim (s',1)$ for all $s, s' \in S^{n-1}$. Thus (at least as sets) there is an obvious bijection between $X \cup_f F^n$ and C_f, and it should be a homeomorphism. Formally, we have to be a little bit careful, as in $X \cup_f D^n$ we first take the quotient of $S^{n-1} \times [0,1]$, and then glue it to X with f, whereas in C_f we first glue to X and then take the quotient. We'll see below that these are the same thing.
The theorem we would like to prove is the following:

Theorem 0.31. *Suppose that $f, g : X \to Y$ are homotopic. Then M_f, M_g are homotopy equivalent, and so are C_f, C_g.*

As a consequence, we get

Corollary 0.32. *The result of attaching a cell depends up to homotopy equivalence only on the homotopy class of the attaching map.*

For the proof of the theorem, I closely follow Bredon, I.14.18.

Lecture 5 (October 29)

One crucial step is the proof following lemma, which is often useful:

Lemma 0.33. *Suppose that $q : X \to Y$ is a quotient map (i.e. Y has the quotient topology with respect to that map), and that K is locally compact Hausdorff. Then the map*

$$q \times \mathrm{id} : X \times K \to Y \times K$$

is a quotient map as well.

Intuitively, taking products (with nice spaces) commutes with taking quotients. The proof is Bredon, I.13.19. It in turn relies on this characterisation of the quotient topology:

Lemma 0.34. *A continuous map $q : X \to Y$ is a quotient map if and only if the following is true: a map $h : Y \to Z$ is continuous if and only if $h \circ q : X \to Z$ is continuous.*

This is Bredon, I.13.5. We also need the following

Lemma 0.35. *Suppose that K is compact and X is any space. Then the projection $\pi_X : X \times K \to X$ is a closed map.*

This is Bredon, I.8.2.

Part III: Covering Space Theory. Our next immediate goal will be to compute the fundamental group of the circle. We would guess that it is $\mathbb{Z}$, via winding number.
How do we prove this? First, here is a convenient way to get all of these basic loops. We think of $S^1 \subset \mathbb{C}$ as the unit norm complex numbers. Then we have the maps

$$\gamma_n(t) = e^{2\pi int}$$

all of which are continuous, $\gamma_n : [0,1] \to S^1$ and $\gamma_n(0) = \gamma_n(1) = 1$.
How can we relate an arbitrary loop $\gamma : [0,1] \to S^1$ to one of these? Let

$$f : \mathbb{R} \to \mathbb{C}, \quad f(t) = e^{2\pi it}$$

be the exponential function.

Lemma 0.36. *Suppose that $\gamma : [0,1] \to S^1$ is an arbitrary loop with $\gamma(0) = 1 = \gamma(1)$. Suppose $m \in \mathbb{Z}$ is arbitrary. Then there is a unique continous map $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ so that $f \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = m$.*

Proof. Note that f has the following property: for any $x \in S^1$ there is an open set $U_x \subset S^1$, and

$$f^{-1}(U_x) = \coprod_{n \in \mathbb{Z}} I_n$$

and each $f : I_n \to U_x$ is a homeomorphism. Call these U_x *good.*
For any t, we can choose δ so that $\gamma(t - delta, t + \delta) \subset U_t$ for a good U_t. By compactness of the interval, there are in fact intervals $J_k = [a_k, b_k], k = 1, \ldots, n$ with $a_1 = 0, b_k = a_{k+1}, b_n = 1$ and so that $\gamma(J_k) \subset U_k$ for good U_k. Now, we will inductively choose local inverses $g_k : U_k \to I_k$ to f. We begin by choosing g_1 so that $g_1(\gamma(0)) = m$. This is uniquely possible by the property above. Now, suppose that g_k is already defined. We then choose $g_{k+1} : U_{k+1} \to I_{k+1}$ so that $g_{k+1}(\gamma(a_{k+1})) \in I_k$ (which is possible since $\gamma(a_{k+1}) = \gamma(b_k) \in U_k$). Observe (also inductively) that these g_k are uniquely determined by this requirement.
But now we can simply define:

$$\alpha(t) = \begin{cases} g_k \circ \gamma(t) & \text{if} t \in [a_k, b_k] \end{cases}$$

and observe that it defines a lift of γ.
For the uniqueness, observe that by uniqueness of g_1, the lift α is unique on $[a_1, b_1]$ by the defining property of f above. Now induction shows uniqueness globally. □

As a conseqeunce, we have

Lemma 0.37. *The map $\mathbb{Z} \to \pi_1(S^1, 1)$ defined by $n \mapsto [\gamma_n]$ is surjective.*

Proof. Let $\gamma : [0,1] \to S^1$ be any loop, and let $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ be the lift guaranteed by the previous lemma. Note that γ is a loop, we have that $\widetilde{\gamma}(1) = n \in \mathbb{Z} = f^{-1}(1)$. Thus, $\widetilde{\gamma}$ is homotopic to the affine map $[0,1] \to [0,n]$. Postcomposing with f yields a homotopy from γ to γ_n as claimed. □

Lecture 6 (October 30)

Next, we want to show injectivity. This will be very similar, except that we will lift *homotopies* now, instead of paths. Here's the result:

Lemma 0.38. *Suppose that $H : [0,1] \times [0,1] \to S^1$ is a homotopy relative to $\{0,1\}$, and suppose that $\widetilde{\gamma}$ is a lift of $H(\cdot, 0)$. Then there is a unique map $\widetilde{H} : [0,1] \times [0,1] \to \mathbb{R}$ so that $\widetilde{H}(x, 0) = \widetilde{\gamma}(x)$ and $f \circ \widetilde{H} = H$. This is again a homotopy rel $\{0,1\}$.*

Proof. The existence and uniqueness of $\widetilde{H}$ as a set-theoretic map follows from the following trick: fixing a t, the map $s \mapsto H(t,s)$ is a path, and so it has a unique lift starting at $\widetilde{\gamma}(t)$.

Thus, we just have to show that this is continuous. This requires some preliminaries.

For any $p = (s,t)$, find a good neighbourhood U_p of $H(s,t)$ (see above), and $\epsilon_p > 0$, so that $H((s - 2\epsilon_p, s + 2\epsilon_p) \times (t - 2\epsilon_p, t + 2\epsilon_p)) \subset U_p$. Covering the unit square with boxes of sidelength $2\epsilon_p$ (inside these $4\epsilon_p$ boxes above), and using compactness of the unit square, we see that there is some ϵ with the following property: for any $(s,t) \in [0,1]^2$ there is a good neighbourhood U, and the 2ϵ–box is mapped by H into U.

This allows us to describe $\widetilde{H}(t,s)$ in a different way. Recall that, by definition, we know that $\widetilde{H}(t,s)$ is the endpoint of a lift $\widetilde{\rho}_0 : [0,s] \to \mathbb{R}$ of $s \mapsto H(t,s)$. Namely, choose points $s_0 = 0 < s_1 < \ldots < s_n = s$ so that $|s_i - s_{i+1}| < \epsilon/2$. Choose good neighbourhoods U_i containing $H(t - \epsilon, s_i - \epsilon) \times (t + \epsilon, s_i + \epsilon)$. There are *unique* sets I_i, so that $f : I_i \to U_i$ is a homeomorphism, and so that $\widetilde{\rho}_0(s_i) \in I_i$ (by the definition of good). Let $g_i : U_i \to I_i$ be the inverses. We claim that for $s \in [s_i, s_{i+1}]$ we have $\widetilde{\rho}_0(s) = g_i(H(t,s))$. This is true, since both the left and right hand side are lifts of $s \mapsto H(s,t), s \in [s_i, s_{i+1}]$ with the same initial point. But, similarly, for any $t', |t - t'| < \epsilon$ we then have that $\widetilde{\rho}(s) = g_i(H(t', s))$ is a lift of $s \mapsto H(s, t')$. In particular, this is well-defined if the choice of i in nonunique, and in fact, we have that for (s', t') ϵ–close to (s,t):

$$\widetilde{H}(t', s') = g_n(H(t', s')).$$

This shows continuity.

The final claim (about homotopies rel endpoints) follows since $\widetilde{H}(t,s)$ for $t \in \{0,1\}$ is continuous and contained in the discrete set $f^{-1}(H(s,t))$, hence constant. □

As a consequence we have

Lemma 0.39. *The map* $\mathbb{Z} \to \pi_1(S^1, 1)$ *defined by* $n \mapsto [\gamma_n]$ *is injective.*

Proof. If γ_n would be nullhomotopic, simply lift the nullhomotopy. By the previous lemma this is a homotopy rel endpoint, showing $n = 0$. ☐

With this in hand, we can show a few nice consequences.

Theorem 0.40. *Every continuous function* $f : D^2 \to D^2$ *has a fixed point.*

Proof. Hatcher, Theorem 1.9 (Chapter 1) ☐

We can also use this to distinguish spaces. To this end, we need

Theorem 0.41. *For any* $n \geq 2$*, we have* $\pi_1(S^n) = 1$.

Proof. Hatcher, Proposition 1.14 (Chapter 1) ☐

Lecture 7 (November 5)

First, we want to show that the fundamental group can actually distinguish spaces up to homotopy equivalence, not just homeomorphism (this is not entirely trivial, since one has to be careful about what happens to the basepoint). Concretely, we proved:

Proposition 0.42 (Hatcher, 1.18)**.** *If* $f : X \to Y$ *is a homotopy equivalence, and* $p \in X$ *is a point, then the induced map*

$$f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$$

is an isomorphism.

[Hatcher, 1.18] Next, we come to the crucial definition for the next few classes:

Definition 0.43. A map $f : X \to Y$ between topological spaces is a *covering map*, if f is surjective, and every point $y \in Y$ has an open neighbourhood U so that

$$f^{-1} = \coprod_{i \in I} V_i$$

for some index set I, and so that $f_{V_i} : V_i \to U$ is a homeomorphism for all $i \in I$.

The core property of coverings is the following:

Theorem 0.44 (Homotopy lifting, Hatcher Proposition 1.30)**.** *Suppose that* $p : X \to Y$ *is a covering map, and suppose that* $f : Z \to Y$ *is a map. Let* $\widetilde{f} : Z \to X$ *be a* lift *of* f*, i.e. a continuous map so that* $p\widetilde{f} = f$*.*
If $H : Z \times [0, 1] \to X$ *is any homotopy starting in* f *(i.e.* $H(z, 0) = f(z)$*), then there is a unique lift* $\widetilde{H}$ *of* H *starting in* $\widetilde{f}$ *(i.e.* $p\widetilde{H} = H$ *and* $\widetilde{H}(0, z) = \widetilde{f}(z)$*)*

We actually proved this when figuring out the fundamental group of the circle. Check the discussion in Hatcher if you are unsure about the details. To see applications of this, we first need a supply of examples. These will be given by graphs.

Definition 0.45. A *Graph* Γ consists of the following data:

(1) A set $V = V(\Gamma)$, (the vertex set)
(2) A set $E = E(\Gamma)$, (the set of oriented edges)
(3) A map $\bar{\cdot} : E \to E$ (inverting oriented edges)
(4) Maps $i, t : E \to V$ (initial and terminal vertex maps)

so that

i) $\bar{\cdot}$ is a fixed point free involution
ii) $t(\bar{\cdot}(e)) = i(e)$ for all e.

An *orientation* on a graph is a choice $E^+ \subset E$ containing exactly one of $e, \overline{e}$ for all $e \in E$.

Out of a graph we can build a topological space as follows. Suppose Γ is a graph and E^+ an orientation (the choice will not matter, but this makes it easier to describe the construction)

$$X_\Gamma = (V \coprod E^+ \times [0,1])/\sim$$

where V, E^+ are equipped with the discrete topology, and $\sim$ is generated by $(e, 0) \sim i(e), (e, 1) \sim t(e)$. This is called the (topological) *realisation* of the graph Γ.
One example is the wedge of circles. To this end, consider the graph with one vertex and $2k$ edges (all other data is then already determined). The realisation then is

$$S^1 \vee \cdots \vee S^1 = \{1, \ldots, k\} \times [0,1]/\sim$$

where $\sim$ is generated by $(i, t) \sim (j, s)$ for all i, j and $s \in \{0, 1\}$.
In fact, we can relate graphs to this example.

Definition 0.46. Suppose that Γ is a graph and E^+ an orientation. An *edge-labelling* by a set S is a map

$$L : E^+ \to S.$$

Now suppose that we have an oriented graph labelled by $\{1, \ldots, k\}$. Then the map

$$V \coprod E^+ \times [0,1] \to \{1, \ldots, k\} \times [0,1]$$

which maps any $v \in V$ to $(1, 0)$ and a point (e, t) to $(L(e), t)$ induces a continuous map

$$p : X_\Gamma \to S^1 \vee \cdots \vee S^1.$$

Lemma 0.47 (Covering Criterion)**.** *In the context above, assume that for every $v \in V$ the following holds. For any $1 \leq j \leq k$ there is exactly one $e \in E^+$ with $L(e) = j, i(e) = v$ and exactly one $e \in E^+$ with $L(e) = j, t(e) = v$. Then the map p above is a covering map.*

Proof. There are two cases. First, consider a point $p \in S^1 \vee \cdots \vee S^1$ which is not the vertex, i.e. defined by a point $(i, t_0), 0 < t_0 < 1$. Then let U be the image of

$$\{(i,t), 0 < t < 1\}$$

The preimage is

$$p^{-1}(U) = \{(e,t) \mid L(e) = i, 0 < t < 1\}/\sim$$

It is easy to see that p restricted to any set of the form $\{(e,t) \mid 0 < t < 1\}$ for a given e is a homeomorphism onto its image. Hence U is a good neighbourhood.
Next, consider the point p which is the vertex of the wedge of circles. In this case, define U be the image of

$$\{(i,t), 1 \leq i \leq k, 0 \leq t < \epsilon \text{ or } 1 - \epsilon < t \leq 1\}$$

The preimage is

$$p^{-1}(U) = \{(e,t) \mid e \in E^+, 0 \leq t < \epsilon \text{ or } 1 - \epsilon < t \leq 1\}/\sim$$

Given a vertex $v \in V$, define V_v to be the image of

$$\{(e,t) \mid i(e) = v \text{ and } 0 \leq t < \epsilon \text{ or } t(e) = v \text{ and } 1 - \epsilon < t \leq 1\}$$

It is clear that

$$p^{-1}(U) = \coprod_{v \in V} V_v.$$

By the condition assumed in the lemma, $p|_{V_v} : V_v \to U$ is a bijection for each v, and it is easy to see that it is in fact a homeomorphism. □

Lecture 8 (November 6)

We now want to use the construction of covering spaces via graphs:

Corollary 0.48. *The group $\pi_1(S^1 \vee S^1)$ is nonabelian.*

Proof. Denote by a, b the loops given by the two obvious circles in $S^1 \vee S^1$. First, we claim that there is a cover $X \to S^1 \vee S^1$ with the following property: there is an embedded segment ρ in X (with distinct endpoints!) which maps under the covering map to $a * b * \overline{a} * \overline{b}$. The existence of this can easily be seen by drawing a suitable graph and appealing to the lemma from last time.
Now, suppose that $[a, b] \in \pi_1(S^1 \vee S^1)$ were trivial. Then there would be a homotopy H starting in $a * b * \overline{a} * \overline{b}$ and ending in the constant path rel endpoints. Lift this homotopy to X starting in ρ. The lift would be a homotopy from ρ to the lift of a constant path rel endpoints. This is impossible since ρ has distinct endpoints. □

Let us emphasise two important properties about covers (compare the discussion after Prop. 1.30 in Hatcher)

- The path lifting property: if $p : X \to Y$ is a covering, and $\gamma : [0,1] \to Y$, then for any $x \in p^{-1}(\gamma(0))$ there is a unique lift $\widetilde{\gamma} : [0,1] \to X$ with $\widetilde{\gamma}(0) = x$ (apply the homotopy lifting theorem for $Z = \{*\}, H = \gamma$)
- Endpoints of path lifts: suppose that $p : X \to Y$ is a covering, $\gamma_1, \gamma_2 : [0,1] \to Y$ are two paths which are homotopic rel endpoints. If $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are two lifts with the same initial point, then they have the same endpoint as well (lift the homotopy to find some lift of γ_2 both of whose endpoints agree and use uniqueness of path lifting)

Next, we study how covering maps interact with the fundamental group.

Lemma 0.49 (Hatcher 1.31)**.** *If $p : X \to Y$ is a covering map, then the induced map*

$$p_* : \pi_1(X, x) \to \pi_1(Y, p(x))$$

is injective. Its image consists exactly of those loops based at $p(x)$ which lift to loops in X at x.

Lemma 0.50 (Hatcher 1.32)**.** *Suppose that $p : X \to Y$ is a covering map, and X is path-connected. Then, the cardinality of $p^{-1}(y)$ is equal to the index of $p_*\pi_1(X, x)$ in $\pi_1(Y, y)$ for any $y \in p^{-1}(y)$.*

Crucial is the following criterion guaranteeing existence of lifts:

Theorem 0.51 (Lifting theorem, Hatcher 1.33)**.** *Let $p : X \to Y$ be a covering, and suppose X is locally path-connected and path-connected. Suppose that $f : Z \to Y$ is a continuos map and $f(z_0) = y_0 = p(x_0)$. Then f has a lift $\widetilde{f}$ with $\widetilde{f}(z_0) = x_0$ if and only if $f_*\pi_1(Z, z_0) \subset p_*\pi_1(X, x_0)$.*

Lecture 9 and 10 (November 12 and 13)

Definition 0.52. A topological space X is *semilocally simply-connected* if every point $x \in X$ has a neighbourhood U so that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Theorem 0.53 (Hatcher, page 63-65)**.** *Let X be path-connected and locally path-connected. Then X is semilocally simply-connected if and only if it has a simply-connected covering space.*

We call simply-connected covering spaces *universal covers.*
There is the following correspondence between covering spaces and subgroups of the fundamental group:

Theorem 0.54 (Hatcher 1.38)**.** *Suppose that X is path-connected, locally path-connected and semilocally simply-connected and $x \in X$. Then the assignment*

$$\{\textit{path-connected covering spaces } p : Y \to X \textit{ with } p(y) = x\}/\sim \to \textit{subgroups of } \pi_1(X, x)$$

defined by $p \mapsto p_*(\pi_1(Y, y)) \subset \pi_1(X, x)$ is a bijection. Here, $\sim$ is isomorphism of coverings, i.e. homeomorphisms lifting the identity.

Lecture 11 (November 19)

Next, we want to study a different way to characterise covering spaces. This uses the notion of a set with group action.

Definition 0.55. Let G be a group, and M a set. A *(right) G–action on M* is a map

$$(m, g) \mapsto m \cdot g$$

so that

$$m \cdot 1 = m$$

for all $m \in M$ and

$$m \cdot (gg') = (m \cdot g) \cdot g'$$

A set with a G action we also call a G–set. A right action of G on a set M is equivalent to a homomorphism $\rho : G \to \mathrm{Bij}(M)$ into bijections of M. The correspondence is given by $\rho(g) = [m \mapsto m \cdot g^{-1}]$ (the inverse is necessary to make ρ a homomorphism).

One core example is given by covering spaces. Namely, suppose $p : Y \to X$ is a covering space, and $x_0 \in X$ a point. Given a loop $\gamma : [0, 1] \to X$ based at $x_0 \in X$, and $y \in p^{-1}(x_0)$, let $\widetilde{\gamma}$ be the lift of γ starting at y. Then define

$$y \cdot [\gamma] = \widetilde{\gamma}(1).$$

We have already seen that this indeed only depends on the homotopy class of γ, and thus it defines a $\pi_1(X, x_0)$–set structure on $p^{-1}(x_0)$.

The other core example is given by sets of cosets. Namely, if G is a group, and H is a subgroup, then the set $H\backslash G = \{Hg\}$ of left cosets is a right G–set in the obvious way: $Hg \cdot g' = Hgg'$.

Given a G–set M and a $m \in M$, define the *stabiliser*

$$G_m = \{g \in G \mid m \cdot g = m\}.$$

In the covering space setup, we can identify the stabilisers:

Lemma 0.56. *Suppose $p : Y \to X$ is a covering space, and let $G = \pi_1(X, x_0)$. Then, for the G–set $M = p^{-1}(x_0)$ we have*

$$G_m = p_*(\pi_1(Y, m))$$

Proof. By definition of the action, G_m is the subgroup of $\pi_1(X, x_0)$ formed by all those loops whose lifts to m are closed. We have already seen that this is $p_*(\pi_1(Y, m))$. □

We call a G–set M *transitive* if for all $m, m' \in M$ there is an element g so that $m \cdot g = m'$.

Lemma 0.57. *Suppose $p : Y \to X$ is a covering space with X path-connected, and let $G = \pi_1(X, x_0)$. Then, the G–set $M = p^{-1}(x_0)$ is transitive if and only if Y is path-connected.*

Proof. First suppose that Y is path-connected. Then, for any $m, m' \in p^{-1}(x_0)$ there is a path $\rho : [0,1] \to Y$ with $\rho(0) = m, \rho(1) = m'$. Then ρ is the lift of the loop $\gamma = p \circ \rho$ based at x_0, and thus $m \cdot \gamma = m'$ by definition.
Conversely, suppose that M is transitive. By definition, this means that any two points of $p^{-1}(x_0)$ can be joined by a path (the lift of a suitable loop). Hence, it suffices to show that any point $y \in Y$ can be joined to the fiber $p^{-1}(x_0)$ by a path. This follows, since X is path-connected, and therefore $p(y)$ can be joined to x_0 by a path ρ. A lift $\widetilde{\rho}$ starting in y then has the desired property. □

A *morphism between G–sets M, M'* is a map $\varphi : M \to M'$ so that $\varphi(m \cdot g) = \varphi(m) \cdot g$ for all m. An *isomorphism of G sets* is a morphism which is bijective, and whose inverse is also a morphism.

Lemma 0.58. *Suppose M is a transitive G–set. Let $m \in M$ be given. Then there is an isomorphism of G–sets*

$$\varphi : G_m \backslash G \to M$$

Proof. First, we define a map $G \to M$ by $g \mapsto m \cdot g$. By definition, this induces a map $G_m \backslash G \to M$. Transitivity of the set M guarantees that this map is surjective. For injectivity, suppose that $\varphi(G_m g) = \varphi(G_m g')$, and thus

$$m = \varphi(G_m) = \varphi(G_m g) \cdot g^{-1} = \varphi(G_m g') \cdot g^{-1} = \varphi(G_m g' g^{-1}) = m \cdot (g' g^{-1}).$$

Thus, $g' g^{-1} \in G_m$ and $G_m g = G_m g'$. Hence, φ is bijective. The inverse is then automatically a G–set morphism. □

Hence, to understand transitive G–sets, it suffices to understand the concrete sets $H \backslash G$. An example of this is the following very useful lemma:

Lemma 0.59. *Suppose that M, N are two transitive G–sets, and $m \in M, n \in N$. Then there is a G–set morphism $\varphi : M \to N$ with $\varphi(m) = n$ if and only if $G_m \subset G_n$. If it exists, the map is unique and surjective.*

Proof. For one direction, suppose the map exists, and $g \in G_m$. Then

$$n \cdot g = \varphi(m) \cdot g = \varphi(m \cdot g) = \varphi(m) = n.$$

For the other direction, we can construct the map as

$$M \to G_m \backslash G \to G_n \backslash G \to N$$

where the first and last map are from the previous lemma, and the middle map exists by the assumption on the stabilisers.
Uniqueness follows from transitivity of M: the value $\varphi(m')$ is determined by $\varphi(m)$ since there is some $g \in G$ with $m' = m \cdot g$. Surjectivity follows from transitivity of N, since for any n' there is a $g \in G$ with $n' = n \cdot g = \varphi(m \cdot g)$. □

Corollary 0.60. *Suppose M is a transitive G–set. Then there is an automorphism $\varphi : M \to M$ with $\varphi(m) = m'$ if and only if $G_m = G_{m'}$.*

Corollary 0.61. *Consider $M = G$ as a G–set (with the usual right multiplication action). Then there is an isomorphism*

$$G \to \mathrm{Aut}_G(M)$$

from G to the group of G–set automorphisms of M, which associates to g the unique automorphism mapping 1 to g.

Next, we want to interpret morphisms between G–sets with covers.

Definition 0.62. Suppose that $p_i : Y_i \to X, i = 1, 2$ be two covers of a space X. A continuous map $f : Y_1 \to Y_2$ *morphism of covers* if $p_2 \circ f = p_1$.

Equivalent (and sometimes useful) is the perspective: a morphism of covers is a lift of the identity map.

Lemma 0.63. *Suppose $p_i : Y_i \to X, i = 1, 2$ be two covers of a space X and $x_0 \in X$. A morphism of covers $f : Y_1 \to Y_2$ induces by restriction a morphism of $\pi_1(X, x_0)$–sets $p_1^{-1}(x_0) \to p_2^{-1}(x_0)$.*

Proof. Let $y \in p_1^{-1}(x_0)$ be given. It is clear from the definition of morphism of covers that $f(y) \in p_2^{-1}(x_0)$. Thus, we just have to check that it commutes with the $G = \pi_1(X, x_0)$ action. In other words, let γ be a loop in X based at x_0 and let $\widetilde{\gamma}$ be a lift at y. Then, observe that $f \circ \gamma$ is a lift of γ based at $f(y)$. By definition of the action, this shows

$$f(y \cdot [\gamma]) = f(y) \cdot [\gamma].$$

□

Much more interestingly, the coverse is also true:

Theorem 0.64. *Suppose that X is path-connected, locally path-connected, and semi-locally simply-connected and $x_0 \in X$. Let $p_i : Y_i \to X, i = 1, 2$ be two covers. Then the assignment $f \mapsto f|_{p_1^{-1}(x_0)}$ induces a bijection*

$$\{\textit{morphisms of covers } Y_1 \to Y_2\} \to \{\textit{morphisms of } G\textit{–sets } p_1^{-1}(x_0) \to p_2^{-1}(x_0)\}$$

for $G = \pi_1(X, x_0)$.

Proof. The first step is a technical observation. Write Y_1

$$Y_1 = \coprod V_i$$

as the union as its path-connected components. We claim that each V_i is open (and so Y_1 has the topology of the disjoint union of the V_i). Namely, let $v \in V_i$ be any point. By assumption, there is a neighbourhood U of v so that $p_1|_U$ is a homeomorphism onto its image. By assumption, $p_1(U)$ contains a path-connected neighbourhood of $p_1(v)$, and therefore U contains a path-connected neighbourhood of v, showing the claim.

Next, observe that each $p|_{V_i}$ is surjective. Namely, if $v \in V_i$ is arbitrary, lift a path from $p_1(v)$ to some x starting in v. The lift (being a path) cannot leave the path-component V_i, and thus its endpoint lies in $V_i \cap p_1^{-1}(x)$. In particular, each $p|_{V_i}$ is itself a covering map.
Now, the lifting theorem guarantees that a lift f_i of the identity (of X) to a (path-connected) cover $p|_{V_i}$ is already determined by the image of a single point in $p_1^{-1}(x_0) \cap V_i$. This shows injectivity of the map.
Finally, suppose that $\varphi : p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ is a morphism of G–sets. By the lemma above we have, for any $m \in p_1^{-1}(x_0)$ the inclusion $G_m \subset G_{\varphi(m)}$ of stabilisers. This implies

$$G_m = (p_1)_*(\pi_1(Y_1, m)) \subset (p_2)_*(\pi_1(Y_2, \varphi(m))).$$

Denoting by V_j again the path-component containing m, the lifting theorem implies that there is a lift f_j of p_1 to $p_2 : Y_2 \to X$ with $f_j(m) = \varphi(m)$.
Since Y_1 is the disjoint union of the V_j as a topological space, $f = \coprod f_j$ is a continuous map $f : Y_1 \to f_2$ which has the desired property. □

Lecture 12 (November 20)

We call the group of automorphisms of a covering $p : Y \to X$ the group of *deck transformations*. By the theorem from last time, this is the same as the group of G–set automorphisms of the fibers. A covering is *normal* if the group of deck transformations acts transitively on the fibers.
A crucial example is the universal covering.

Lemma 0.65. *Let X be path-connected, locally path-connected, and suppose that $p : Y \to X$ is a universal covering. Then p is normal.*

Proof. Let $G = \pi_1(X, x_0)$. The fiber $p^{-1}(x_0)$ is a transitive G–set, since Y is path-connected. Further, the stabilisers $G_y = p_*(\pi_1(Y, y)) = e$ since the covering is universal. Thus, $p^{-1} = G$ as G–sets. By a corollary from last time, the automorphism group of G is G itself, in particular it acts transitively. Thus, by the theorem from last time, the deck transformation group does as well. □

Lemma 0.66. *Suppose X is path-connected and locally path-connected. Let $p : Y \to X$ be a covering, and suppose Y is path-connected. Then p is normal if and only if $p_*(\pi_1(Y, y))$ is normal in $\pi_1(X, p(y))$.*

Proof. Fix some x_0, and $y_0 \in p^{-1}(x_0)$. Given any other $y \in p^{-1}(x_0)$, suppose that ρ is a path from y_0 to y. Observe that then $\gamma = p \circ \rho$ is a loop based at x_0. Also, as we have seen earlier, the map $\alpha \mapsto \rho * \alpha * \overline{\rho}$ gives an isomorphism $\pi_1(Y, y) \to \pi_1(Y, y_0)$. Now, there is a G–set isomorphism mapping y to y_0 if and only if the stabilisers are the same. In this setting, this is equivalent to

$$G_y = p_*(\pi_1(Y, y)) = [p \circ \rho] p_*(\pi_1(Y, y_0)) [p \circ \rho]^{-1}.$$

Hence, if $p_*(\pi_1(Y, y_0))$ is normal, then for any y there is a deck transformation mapping it to y_0, showing normality of p. Conversely, since $[p \circ \rho]$

above can be in any homotopy class, for a normal covering p the group $p_*(\pi_1(Y, y_0))$ is normal. □

At this point, a **warning**: the fiber of a normal cover has two different actions – by the deck group and by the fundamental group of the base. These are not the same, even in cases where one might think that they might be related. E.g. in general for the universal cover (where the two groups are the same), the two actions are different!

Our next goal is to construct covers from G–sets. To do so, we will use group actions by homeomorphisms.

Definition 0.67. An action of G on Y by homeomorphisms is called a *covering space action* if the following holds: for any $y \in Y$ there is an open set $U, y \in U$, so that $gU \cap U \neq \emptyset$ implies $g = \mathrm{id}$.

The importance and name of this is given by the following:

Lemma 0.68. *Suppose that Y is a space, and G acting on Y as a covering space action.*

i) The quotient map $p : Y \to Y/G$ is a normal covering space.
ii) If Y is path-connected, then G is the group of deck transformations of p.

Proof. To see the first claim, suppose that $x = p(y)$ is any point. Take a neighbourhood U of y as in the definition of covering space action. Then we have

$$p^{-1}(p(U)) = \coprod_{g \in G} gU$$

by the defining property. Further, for any g, we have

$$p|_{gU} : gU \to p(U)$$

is a continuous bijection, and by definition of the quotient topology it is actually a homeomorphism. Thus, p is a covering. Since G acts as covering automorphisms, and transitive on fibers, it is normal, showing i).
Part ii) is immediate, since for a connected Y deck transformations are uniquely determined by the value of a single point. □

Lecture 13 (November 26)

Lemma 0.69. *Let X be path-connected, locally path-connected, and semi-locally simply-connected, and put $G = \pi_1(X, x_0)$. Suppose that M is any G–set. Then there is a cover $p : Y \to X$ whose fiber $p^{-1}(x_0)$ is isomorphic to M as a G–set.*

Proof. Let $q : Z \to X$ be the universal cover, and let $\delta : G \to \mathrm{Homeo}(Z)$ be the action by deck transformations. Similarly, let $\rho : G \to \mathrm{Bij}(M)$ be the action corresponding to the G–set M.

Define the space $Z \times M$ (M has the discrete topology), and observe that there is a (diagonal) action by homeomorphisms

$$G \to \text{Homeo}(Z \times M)$$

We define

$$Y = Z \times M/G.$$

Observe that $Z \times M$ is a cover of Y as the action is a covering space action. There is a natural map

$$p : Y \to X$$

induced by the projection to Z and the covering map q. We claim that this is a covering map. To this end, suppose that $x \in X$ is a point, and $U \subset Z$ be a set so that $q^{-1}(q(U)) = \coprod_{g \in G} gU$. We then claim that

$$p^{-1}(U) = U \times M,$$

so that p is the projection to the first factor. This shows that p is a covering map, and that $p^{-1}(x_0)$ is M as a set. It remains to check that the G–set structure is the correct one.
To see this, suppose that $g = [\gamma] \in \pi_1(X, x_0)$ is a loop, and $\widetilde{\gamma}$ is a lift to Z, based at z_0. Its endpoint is $\delta(g)(z_0)$ (this is how the identification of the deck group with the fundamental group works). Hence, lifting at (z_0, m) in $Z \times M$ ends at $(\delta(g)(z_0), m)$, which is equivalent to $(z_0, \rho(g^{-1})(m))$. Hence, it acts on the fiber M as $m \cdot g$.

□

Together with the result from last time, this shows: *covers (and their morphisms) of X are in 1-1 correspondence to G–sets (and their morphisms).* Based covers (and their morphisms) are in 1-1 correspondence to based G–sets.
The *free group on a set of symbols* is the "biggest" group generated by those symbols. There are two flavors of saying this.

Definition 0.70 (Free groups, flavor 1)**.** Let S be a set. The *free group* $F\langle S\rangle$ is the group whose elements are words with letters $s^{\pm}$ for $s \in S$ up to obvious cancellation.

This is easy to imagine and play with, but something of a formal annoyance (checking that things are well-defined or unique...)

Definition 0.71 (Free groups, flavor 2)**.** Let S be a set. The *free group* $F\langle S\rangle$ is the group together with a subset $S \subset F\langle S\rangle$ with the following universal property: if G is any group and $g_s, s \in S$ are any collection of elements, then there is a unique homomorphism $\varphi : F\langle S\rangle \to G$ so that $\varphi(s) = g_s$ for all s.

In other words, free groups are groups which are easy to map out of.
It is sort of clear that version 1 has the property of version 2 (once you know that you can actually compute with words the way you want it is actually

easy). It is a standard algebraic trick that things defined as in version 2 are unique once they exist. Hence, they are actually equivalent.

Lemma 0.72. *Let $X = S^1 \vee \cdots \vee S^1$ be the wedge of n circles. Then $\pi_1(X) = F\langle\{1, \ldots, n\}\rangle = F_n$.*

We have a map $F_n \to \pi_1(X)$ mapping i to the i–th petal. One could now directly check that this is surjective and injective with various means (e.g. by dealing with paths, homotopies and being careful about the basepoint, or by "guessing a universal cover") – and it is sort of instructive that you try!

Lecture 14 (November 27)

We now discuss an important special case: suppose X is as before path-connected, locally path-connected and semi-locally path-connected. Suppose $x_0 \in X$ is a point, $G = \pi_1(X, x_0)$ and Q is any group.
A homomorphism $\varphi : G \to Q$ turns Q into a G–set *with a preferred point* $\varphi(e)$ by right-multiplication by $\varphi(g)$. Further, this right action commutes with the left action of Q on itself. We now build out of this G–set a cover $p : Y \to X$. This cover furthermore comes with an action of Q by deck transformations (since the Q–action on the left of Q gives G–set automorphisms), and this action is free and transitive, showing in particular that the cover is normal.
Now, conversely, suppose that we have a normal cover $p : Y \to X$, together with a preferred point $y \in p^{-1}(x_0)$, and a free transitive Q–action by deck transformations. The free transitivity of the action allows to identify $p^{-1}(x_0)$ with Q, so that the deck group action is the multiplication action on the left, and the preferred point y corresponds to the identity. Since the G–action commutes with the deck transformation action, the G–action is multiplication by elements on the right under this identification (think of the G–action as an automorphism of the left–Q–set structure of Q), and therefore we get a homomorphism $\varphi : G \to Q$.
On the level of (based) G–sets these two operations are inverses to each other, and so by the results from last time we have shown:

Corollary 0.73. *In the context above, there is a 1–1 correspondence between homomorphisms $\pi_1(X, x_0) \to Q$ and based regular covers $(Y, y) \to (X, x_0)$ with a free transitive Q–action (which we'll call Q–regular for short).*

This allows to transfer from topology to algebra and vice versa. We will see a powerful application of this next, the Seifert-van Kampen theorem.
As a warm-up, we will discuss a different proof of the following lemma using the machinery above.

Lemma 0.74. *Let $X = S^1 \vee \cdots \vee S^1$ be the wedge of n circles. Then $\pi_1(X) = F\langle\{1, \ldots, n\}\rangle = F_n$.*

So, suppose we have any group G and elements $g_1, \dots, g_k$. Clearly there are unique homomorphisms $\rho_i : \pi_1(S^1, 1) = \mathbb{Z} \to G$ which map 1 to g_i.
By the correspondence above, these correspond to unique based regular G–covers $p_i : (Y_i, y_i) \to (S^1, 1)$ of the circle.
We now define a space

$$Y = \coprod Y_i / \sim \qquad y_i \cdot g \sim y_j \cdot g,$$

i.e. we glue the Y_i identifying the preimages of the basepoint in a G–equivariant way.
The space Y has a natural map $p : Y \to S^1 \vee \cdots \vee S^1$ (induced by the p_i), and this map is a covering space: at points not glued this follows from the fact that the p_i are coverings, and at the basepoint it follows observing that gluing evenly covered neighbourhoods is an evenly covered neighbourhood. The space also has a natural G–action (since the gluing is G–equivariant), making Y a G–regular cover.
Hence, $p : (Y, y) \to (S^1, 1)$ corresponds, by the corollary again, to a unique homomorphism $\varphi : \pi_1(X, 1) \to G$. This maps the loops represented by the petals to g_i. Why? Because restricted to the i–th petal C_i the map p is exactly p_i, and so the fibers are the same $\pi_1(C_i, 1)$–sets. Hence, the corollary tells us that φ restricted to the subgroup $\pi_1(X_i, 1)$ is ρ_i.
The same argument also yields uniqueness of φ – given the cover Y' corresponding to any such φ', restricting to the petals needs to give the same covers p_i using the corollary the other way around, and so Y' is also obtained by gluing the Y_i equivariantly.

Lecture 15 (December 4)

Sometimes, it is useful to have the other definition of a free group.

Lemma 0.75. *The free group $F\langle S\rangle$ can be identified with the set of words in $S^{\pm}$ (up to obvious cancellation), with concatenations of words as the group operation.*

One useful thing we can do with this is give *presentations* of groups. Namely, suppose we have a set S and a set $R \subset F\langle S\rangle$. We can then form

$$G = \langle S | R\rangle = F\langle S\rangle / N(R)$$

We call the data S, R a *presentation of the group G*.
Some examples: $\mathbb{Z}/k = \langle a | a^k\rangle$.
$\mathbb{Z}^2 = \langle a, b | [a, b]\rangle$.
These claims are actually slightly nontrivial. We have to check that the chosen relations suffice to completely determine the group.
How does one do this? The key is that it is easy to describe homomorphisms mapping out of groups with presentations: a homomorphism $\rho : F\langle S\rangle \to Q$ induces $\hat{\rho} : G \to Q$ if and only if $\rho(r) = 1$ for any $r \in R$.

Thus, e.g. we have a homomorphism $\varphi : \langle a | a^k \rangle \to \mathbb{Z}/k$. It is clearly surjective, but injectivity requires some minimal care. Similarly for the $\mathbb{Z}^2$ example.

Warning: In general, it is impossible to determine what group is given by a presentation...

The Seifert-van Kampen theorem is a generalisation of this argument where we don't just glue circles. On the algebraic side, we need the notion of *amalgamated free products*. The input data is three groups A, B, C and two homomorphisms $\iota_A : C \to A, \iota_B : C \to B$. The output is a group $A *_C B$ and maps $f_A : A \to A *_C B, f_B : B \to A *_C B$.

We will give three different descriptions, all of which are useful:

(1) *Universal Property* Given $g_A : A \to Q, g_B : B \to Q$ so that $g_A \iota_A = g_B \iota_B$ there is a unique $g : A *_C B \to Q$ with $g f_A = g_A, g f_B = g_B$.

(2) *Presentation*

$$A *_C B = \langle A \cup B | a \cdot a' \cdot (aa')^{-1}, b \cdot b' \cdot (bb')^{-1}, \iota_A(c)\iota_B(c^{-1}) \rangle$$

(3) *Words* $A *_C B$ consists of formal words in A, B where $\iota_A(c) \in A$ is equal to $\iota_B(c) \in B$.

The third has the same "issues" as with the free group. To see that the second has the property of the first is not hard. The standard trick shows uniqueness.

With this in hand, we can now state the Seifert-van Kampen theorem.

Theorem 0.76. *Suppose X is reasonable, $X = U_1 \cup U_2$, and all three of $U_1, U_2, U_1 \cap U_2$ are reasonable. Then for any $x \in U_1 \cap U_2$,*

$$\pi_1(X, x) = \pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x)$$

for the maps induced by inclusion.

The proof is actually very simple to the proof from last time, and relies on the following:

Corollary 0.77. *For a reasonable space X there is a 1–1 correspondence between homomorphisms $\pi_1(X, x_0) \to Q$ and based regular covers $(Y, y) \to (X, x_0)$ with a free transitive Q–action (up to equivariant covering isomorphism).*

We also recall the following observation

Corollary 0.78. *Suppose that $p : (Y, y) \to (X, x_0)$ corresponds to a map $\varphi : \pi_1(X, x_0) \to G$. Suppose that $U \subset X$ is an open subspace with $x_0 \in U$. Then the cover*

$$p|_{p^{-1}(U)} : p^{-1}(U) \to U$$

corresponds to $\varphi \circ \iota$, where $\iota : \pi_1(U, x_0) \to \pi_1(X, x_0)$ is induced by inclusion.

Proof. The correspondence from cover to group homomorphism comes from considering the fiber as a G–set, i.e. by path-lifting. If γ is a path in U, then it acts exactly as $\iota([\gamma])$ on the fiber of p, showing the claim. $\square$

Proof. We will check the universal property. So, suppose we are given two maps $\varphi_i : \pi_1(U_i, x) \to G$ for some group G as in the universal property. Using the classification, associated are covers $p_i : Y_i \to U_i$.
The restrictions $p_i|_{p_i^{-1}(U_1 \cap U_2)} : Z_i = p_i^{-1}(U_1 \cap U_2) \to U_1 \cap U_2$ correspond, via the classification, to the maps $\varphi_i \iota_i$ (where $\iota_i : \pi_1(U_1 \cap U_2, x) \to \pi_1(U_i, x)$) are induced by the inclusion map). By the setup of the universal property, these are equivariantly homeomorphic. This means that we have a homeomorphism

$$F : Z_1 \to Z_2$$

which commutes with the G–action on both sides.
Hence, we can form a cover $Y \to X$ in the following way:

$$Y = Y_1 \coprod Y_2 / \sim \quad z \sim F(z) \forall z \in Z_1$$

Since the gluing is done by a G–equivariant covering morphism, the map $p_1 \coprod p_2$ induces a map $p : Y \to X$, and the G–action extends. It is clearly a covering map, and G acts transitively on fibers (since it did so for the Y_i).
Using the correspondence, we get a morphism $\varphi : \pi_1(X) \to G$. It restricts to the correct maps by the corollary before the proof.
It remains to show uniqueness. To this end, we just read the construction backwards. If φ' is any other map, we obtain a cover $X' \to X$. Restricting to $U_1, U_2, U_1 \cap U_2$ we then see covers which are equivariantly isomorphic to $Y_1, Y_2, Z_1 = Z_2$. Hence, X' is isomorphic to X. □

Lecture 16 (December 10)

The next big topic is *homology.* This will allow us to probe different information than is captured by the fundamental group – it is not better or finer or coarser, but different.
A main difference that will be clear immediately is that homology is much harder to define, and seemingly harder (impossible) to compute, making it a dubious tool. However, the definition just needs to be digested, and we will later develop very powerful tools that allow very quick computation, making homology (in the long run) the most accessible and computable invariant we have.
The basic roadmap is similar to all other invariants we have seen before: given a space, we will associate algebraic data in a functorial way. Here, we will split the algebraic step into two parts. From the space, we will first build a so-called *chain complex*, and out of that chain complex we will then compute *homology.*
We begin with the details of the second step.

Definition 0.79. A *chain complex* consists of Abelian groups $C_n, n \geq 0$ and homomorphisms $\partial_n : C_n \to C_{n-1}, n > 0$ so that $\partial^2 = 0$. The ∂ are called *boundary maps.*

Chain complexes can be defined for other things apart from groups. E.g. if the C_n are k–vector spaces over some field k, and ∂ are vector space homomorphisms, then we say that C_n is a chain complex of k–vector spaces etc.
Here is an example, which in a sense is the only crucial example you ever really need to understand.
First, we consider the *standard n–simplex*:

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}, \sum x_i = 1, x_i \geq 0\}.$$

More generally, for any $v_0, v_1, \ldots, v_n$ with $v_i - v_0$ linearly independent, we also call the set

$$\{\sum x_i v_i \in \mathbb{R}^{n+1}, \sum x_i = 1, x_i \geq 0\}$$

a n–simplex (together with the numbering of the v_i). There is a unique affine linear map from any simplex to a standard simplex (this is where we use the ordering) which maps the v_i to the standard basis vectors (respecting numbering). To have some notation, we denote by $[v_0, \ldots, v_n]$ the simplex defined by the v_i.
A (dimension $n-1$) *face* of the standard n–simplex is a set

$$F_k = \{(x_0, \ldots, x_n) \in \Delta^n, \quad x_k = 0.\}$$

Observe that in a natural way F_k is an $(n-1)$–simplex. Similarly, we define faces of general simplices, and also faces of smaller dimension (i.e. faces of faces). Again, to have notation, we denote by $[v_0, \ldots, \hat{v_i}, \ldots, v_n]$ the face defined by omitting v_i (and taking the induced numbering).
Now we are ready for our definition. Namely, we let $C_k(\Delta^n)$ be the free Abelian group with basis the k–dimensional faces of Δ^n.
Draw some pictures here!
Now, the boundary operator will send a face to its boundary. We just have to be careful with the orientations. Again, pictures will help.
We define:

$$\partial([w_0, \ldots, w_k]) = \sum (-1)^i [w_0, \ldots, \widehat{w_i}, \ldots, w_k]$$

Lemma 0.80. *This ∂ is a boundary operator, i.e. $\partial^2 = 0$.*

Proof. Hatcher, Lemma 2.1 □

Back to abstract algebra. We are now given any chain complex $(C., \partial.)$, and we define

$$H_i(C) = \ker \partial_n / \mathrm{im} \partial_{n+1}.$$

Some names: we call elements of $\ker \partial_n$ *n–cycles* and elements of $\mathrm{im}\partial_n$ *n–boundaries*. The elements of H_i are *homology classes*, cycles defining the same homology class are called *homologous*.
Again, if the C_n had extra structure, and that structure passes to quotients, then homology inherits it. For example, if the C_n was a chain complex of k–vector spaces, then the H_n are k–vector spaces.

Why might one expect that this has anything to do with anything? Here are two hopefully guiding examples, although we don't formally understand what's really going on there yet.

(1) Picture: The circle as a "1–simplex with endpoints glued". Define
$$C_1 = \mathbb{Z}, C_0 = \mathbb{Z}$$
and all $\partial = 0$. We get nothing over degree 1 (dimension?) and $\mathbb{Z}$ both in degree 1 and 0.

(2) Picture: The sphere as "two 2–simplices with boundary glued". Define
$$C_2 = \mathbb{Z} \oplus \mathbb{Z}, C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, C_0 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$
where the ∂ are defined as in the initial simplex example. Then, $H_2 = \mathbb{Z}, H_1 = 0, H_0 = \mathbb{Z}$.

(3) Picture: The torus with "two 2–simplices making a square, then glued".

Why guiding? Because, out of chain complexes looking very similar to spaces we understand, we extract homology groups that seem to be saying something. Why not understood? How do we actually get the complex out of the space?

Lecture 17 (December 11)

For the latter question there are two approaches. One (seemingly nice) is to make precise what happens in the examples. That leads to the definition of a *simplicial complex* (a space glued in a very nice way from simplices), and *simplicial homology*. While this definition is pretty easy to understand, and really nice to compute, it is sort of a nightmare to work with from the formal side – e.g. why should homeomorphic spaces have the same homology?
We go down a different route: we define a (seemingly horrible...) chain complex out of a space, which gives the formal properties much easier. Later we will care about how to compute.

Given a space X, we call a continuous map $\sigma : \Delta^n \to X$ a *singular n–simplex*. Now, let $C_n(X)$ be the *free* Abelian group with basis *all* of the singular n–simplices. (Observe: this is really big in general, uncountably dimensional). More explicitly, an element of $C_n(X)$ is a formal sum
$$\sum_i n_i \sigma_i$$
where the $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ are continuous maps. How to define a boundary map? Write $\Delta^n = [v_0, \ldots, v_n]$ as above (the v_i are the standard basis vectors). Observe that there is a canonical ordering-preserving affine linear map A from $[v_0, \ldots, \hat{v}_i, v_n]$ with Δ^{n-1}. We define
$$\sigma_i | [v_0, \ldots, \hat{v}_i, v_n] = \sigma_i \circ A^{-1} : \Delta^{n-1} \to X$$

and put

$$\partial(\sigma_i) = \sum_i (-1)^i \sigma_i|[v_0, \ldots, \hat{v_i}, v_n],$$

extending linearly.

Lemma 0.81. *∂ is a boundary map.*

We have already proved that (it is verbatim the same computation as above). Hence, we can define the *singular homology groups*

$$H_n(X) = \ker \partial_n / \mathrm{im}\partial_{n+1}.$$

With this definition, it is clear that H_n is a homeomorphism invariant. Let's be a little bit more precise. First, observe

Lemma 0.82. *Let $f : X \to Y$ be a continuous map. Then there is an induced homomorphism*

$$f_\sharp : C_n(X) \to C_n(Y)$$

determined by $f_\sharp(\sigma) = f \circ \sigma$, which descends to a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

These constructions respect the identity and composition (functoriality), and in particular if f is a homeomorphism, then $f_\sharp$ and f_* are isomorphisms.

Proof. The existence of the map $f_\sharp$. is obvious (on a free Abelian group we can freely prescribe images on basis vectors). To show that it descends, we observe that

$$\partial^Y(f_\sharp \sigma) = f_\sharp \partial^X \sigma,$$

and note that this implies $f_\sharp(\ker \partial_n^X) \subset \ker \partial_n^Y$, and $f_\sharp(\mathrm{im}\partial_n^X) \subset \mathrm{im}\partial_n^Y$, showing the second claim.
Functoriality of $f_\sharp$ is obvious from the definition, and descends to functoriality of f_*. The last claim we have done a couple of times already. □

Let's try to prove some basic properties. First, let's try to compute at least one homology.

Lemma 0.83. *The homology of the one-point space has $H_0 = \mathbb{Z}, H_i = 0, i > 0$.*

Proof. We have exactly one singular n–simplex σ_n in each degree. Hence, all $C_n = \mathbb{Z}$. The boundary maps are trivial for n odd (odd-dimensional simplices have an even number of faces), and isomorphisms for n even. □

Also, note that if $X = \bigcup X_i$ is the decomposition into path-connected components, then

$$C_n(X) = \bigoplus C_n(X_i)$$

as the image of any simplex is path-connected. Further, the boundary respects this decomposition, showing

Lemma 0.84. *If $X = \bigcup_i X_i$ is the decomposition into path-connected components, then*

$$H_n(X) = \bigoplus H_n(X_i).$$

Finally, we can always compute the zeroth homology group:

Lemma 0.85. *If X is path-connected and nonempty, then $H_0(X) = \mathbb{Z}$.*

Proof. Hatcher Proposition 2.7. □

Our first real goal is the following:

Theorem 0.86. *The map $f_* : H_n(X) \to H_n(Y)$ depends only on the homotopy class of $f : X \to Y$.*

In order to do this, we will show that a homotopy F between f and g induces a certain kind of map between the chain complexes $C_\bullet(X), C_\bullet(Y)$.
The kind of map we will extract is a map $P : C_n(X) \to C_{n+1}$ (sometimes called a *chain homotopy*) satisfying

$$\partial P = g_\sharp - f_\sharp - P\partial$$

which in particular shows that $g_* = f_*$.

Lecture 18 (December 17)

So, suppose that a homotopy $F : X \times [0,1] \to Y$ between f, g is given, and suppose that $\sigma : \Delta^n \to X$ is a singular simplex. Relating $f_\sharp\sigma, g_\sharp\sigma$ is the map $\Delta^n \times [0,1] \to Y$ induced by the homotopy.
To connect this to singular homology, we first need to decompose the "prism" $\Delta^n \times [0,1]$ into simplices. To do this, embed this space (in the obvious way) into $\mathbb{R}^{n+2}$, and let v_i, w_i be defined so that $\Delta^n \times \{0\} = [v_0, \ldots, v_n], \Delta^n \times \{1\} = [w_0, \ldots, w_n]$ and the labels are so that v_i, w_i differ only in the last coordinate.
Now, consider the simplex $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$. This is in fact the graph of the function $\varphi_i(t_0, \ldots, t_n) = t_{i+1} + \ldots + t_n$, defined on the simplex $[v_0, \ldots, v_n]$. Namely, the last coordinate of a point in the simplex $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ is exactly the sum of the coefficients of the $w_{i+1}, \ldots, w_n$; whereas the first coordinates are the same as when one replaces the w's with v's. As i increases, the φ_i increase, and so the region bounded by the graphs of φ_i, φ_{i+1} is in fact a $(n+1)$–simplex.
Together, this shows that $\Delta^n \times [0,1]$ is the union of the $(n+1)$–simplices $[v_0, \ldots, v_i, w_i, \ldots, w_n], i = 0, \ldots, n+1$. Now, we define

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathrm{id})|[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

and we claim that this is a chain homotopy. This computation is in Hatcher, bottom of page 112, and it shows the theorem about homotopy invariance.

From now on, we follow Hatcher very closely. Our next big goal is Theorem 2.13, which allows to compute homology of a space by collapsing a subspace. Before we enter the proof (which will take us quite some time...) we want to highlight some applications. Namely, Corollary 2.14 (homology groups of spheres) and Corollary 2.15 (Brouwer fixed point theorem)

Lecture 19 (December 18)

We now start our journey towards Theorem 2.13 from Hatcher. The first step is the long exact homology sequence for a pair. This is discussed as Theorem 2.16 in Hatcher.

Lecture 20 (January 7)

Next is a lengthy chain of results leading towards the "theorem of small simplices".

- **Barycentric subdivision** Given a simplex $[v_0, \dots, v_n]$ we define the *barycenter* as

$$b = \frac{1}{n+1} \sum v_i$$

The barycentric subdivision of a 1–simplex $[v_0, v_1]$ consists of the two simplices $[v_0, b], [b, v_1]$. Their union is indeed the original simplex, and they intersect (if at all) in faces.

Now, suppose we know how to barycentrically subdivide $(n-1)$–simplices, and we are given a n–simplex $[v_0, \dots, v_n$. Then, the barycentric subdivision consists of all simplices of the form $[b, w_1, \dots, w_n]$ where $[w_1, \dots, w_n]$ is a simplex of a barycentric subdivision of any face. Draw some pictures.

Lemma 0.87. *With respect to the Euclidean metric, and for any simplex Δ' in the barycentric subdivision of Δ, we have*

$$\operatorname{diam}(\Delta') \leq \frac{n}{n+1} \operatorname{diam}(\Delta).$$

Proof. Observe that the diameter of a simplex is always realised by the distance between vertices:

$$\|v - \sum t_i v_i\| = \| \sum t_i (v - v_i) \| \leq \max \|v - v_i\|.$$

We prove the result by induction on dimension. For $n = 1$ this is clear. This also means that the distance between vertices on a face of Δ' not involving the barycenter of Δ is controlled (by induction).

Thus, the only case that is left is estimating the distance of the barycenter b of Δ to one of the vertices v_i of Δ. But then we have

$$\|b - v_j\| = \frac{1}{n+1} \| \sum_i v_i - v_j \| \leq \frac{n}{n+1} \operatorname{diam}(\Delta) \|$$

since one of the terms in the sum is 0, and so only n remain. $\square$

In other words, repeated barycentric subdivision makes the simplex arbitrarily small, independent of shape.

- Now, suppose that $Y \subset \mathbb{R}^k$ is a convex set. We define a chain complex $\mathrm{LC}_n(Y)$ formed by the *linear chains*, i.e. maps $\Delta^n \to Y$ which are (affine) linear. This is a subcomplex of $C_n(Y)$, since it is compatible with the boundary operator. Also recall that such a map is completely determined by the images $w_0, \ldots, w_n$ of the vertices, so we can denote (basis) elemenents of LC_n by $[w_0, \ldots, w_n]$.

  Given any point $b \in Y$, we can define a (coning) homomorphism

$$c_b : \mathrm{LC}_n(Y) \to \mathrm{LC}_{n+1}(Y)$$

defined by

$$c_b([w_0, \ldots, w_n]) = [b, w_0, \ldots, w_n].$$

Lemma 0.88. *We have*

$$\partial c_b = \mathrm{id} - c_b \partial$$

Proof. It suffices to check this on basis elements. There, it is immediate form the usual formula for boundary (and c_b). □

In other words, c_b is a chain homotopy between the identity and the zero map on the augmented chain complex $\mathrm{LC}_* \to \mathbb{Z} \to 0$ (this shouldn't come as a shock – a convex set Y is contractible, so one might expect that reduced homology associated to it is trivial)

Lecture 21 (January 8)

- Next, we define a subdivision operator

$$S : \mathrm{LC}_n(Y) \to \mathrm{LC}_n(Y)$$

inductively. We start with $S([\emptyset]) = [\emptyset]$ in LC_{-1}, and then define

$$S([w_0, \ldots, w_n]) := c_{b[w_0,\ldots,w_n]}(S\partial[w_0, \ldots, w_n])$$

where $b[w_0, \ldots, w_n]$ is the barycenter. Observe that the result is a sum of simplices in the barycentric subdivision of $[w_0, \ldots, w_n]$ (with some signs, which we don't care about right now) supposing that it is indeed a simplex.

Lemma 0.89. *We have*

$$\partial S = S\partial.$$

In other words: S is a chain map.

Proof. In degree -1 and 0 this is clear, since $S = \mathrm{id}$ there.

Otherwise, we compute

$$\partial S\lambda = \partial c_b S\partial\lambda = S\partial\lambda - c_b\partial S\partial\lambda$$

where we have used the inductive definition of S and the previous lemma. Now, we can use induction to find

$$c_b\partial S\partial\lambda = c_b\partial\partial S = 0$$

which gives the claim. □

- Next, we build a chain homotopy

$$T : \mathrm{LC}_n(Y) \to \mathrm{LC}_{n+1}(Y)$$

between S and the identity. Again, this is done inductively. We start with $L_{-1} = 0$ and then

$$T\lambda := c_b(\lambda - T\partial\lambda)$$

Lemma 0.90. *We have*

$$\partial T + T\partial = \mathrm{id} - S$$

Proof. This is obvious on LC_{-1} ($T = 0, S = \mathrm{id}$). Otherwise, we compute

$$\partial T = \partial c_b(\lambda - T\partial\lambda) = \lambda - T\partial\lambda - c_b(\partial\lambda - \partial T\partial\lambda)$$

(using the lemma above again) and then by induction (and $\partial^2 = 0$)

$$c_b(\partial\lambda - \partial T\partial\lambda) = c_b(\partial\lambda - \partial\lambda + S\partial\lambda) = S\lambda$$

by the inductive definition of S. □

Also observe that this lemma also holds without the (auxiliary) LC_{-1}.

- Finally, we are ready to define the objects we actually care about. Given any space X, we define a subdivision operator $S : C_n(X) \to C_n(X)$ by

$$S\sigma = \sigma_\sharp S\Delta^n$$

Lemma 0.91. *S is a chain map.*

Proof.

$$\partial S\sigma = \sigma_\sharp \partial S\Delta^n = \sigma_\sharp S\partial\Delta^n$$

where, with abuse of notation as above, we denote by Δ^n the obvious n-simplex with image in $\mathbb{R}^{n+1}$. Denoting the faces of Δ^n by Δ_i (suitably numbered), we then have

$$\sigma_\sharp S\partial\Delta^n = \sigma_\sharp S\sum_i(-1)^i\Delta_i = \sum(-1)^i S(\sigma|\Delta_i) = S\partial\sigma$$

□

We also define $T: C_n(X) \to C_{n+1}(X)$ by

$$T\sigma = \sigma_\sharp T\Delta^n$$

and a completely analogous computation shows

$$\partial T + T\partial = \mathrm{id} - S$$

- **Iterating barycentric subdivision**. Define

$$D_m = \sum_{i=0}^{m-1} TS^i.$$

Lemma 0.92. *D_m is a chain homotopy between S^m and id.*

Proof.

$$\partial D_m + D_m\partial = \sum \partial T S^i + T S^i \partial = \sum \partial T S^i + T\partial S^i = \sum (\partial T + T\partial) S^i$$

by definition and the fact that S is a chain map. Now, using that T is a chain homotopy, and telescope sum, we are done. □

- **Connecting to covers** Suppose now that $\mathcal{U} = \{U_i\}$ is a cover of X by open sets, and $\sigma : \Delta^n \to X$ is a singular simplex.

Lemma 0.93. *There is a number $\epsilon > 0$ so that if $M \subset \Delta^n$ is any set of diameter $< \epsilon$, then there is an index i so that*

$$M \subset \sigma^{-1}(U_i)$$

Proof. This follows by compactness of the simplex, and the fact that if there is a δ–ball around y contained in U_i, then a $(\delta - d(x, y))$–ball around x is contained in U_i. □

Using the contraction lemma from above, this implies that given σ there is a smallest number $m(\sigma)$ so that $S^{m(\sigma)}\sigma \in C_n^{\mathcal{U}}(X)$.

- We define $D\sigma = D_{m(\sigma)}\sigma$. What property does this have? We begin by noting

$$\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma = \sigma - S^{m(\sigma)}\sigma$$

This is not quite saying that D is a chain homotopy, since $D_{m(\sigma)}\partial\sigma$ is not the same as $D\partial\sigma$. We therefore simply add the missing term on both sides, and get

$$\partial D\sigma + D\partial\sigma = \sigma - (S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D(\partial\sigma)).$$

Let's denote the term in the bracket by $\rho(\sigma)$, and observe that it is a chain map, by the equality we have shown. This shows:

Lemma 0.94. *D is a chain homotopy between* id *and ρ.*

Lemma 0.95. *ρ has image in $C_n^{\mathcal{U}}(X)$, and so it defines a chain homotopy inverse for the inclusion $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$.*

Proof. The first claim is clear by definition for the $S^{m(\sigma)}$ contribution. To analyse the other, observe that if σ_i is a face of σ, then $m(\sigma_i) \leq m(\sigma)$. Thus, $D_{m(\sigma)}\partial\sigma - D(\partial\sigma)$ is a sum of simplices of the form $TS^k\sigma_i$ with $k \geq m(\sigma_i)$ and all these are small (as T preserves smallness).

Thus, we can interpret the equation from above as

$$\partial D + D\partial = \mathrm{id} - \iota\rho$$

Finally, the last claim follows by noting that $D = 0$ on $C_n^{\mathcal{U}}(X)$ (since there $m(\sigma) = 0$), and thus $\rho\iota = \mathrm{id}$. □

Lecture 22 (January 14)

Now, we are finally ready for the proof of excision:

Proof of the excision theorem. Let A, B be the cover from the theorem: $X = A \cup B$. We can then identify $C^{\mathcal{U}}(X)$ with the chains which are sums of chains in A and chains in B (not a direct sum!). Take D, ρ from the small simplices theorem. That means:

$$\partial D + D\partial = \mathrm{id} - \iota\rho, \quad \rho\iota = \mathrm{id}.$$

Consider the inclusion

$$\iota_1 : C^{\mathcal{U}}(X)/C(A) \to C(X)/C(A)$$

induced by ι. Recall that both D and ρ were defined by a subdivision process. In particular, they preserve chains with image in A, and therefore define maps on the quotients by $C(A)$, satisfying the same formulas. Thus, ι_1 induces an isomorphism in homology.
On the other hand, both groups

$$C(B)/C(A \cap B), \qquad C^{\mathcal{U}}(X)/C(A)$$

are free Abelian with basis the simplices in B which do not lie in A. Thus, the inclusion

$$\iota_2 : C(B)/C(A \cap B) \to C^{\mathcal{U}}(X)/C(A)$$

is actually an isomorphism. Together they show the theorem. □

Now, we quickly recall two results you show on the problem sets:

(1) The triple sequence in homology: if $A \subset B \subset X$, then the inclusions give an exact sequence:

$$\ldots \to H_n(B, A) \to H_n(X, A) \to H_n(X, B) \to H_{n-1}(B, A) \to \ldots$$

(2) If (V, A) is a pair in X so that V deformation retracts to A, then $H_n(V, A) = 0$ and $H_n(X, A) \to H_n(X, V)$ is an isomorphism.

Now, we are ready to prove the exact sequence for good pairs. This just involves one diagram now, see Prop. 2.22 in Hatcher.

Next, we prove another useful tool in computing homology: the Mayer-Vietoris sequence.

Theorem 0.96 (Mayer-Vietoris)**.** *Suppose that $X = A \cup B$ the union of two open sets. Then there is a long exact sequence*

$$\ldots H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

To prove this, consider the short exact sequence of chain complexes:

$$0 \to C(A \cap B) \to C(A) \oplus C(B) \to C^{\mathcal{U}}(X) \to 0$$

where $\mathcal{U} = \{A, B\}$, the first map is $\sigma \mapsto (\sigma, -\sigma)$ and the second map is $(\sigma, \tau) \mapsto \sigma - \tau$.
What is the connecting map? Suppose $x \in H_n(X)$ is given. By small simplices, we can write it (nonuniquely) as $x = a + b$ for chains supported

in A, B. Since x is a homology class, we have $\partial a = -\partial b$; in particular it is an element of $C(A \cap B)$. Thus, we can set $\delta(x) = [\partial a]$.

Lecture 23 (January 15)

We now perform two computations of homology using the sequences we have developed.

The first concerns the 2–torus $T = S^1 \times S^1$, and we will use the Mayer-Vietoris sequence. To this end, we interpret T as the quotient $T = [0,1]^2/\sim$ as the space obtained from a square by identifying opposite sites.

Now, let B be a small disk in Q, seen as a subset of T, and let A be the complement of a smaller disk. We then have:

- B is homotopy equivalent to a point.
- A is homotopy equivalent to $S^1 \vee S^1$ – indeed you have shown on the homework this is true for the torus with a point removed; this is proved in exactly the same way.
- $A \cap B$ is homeomorphic to a annular domain (a ring-shaped region), hence homotopy equivalent to S^1.

Now, consider a part of Mayer-Vietoris:

$$H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(T) \to H_{k-1}(A \cap B)$$

If $k > 2$ then both outer terms are zero, hence $H_k(T) \simeq H_k(A) \oplus H_k(B) = 0 \oplus 0$.

If $k = 2$, then the left term is zero, but the right is not. We thus consider how the sequence continues:

$$H_2(A) \oplus H_2(B) \to H_2(T) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B).$$

We want to understand the rightmost map. To this end, note that it is induced by inclusion. Hence, consider a loop δ which represents a generator of $H_1(A \cap B)$. Then, arguing as above, it is homotopic (in A) to the loop which is the image of the boundary of the square in T. However, this is zero in homology (not π_1), and therefore the rightmost map of the sequence above is zero. As a consequence, we have that

$$H_2(T) \to H_1(A \cap B)$$

is an isomorphism.

For $k = 1$ we have

$$H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(T) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B)$$

We already know that the leftmost map is zero. The rightmost map is injective: a generator of $H_0(A \cap B)$ is given by any point in $A \cap B$, which maps to generators of $H_0(A), H_0(B)$. Thus, we again have that

$$H_1(A) \oplus H_1(B) \to H_1(T)$$

is an isomorphism.

Next, we want to do the same with the three-torus $T = S^1 \times S^1 \times S^1$, or the quotient of the cube $[0,1]^3$ by identifying opposite faces. Similarly to above, we let B be a small cube in $[0,1]^3$, and A the complement of a smaller cube. We then have

- B is homotopy equivalent to a point.
- A is homotopy equivalent to the space X obtained from $\partial[0,1]^3$ by identifying opposite faces.
- $A \cap B$ is homotopy equivalent to a 2–sphere S^2.

As before, we have the Mayer-Vietoris sequence:

$$H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(T) \to H_{k-1}(A \cap B)$$

If $k \neq 3, 2, 1$, then both outer terms are zero. As a consequence $H_k(T) = 0, k > 3$. For the others, we need to understand the inclusion maps:

$$H_2(A \cap B) \to H_2(A) \oplus H_2(B)$$

For this, the generator of $H_2(A \cap B)$ is represented by the sum of all faces of the small cube, which in B is homotopic to the quotient of the boundary. Similar to above, this is zero in $H_2(B)$ as opposite sides cancel. Hence, this map is zero, showing that $H_2(B) \to H_2(T)$ is an isomorphism.

$$H_0(A \cap B) \to H_0(A) \oplus H_0(B)$$

is again an isomorphism, exactly as above, and thus $H_1(B) \to H_1(T)$ is also an isomorphism.

Lecture 24 (January 21)

To finish the computation of the homology of the 3–torus, we need to compute the homology of the space X above. X can be obtained by gluing three 2–tori T_1, T_2, T_3 (the images of the individual faces of the cube), glued along curves. To compute the homology, we perform a similar trick as for the 2–torus as above: let B be the disjoint union of three small disks, one in each T_i, and let A be the complement of a smaller disks. We then have:

- B is homotopy equivalent to the disjoint union of three points.
- A is homotopy equivalent to $S^1 \vee S^1 \vee S^1$ – indeed each single T_i minus the part of B in it is homotopy equivalent to $S^1 \vee S^1$, and the gluing identifies the S^1's pairwise.
- $A \cap B$ is homeomorphic to a disjoint union of three annular domains, hence homotopy equivalent to $S^1 \coprod S^1 \coprod S^1$.

Now, consider a part of Mayer-Vietoris:

$$H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(T) \to H_{k-1}(A \cap B)$$

We are interested in the sequence for $k = 1, 2$. For $k = 2$ we have

$$0 = H_2(A) \oplus H_2(B) \to H_2(T) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B)$$

The rightmost map is zero again: generators of $H_1(A\cap B)$ map to zero homology classes in the T_i (as in the 2–torus). Thus, $H_2(T)\to H_1(A\cap B)=\mathbb{Z}^3]$ is an isomorphism.
For $k=1$ we have

$$H_1(A\cap B)\to H_1(A)\oplus H_1(B)\to H_1(T)\to H_0(A\cap B)\to H_0(A)\oplus H_0(B)$$

The left map is zero as we just saw, and the right map is injective, as $H_0(A\cap B)\to H_0(B)$ is injective. Thus, $\mathbb{Z}^3=H_1(B)\to H_1(T)$ is an isomorphism.

We now need to briefly discuss a very useful class of topological spaces.

- Recall: if X is any space, and D^n is the n–ball (from now on often called: *n–cell*). Suppose we also have a continuous map $f:\partial D^n\to X$ (called: *attaching map*). We can then form the space
  $$X'=X\cup_f D^n$$
  and say that it is obtained from X by *attaching an n–cell*.
- Recall: the homotopy type of X' depends only on the homotopy type of f.
- Examples of attaching: attaching D^n to a single-point space yields the sphere S^n. Attaching D^1 to S^1 has two possibilities, depending on the attaching map.
- More generally, suppose that J is any index set, and
  $$f:J\times\partial D^n\to X$$
  is a continous map (where J has the discrete topology). Then we say that
  $$X'=X\cup_f J\times D^n$$
  is obtained from X by *attaching n–cells*. J is explicitly allowed to be infinite.
-

  Lemma 0.97. *Denote by $p:X\coprod J\times D^n\to X'$ the quotient map.*
  - *The restricted map $p|_X:X\to p(X)$ is a homeomorphism; in particular X is closed in X'.*
  - *The restricted map $p|_{J\times(D^n)^\circ}:J\times(D^n)^\circ\to p(J\times(D^n)^\circ)$ is a homeomorphism; in particular $J\times(D^n)^\circ$ is open in X'.*
  - *If, for each j, there is an open set $V_j\subset D^n$ containing ∂D^n, then $X\cup\bigcup_j V_j$ is open.*
  - *There is an open neighbourhood V of X in X' which deformation retracts to X.*

  Proof. – p is continuous (restriction of continous), and bijective (by definition of the equivalence relation). We want to show that p is also closed. So: suppose that $A\subset X$ is closed. We have
  $$p^{-1}(p(A))=X\cup f^{-1}(A)$$

which is closed (disjoint union topology and continuity of f)

- Again, the restriction of p is actually a homeo onto the image. We show that it is open. Take $O \subset J \times (D^n)^\circ$ open. Then

$$p^{-1}(p(O)) = O$$

and that is clearly open.
- The preimage under p is open.
- Put V to be the image of $X \cup J \times D^n - \{0\}$. This is open by the previous. The retraction is given by radial retraction on each cell.

□

- If X is Hausdorff, then so is any space X' obtained by attaching cells.

  Namely: If x, y are both in the interior of a cell, we can clearly seperate them inside that cell. If $x \in X$ and y is in the interior of a cell, then we can take a small neighbourhood of ∂D^n not containing y, together with X as one of the sets, and a small ball around y as the other. For the last case, we can use seperating sets O_x, O_y and then take the preimage under the retraction r from above.
- If X is compact, then so is any space obtained by attaching only finitely many cells.
- If X' is obtained from X by attaching cells, and $K \subset X'$ is compact, then K intersects only finitely many of the new cells. You'll discuss this on the problem set.

We now come to the core definition:

Definition 0.98. Let A be any topological space (possibly empty). A *CW complex relative to* A consists of a space X together with a sequence of subspaces

$$A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X,$$

with the following:

(1) For any $n \geq 0$ the space X_n is obtained from X_{n-1} by attaching n–cells.
(2) We have $X = \cup_n X_n$, and a set $O \subset X$ is open if and only if all intersections $O \cap X_n$ are open.

If $A = \emptyset$, we simply say that X is a CW complex. The choice of the X_i is called a CW-structure. The X_i are called skeleta, and X is finite-dimensional, if $X = X_n$ for some n. It is called finite, if it is finite-dimensional, and in every stage only finitely many cells are attached.

Lecture 25 (January 22)

Some remarks on CW complexes

- The condition on the topology is equivalent to: a set-map $f : X \to Y$ is continuous if and only if all the restrictions $f : X_n \to Y$ are continuous.
- The condition on the topology is nontrivial (if the dimension is infinite).
- The complements $X_n - X_{n-1}$ are disjoint unions of open balls, also called the n–cells of the complex. Observe also that the inclusion of any of these balls extends to the boundary of the ball. However, these characteristic maps (or attaching maps) are not part of the data of a CW structure.

Let's discuss some examples:

(1) The n-sphere has various CW structures. There is one with one 0–cell and one n–cell, or one could build an inductive one by always adding two n–cells to the structure of the $n-1$ sphere.
(2) From pictures: the cube or tetrahedra, the torus, ...

We record from before:

Lemma 0.99. *Suppose that (X, A) is a relative CW-complex. If A is Hausdorff, then so is X. If A is compact, and X is finite, then X is compact.*

Proof. There is one small thing to check. Given two points $x, y \in X_n$, we know that there are separating sets U, U' in X_n. We need to promote these to open separating sets in X. To do so, observe that there were retractions $r_n : V_{n+1} \to X_n$ for open $V_{n+1} \subset X_{n+1}$. Now, inductively set $U_{i+1} = r_{i+1}^{-1}(U_i)$ and similarly for U_i'. These stay open and disjoint, and by the defining property of CW complexes they are also open. □

Lemma 0.100. *Suppose that (X, A) is a Hausdorff relative CW complex.*

(1) The closure of any cell is compact.
(2) A set $U \supset A$ is closed if and only if the intersection of A with the closure of any cell is closed.

Proof. Take an open cell C, and an attaching map $f : D^n \to X$ so that $f(\text{int}D^n) = C$. Then, the image $f(D^n)$ is compact, hence closed. Since $f(D^n)$ is certainly contained in the closure of C it is therefore equal to the closure, showing the first point.

One direction of the second point is clear: intersections of closed sets are closed. For the other direction, we need to show that $U \cap X_n$ is closed for all n. For $n = -1$ this is clear. Now, consider the quotient map

$$p_n : X_{n-1} \coprod J_{n-1} \times D^n \to X_n,$$

and the preimage $p_n^{-1}(U)$. The part in X_{n-1} is exactly the intersection $X_{n-1} \cap U$, so we can assume by induction that it is closed. By assumption, the intersection $U \cap p(j \times D^n)$ is closed, and thus the same is true for the part in $J_{n-1} \times D^n$. This implies that $p_n^{-1}(U)$ is closed, which by the definition of the quotient topology proves what we want. □

Definition 0.101. We call a subspace $Y \subset X$ of a CW complex (X, A) a subcomplex, if there is a CW structure on Y so that $Y_n = Y \cap X_n$ and furthermore $Y_n - Y_{n-1}$ is a union of open n-cells of X.

One can show that subcomplexes are always closed subsets, but we don't need this (yet). You will show in the homework

Lemma 0.102. *Suppose that (X, A) is a CW-complex. Then the closure of any cell is contained in a finite subcomplex. More generally, any compact set $K \subset X$ is contained in a finite subcomplex.*

Whitehead (who invented CW complexes) states that the C means "closure finite" (in the sense of that lemma), and the W means "weak topology" (i.e. that sets are open iff their intersections with the skeleta are open). Then again, his full name was JH*C W*hitehead, so that may or may not be completely accurate reasoning.

CW complexes, homologically. Next, we want to study the homology of CW complexes.

Lemma 0.103. *Let X be a CW complex, and X^i its skeleta. We then have*

(1) $H_k(X^n, X^{n-1})$ is zero unless $k = n$, and is the free Abelian group with basis the set of n–cells for $k = n$.
(2) $H_k(X^n) = 0$ if $k > n$.
(3) The inclusion $i : X^n \to X$ induces isomorphisms $H_k(X^n) \to H_k(X)$ for all $k < n$.

Proof. (1) Observe that (X^n, X^{n-1}) is a good pair (previous lemma), and thus we have

$$H_k(X^n, X^{n-1}) = \widetilde{H}_k(X^n/X^{n-1}).$$

But X^n/X^{n-1} is a wedge of n–spheres, one for each n–cell, which implies the claim.

(2) Consider the long exact sequence for (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

If $k \neq n, n-1$ then both outer terms are zero. Thus, we have, for $k > n$:

$$H_k(X^{n-1}) \simeq H_k(X^n)$$

via inclusion. Inductively, this shows $H_k(X^n) \simeq H_k(X^0) = 0$.

(3) If X is finite-dimensional, then the argument from the previous point suffices. In the infinite-dimensional case a little bit more work is needed.

We will show surjectivity and injectivity individually. So, suppose that $[x] \in H_k(X)$ is given. We can realise $x = \sum n_i\sigma_i$ as a cycle. The union of all the images of the σ_i is a compact set, and so, by homework, is contained in a finite subcomplex. In particular, there is some m so that the image is contained in X^m. But then, we have that $[x]$ is in the image of the map $H_k(X^m) \to H_k(X)$ (since it is already in the image of the chain level map). Now, using the fact that $k > n$ we have that $H_k(X^m) = H_k(X^n)$ via inclusion, showing that $[x]$ is also in the image of the desired inclusion.

Injectivity is very similar. Suppose that x is some chain so that $[x] = 0$ in $H_k(X)$. This means that $x = \partial y$ for some other chain y. Again, y lives in some skeleton X^m, and so $[y] = 0$ in $H_k(X^m)$, and therefore the same is true in $H_k(X^n)$.

□

Lecture 26 (January 28)

Now, we construct a (fairly big) diagram by combining the pair sequences of (X_i, X_{i-1}) for all i. Compare Hatcher, before Theorem 2.35 for this. The result is the *cellular chain complex* $H_i(X^i, X^{i-1})$ with boundary maps d_i. From this diagram we conclude using diagram chasing:

Theorem 0.104. *The cellular chain complex computes singular homology.*

Proof. See Hatcher, Thm. 2.35. □

Why would this be useful? We've identified $H_i(X^i, X^{i-1})$ with something very concrete before: the free Abelian group with basis the i–cells of X. Our next long-term goal will be to describe the boundary maps d_i of the cellular chain complex under this identification, as this will give us a very good tool to compute homology.

This will require a quick discussion of *mapping degree*. Before beginning with the real discussion of mapping degree, we revisit generators for the homology of a sphere. In particular, we show

Lemma 0.105. *The identity map* id : $\Delta^n \to \Delta^n$*, seen as a singular n–simplex, generates the group $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. Hatcher, the first part of Example 2.23 □

Lemma 0.106. *Identifying the n–sphere S^n with the gluing of two n–simplices $\Delta^n_+ \cup \Delta^n_-$ along their common boundary, the group $H_n(S^n)$ is generated by $[\Delta^n_+ - \Delta^n_-]$.*

Proof. Hatcher, the second part of Example 2.23 □

In fact, this same proof also applies more generally. If we *triangulate* S^n, i.e. write it as a union of simplices, then the sum of all those simplices (with appropriate signs, so that the boundary is zero) will be another generator of $H_n(S^n)$.

Lecture 27 (January 29)

Now, we collect some basic facts on mapping degree. This is in Hatcher, Section 2.2. The basic properties (a)–(g) as well as the results 2.28, 2.29. are important.

To compute the mapping degree, one often uses a local constuction, which is described in Proposition 2.30 of Hatcher (and the preceding diagram).

Lecture 28 (February 4)

Examples of degree computations.

Lecture 29 (February 5)

The *cellular boundary formula* (Hatcher, page 140), and some example computations (points i)–iii) on page 140, Example 2.36, 2.39)