Prof. Dr. Sebastian Hensel Sommersemester 2019

Solution to Set 3, Problem 2

(a) Let
$$(p(t), v(t))$$
be a path in TM, where p is a path in M, and v is a vector field along p. Take a chart φ for M around $p(0)$, and write
$$p = \varphi \circ c.$$
We can write
$$v(t) = \sum_i v^i(t) \frac{\partial \varphi}{\partial x^i}_{c(t)},$$
and thus
$$\frac{\nabla}{dt} v(0) = \sum_i \dot{v}^i(0) \frac{\partial \varphi}{\partial x^i}_{c(t)} + v(0) \frac{\nabla}{dt} \frac{\partial \varphi}{\partial x^i}_{c(t)}.$$
Further,
$$\frac{\nabla}{dt} \frac{\partial \varphi}{\partial x^i}_{c(t)} = \nabla_{p'(0)} \frac{\partial \varphi}{\partial x^i}$$
This shows that $\frac{\nabla}{dt} v(0)$ depends only on $v'(0)$ and $p'(0)$, and thus the formula is well-defined.

The fact that it is a Riemannian metric is clear.

(b) A curve α in TM is contained in a fiber, exactly if $\pi \circ \alpha$ is constant, which happens exactly if $\alpha'(t) \in \ker d\pi$ for all t. Hence, the tangent vectors parallel to the fiber are exactly those where $d\pi(V) = 0$ (in the description of a)). Such tangent vectors are those which can be realised as derivatives of paths $(p, w(t))$ where p is a point, and $w(t)$ is a path in T_pM. Observe that $\frac{\nabla}{dt} w = w'$ in that case (the usual derivative in the real vector space T_pM).

Let $(p(t), v(t))$ be a path in TM. The scalar product with the derivative of the path $(p(t_0), w(t))$ is therefore
$$\langle p'(t_0), 0 \rangle + \langle \frac{\nabla}{dt} v(t_0), w'(t_0) \rangle$$
As $w'(t_0)$ can be arbitrary, this is zero for all tangent vectors to the fiber exactly if $\frac{\nabla}{dt} v(0) = 0$.

(c) We have seen that the trajectories to the geodesic field are exactly the curves (γ, γ') for γ a geodesic. Since γ' is parallel along γ for geodesics, this shows c), using b).

(d) Let $(\alpha(t), v(t)) = \overline{\alpha}(t)$ be a path in TM. We have
$$l(\overline{\alpha}) = \int \sqrt{\langle d\pi(\alpha'(t)), d\pi(\alpha'(t)) \rangle + \langle \frac{\nabla}{dt} v(t), \frac{\nabla}{dt} v(t) \rangle}.$$

Hence,

$$l(\overline{\alpha}) \geq \int \sqrt{\langle d\pi(\alpha'(t)), d\pi(\alpha'(t))\rangle} = l(\alpha)$$

with equality if and only if $\frac{\nabla}{dt}v = 0$.

Now, suppose that $(\gamma(t), \gamma'(t)) = \overline{\gamma}$ is a trajectory of the geodesic field, and suppose that γ is length-minimising between $\gamma(0), \gamma(\epsilon)$ (we know from class that any short enough geodesic segment has this property). We then have

$$l(\overline{\gamma}) = l(\gamma)$$

Suppose $\overline{\alpha} = (\alpha, w)$ is any path in TM joining $\overline{\gamma}(0)$ and $\overline{\gamma}(\epsilon)$ for some ϵ. Then $\alpha = \pi \circ \overline{\alpha}$ joins $\pi\overline{\gamma}(0) = \gamma(0)$ and $\overline{\gamma}(\epsilon) = \gamma(\epsilon)$. We thus have, since γ is length-minimising,

$$l(\overline{\gamma}) = l(\gamma) \leq l(\alpha) \leq l(\overline{\alpha}),$$

and therefore $\overline{\gamma}$ is length-minimising. By a result from class, it is therefore a geodesic. Since being a geodesic is a local property, this shows that trajectories of the geodesic field are geodesics in TM.