Prof. Dr. Sebastian Hensel Sommersemester 2019

Riemannian Geometry

Preview

Warnings about this document

This document contains the lecture notes for *upcoming* lectures. All warnings about the script apply here doubly so.

Upcoming Material

0.1. **The Sphere Diameter Rigidity theorem (not covered in class).**

Theorem 0.1. *Suppose M^n is complete and $K_M \geq H > 0$ and* $\operatorname{diam}(M) = \pi/\sqrt{H}$. *Then M is isometric to N_H.*

Proof. Pick p, q realise the diameter. Take $\gamma_1 : [0, t_0] \to M$ be any geodesic segment starting in p, and let γ_2 be a minimal geodesic from p to q. Consider a comparison hinge on N_H. By the length assumption on γ_2, it connects antipodal points, which means that the geodesic closing the hinge in N_H has length $\pi/\sqrt{H} - t_0$. The actual geodesic closing the hinge in M is shorter. But, since p, q realise diameter, this implies that the length is exactly $\pi/\sqrt{H} - t_0$. Hence, γ_1 extends to time $\pi/\sqrt{H}$ and connects p to q. In particular, any Jacobi field along any minimal geodesic starting in p and which vanishes at 0, also vanishes at $\pi/\sqrt{H}$ the next time. Together with the curvature condition this implies that all curvatures spanned by $\gamma'(0)$ and any other vector are exactly H, and $\exp_p$ is nonsingular on the ball of radius $\pi/\sqrt{H}$.
By the previous lemma, this implies that the ball of radius $\pi/\sqrt{H}$ is actually isometric to the corresponding ball in S^n. This isometry extends to a distance-preserving map $M \to S^N$, which is then the desired isometry. □

0.2. **More about the index form.** As the final topic, we will relate conjugate points to minimisers.

Lemma 0.2. *Let $\gamma : [0, l] \to M$ be a geodesic starting in $p = \gamma(0)$. If $q = \gamma(t)$ is not conjugate to p along γ, then for any V, V' there is a unique Jacobi field J with $J(0) = V, J(t) = V'$.*

Proof. Let $\mathcal{J}$ be the space of Jacobi fields with $J(0) = 0$. This is a $n-$dimensional vector space, and the evaluation map $J \mapsto J(t)$ is an injective linear map (as q is not conjugate to p along γ). Hence, it is an isomorphism, showing the lemma in the special case where $V = 0$. The same argument (reversing the geodesic) shows the special case where $V' = 0$. This shows the existence of the J in the general case. For dimension reasons this gives uniqueness as well. □

We return to studying the index form and Jacobi fields.

Now, take a subdivision of the interval $0 = t_0 < t_1 < \ldots < t_k = l$ on which the geodesic is defined, and so that $\gamma[t_i, t_{i+1}]$ is contained in a totally normal neighbourhood. In particular, there are no conjugate points on $\gamma[t_i, t_{i+1}]$.
Let $\mathcal{V}^-$ be the subspace of $\mathcal{V}$ of those fields V so that $V|[t_i, t_{i+1}]$ is a Jacobi field. Let $\mathcal{V}^+$ be the subspace of those W which are zero at all t_i.

Lemma 0.3. *$\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ and the decomposition is orthogonal with respect to I. The form I is positive definite restricted to $\mathcal{V}^+$.*

Proof. The direct sum claim is a direct consequence of the fact that since $\gamma(t_{i+1})$ is not conjugate to $\gamma(t_i)$, the endpoints determine a unique Jacobi field and vice versa. Orthogonality is clear from the definition of I.
Since the $\gamma[t_i, t_{i+1}]$ are minimising geodesics, they are a minimum of any variation. By the second variation formula, this implies that I is positive semidefinite of $\mathcal{V}^+$.
If $I(V, V) = 0$ for $V \in \mathcal{V}^+$, then note that for $W \in \mathcal{V}^+$

$$0 \leq I(V + cW, V + cW) = 2cI(V, W) + c^2 I(W, W)$$

for all c. This implies $I(V, W) = 0$. In fact, V is in the nullspace of I, by the orthogonality. Thus V is a Jacobi field, vanishing at all the t_i, hence zero. □

In particular, the index (or nullity) of I is the index (or nullity) of I restricted to $\mathcal{V}^-$, which is finite.

Theorem 0.4 ((Morse) Index theorem)**.** *The index of I is finite, and the number of conjugate points on $\gamma[0, t]$ counted with multiplicity.*

Before/Instead giving the proof, we note corollaries:

Corollary 0.5. *Suppose $\gamma : [0, a] \to M$ is a geodesic segment so that $\gamma(a)$ is not conjugate to $\gamma(0)$ along γ. Then γ has no conjugate points on $(0, a)$ if and only if for all proper variations of γ energy can be reduced.*

Proof. By the Morse index theorem there are conjugate points exactly if there is a proper variation field V with $I_a(V, V) < 0$. By the variational formula for energy this implies the second variation of energy for that variation is negative. □

Corollary 0.6. *After the first conjugate point, geodesics stop to be minimising.*

Proof of the index theorem.

- Let γ_t be the restriction of γ to $[0, t]$, I_t the corresponding index form and $i(t)$ its index.
- Since the initial segment of γ has no conjugate points, i is 0 close to 0.
- Further, i is nondecreasing: one can just extend every vector field in the negative definite subspace of I_r to $[0, s], s > r$ by 0.

- $i(t)$ does not depend on the chosen subdivision of the interval. Thus, to study i near a fixed t, we may assume $t \in (t_{j-1}, t_j)$.
- We know that the index of I_t is the same restricted to $\mathcal{V}^-(0,t)$, and since elements in that space are determined by their values on the breaks we have
$$\mathcal{V}^-(0,t) \cong \bigoplus_{i<j} T_{\gamma(t_i)} =: S_j$$
in particular the index forms $I_t, t \in (t_{j-1}, t_j)$ can all be interpreted as forms on S_j, and these vary continuously in t (Jacobi solutions vary continuously).
- $i(t-\epsilon) = i(t)$ for small ϵ, since: it could only go down, but by continuity negatively definite subspaces stay negatively definite.
- $i(t+\epsilon) \leq i(t) + d$ for small ϵ and d the nullity of $\gamma(t)$: $\dim(S_j) = n(j-1)$ and I_t is positive definite on a subspace of dimension $n(j-1) - i(t) - d$ (total dim minus neg def minus nullity). By continuity this stays positive definite for small values above t, which shows the claim.
- Suppose that $V \in S_j$ satisfies $V(t_{j-1}) \neq 0$. Let V_{t_0} be the piecewise Jacobi field which agrees with V on the $t_i, i < j$ and vanishes at $t_0 \in (t_{j-1}, t_j)$. Then
$$I_{t_0}(V_{t_0}, V_{t_0}) > I_{t_0+\epsilon}(V_{t_0+\epsilon}, V_{t_0+\epsilon})$$
Namely: If we define W the field which is equal to V_{t_0} up to t_0 and then becomes zero, then by the Index Lemma
$$I_{t_0}(V_{t_0}, V_{t_0}) = I_{t_0+\epsilon}(W, W) > I_{t_0+\epsilon}(V_{t_0+\epsilon}, V_{t_0+\epsilon})$$
since W on the last segment is not a Jacobi field.
- $i(t+\epsilon) \geq i(t) + d$, since if $I_t(V,V) = 0$, then $I_{t+\epsilon}(V,V) < 0$, so the null space becomes negative definite.

□

0.3. **Cut points.** To understand minimisers versus conjugate points in more detail, we use

Definition 0.7. Given a geodesic $\gamma : [0,l] \to M$. We say that $q = \gamma(t_0)$ is a *cut point of p along γ* if
$$t_0 = \sup\{t | d(p, \gamma(t)) = t\}$$
Given a point $p \in M$, the *cut locus* $C_m(p)$ is the set of all cut points of p.

Proposition 0.8. *Suppose that $q = \gamma(t_0)$ is a cut point of $p = \gamma(0)$ along γ. Then*

- *either $\gamma(t_0)$ is the first conjugate point of p along γ.*
- *or there is a geodesic $\sigma \neq \gamma$ joining p to q of the same length as γ.*

Conversely, if one of these hold, then $\gamma(t') = q'$ is a cut point of p along γ for some $t' \leq t_0$.

Proof. First the converse: non-minimising after the first cutpoint we did already. If we had two geodesics, we could find a broken arc of length $t_0+\epsilon$ connecting to $\gamma(t_0+\epsilon)$ (follow along σ and then shortcut in a geodesic ball). Since broken paths are never geodesic, this means that the minimiser is actually shorter thatn $t_0+\epsilon$.
Now suppose that t_0 is as in the assumption.

- Find $t_0+\epsilon_i \to t_0$ and σ_i minimisers from p to $q_i = \gamma(t_0+\epsilon_i)$.
- Up to subsequence, we can let the σ_i converge, and the limit σ is a minimiser from p to q.
- If $\sigma = \gamma$ we are done.
- Otherwise, suppose that $\sigma'(0) = \gamma'(0)$ and that $d\exp_p$ is not singular at $t_0\gamma'(0)$. Hence, there is a neighbourhood U of that point where $\exp_p$ is a diffeomorphism.
- We have
  $$\gamma(t_0+\epsilon_j) = \sigma_j(t_0+\epsilon_j')$$
  with $\epsilon_j' < \epsilon_j$ (as the σ_j are minimisers and γ is not anymore). We may assume that the $\sigma_i(t_0+\epsilon_j')$ are in the neighbourhood U.
- Then
  $$\exp_p(t_0+\epsilon_j)\gamma'(0) = \gamma(t_0+\epsilon_j) = \sigma_j(t_0+\epsilon_j') = \exp_p(t_0+\epsilon_j')\sigma_j'(0)$$
  and by our assumption on U this means
  $$(t_0+\epsilon_j)\gamma'(0) = (t_0+\epsilon_j')\sigma_j'(0)$$
  which implies $\gamma'(0) = \sigma_j'(0)$ as both are unit norm
- This would mean that γ is minimising after t_0, contraditing cut point.

□