Proposition Let $E \to M$ be a vector bundle. The set of connections on E is an affine space:

- For $\nabla, \tilde{\nabla}$ connections,

$$\nabla - \tilde{\nabla} \in \Gamma(T^*M \otimes \operatorname{End} E)$$

- For $A \in \Gamma(T^*M \otimes \operatorname{End} E)$ and a connection ∇, $\tilde{\nabla} := \nabla + A$ is also a connection.

pf This follows from tensoriality: $V \in \Gamma(TM)$, $f \in C^\infty(M)$, $W \in \Gamma(E)$.

$$\nabla_V(fW) - \tilde{\nabla}_V(fW) = f\nabla_V W + df(W) - f\tilde{\nabla}_V W - df(W)$$
$$= f(\nabla_V W - \tilde{\nabla}_V W)$$

i.e. $\nabla_V - \tilde{\nabla}_V$ is $C^\infty(M)$-linear. As before, this implies that at $p \in M$, it only depends on $W(p)$ (and not on a germ or derivatives). So $(\nabla_V - \tilde{\nabla}_V) \in \Gamma(\operatorname{End}(E))$.

$\nabla + A$ is a connection by direct check of conditions. $\square$

Corollary As we have seen that $\nabla_v = v^i \frac{\partial}{\partial x^i}$ is a connection on $\mathbb{R}^n$, it follows that every connection in a chart is of the form

$$\nabla_v = v^i \partial_i + A(v)$$

for a End E valued 1-form $A \in \Gamma(T^*U \otimes \operatorname{End} E)$

If we choose a local trivialization (σ^a) of E, $A(v)$ is a matrix

$$(A(v)\, W)^a = \sum_{b,i} v^i A_{i\,b}^{a} W^b$$

The components depend on the chart and the trivialization!

$$\nabla_v W = \sigma^a v^i \partial_i (\langle \sigma_a, W \rangle) + \sigma v^i A_{i\,b}^{a} \langle \sigma_a, W \rangle$$

In particular, a different trivialization (τ^a) is related via $\sigma^a(p) = \Lambda^a{}_b(p) \tau^b(p)$

$$\nabla_v W = \tau^b \Lambda^a{}_b v^i \partial_i (\langle \Lambda^{-1}{}_a{}^b \tau_b, W \rangle)$$
$$+ \Lambda^b{}_c \tau^c v^i \Lambda^{-1}{}_b{}^d \Lambda^a{}_e A_{i\,d}^{e} \langle \Lambda^{-1}{}_a{}^f \tau_f, W \rangle$$

$$= \tau^b v^i \langle \tau_b, W \rangle$$

$$+ \tau^c v^i \left(A_i{}^b{}_c \pm \Lambda^b{}_a \partial_i \Lambda^{-1a}{}_c \right) \langle \tau_b , w \rangle$$

So, under a change of trivialization, A <u>does not</u> transform tensorially but picks up an additional term $\Lambda \partial_i \Lambda^{-1}$!

<u>example</u> Wave functions: These are sections in a $\mathbb{C}$-bundle over M. Since $|\psi(x)|^2$ has a physical meaning, we only allow changes of trivialization that preserve $|\psi(x)|^2$. (technically, we are dealing with a "hermitean bundle"). This implies A to be imaginary and we write it as $A = iA_i dx^i$.
Thus $\Lambda(x) = e^{i\lambda(x)}$. Then

$$\Lambda \partial_i \Lambda^{-1} = e^{i\lambda(x)} \partial_i e^{-i\lambda(x)}$$

$$= -i \partial_i \lambda$$

And A_i goes to $A_i - i\partial_i \lambda$

We recovered the gauge transformations of electromagnetism. The geometric role of the vector potential is thus that $D = \partial + iA$ is a connection acting on wave functions.

Proposition There are connections for every manifold.

Pf If $(\nabla^{(i)})_i$ are connections and $(\rho^i)_i$ are real functions with $\sum_i \rho^i = 1$. Then $\sum_i \rho^i \nabla^{(i)}$ is a connection. We only need to check

$$\sum \rho^i \nabla_V^{(i)} (f W) = \sum_i \rho^i f \nabla_V^{(i)} W + \sum_i \rho^i \, df(W)$$

$$= f \sum_i \rho^i \nabla_V^{(i)} W + df(W)$$

Since in every chart $\nabla_V = V^i \partial_i$ is a connection, we can glue those together with a subordinate partition of unity.

Curvature

Definition Let $E \to M$ be a vector bundle and ∇ a connection. Then the "curvature of ∇" is defined by

$$\Omega : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

$$(V, W, \psi) \mapsto \nabla_V \nabla_W \psi - \nabla_W \nabla_V \psi - \nabla_{[V,W]} \psi$$

<u>Prop</u> $\Omega(V,W)\psi$ is anti-symmetric in V, W by inspection.

<u>Prop</u> $\Omega(V,W)\psi$ is tensorial in V and W.

<u>Pf</u>

$$\begin{aligned}
\Omega(V, fW)\psi &= \nabla_V (f \nabla_W \psi) - f \nabla_W \nabla_V \psi - f \nabla_{[V,W]} \psi - df(V) \nabla_W \psi \\
&= df(V) \cancel{\nabla_W} \psi + f \nabla_V \nabla_W \psi - f \nabla_W \nabla_V \psi - f \nabla_{[V,W]} + df(V) \cancel{\nabla_W} \psi \\
&= f\, \Omega(V,W)\psi
\end{aligned}$$

In V by anti-symmetry.

<u>Prop</u> $\Omega(V,W)\psi$ is tensorial in ψ.

<u>Pf</u> $\Omega(V,W)(f\psi) =$

$$\nabla_V \nabla_W (f\psi) - \nabla_W \nabla_V (f\psi) - \nabla_{[V,W]}(f\psi) =$$

$$\nabla_V \left(f \nabla_W \psi + df(W)\psi \right)$$
$$- \nabla_W \left(f \nabla_V \psi + df(V)\psi \right)$$
$$- f \nabla_{[V,W]}\psi - df([V,W]) =$$

$$\underline{f \nabla_V \nabla_W \psi} + df(V) \cancel{\nabla_W} \psi + df(W) \cancel{\nabla_V} \psi$$
$$+ d(df(W))(V)$$
$$\underline{- f \nabla_W \nabla_V \psi} - df(W) \cancel{\nabla_V} \psi - df(V) \cancel{\nabla_W} \psi$$
$$- d(df(V))(W) \underline{- f \nabla_{[V,W]} \psi} - df([V,W]) =$$

$$f\,\Omega(V,W)\psi + V[W[f]] - W[V[f]]$$
$$- V[W[f]] + W[V[f]] =$$

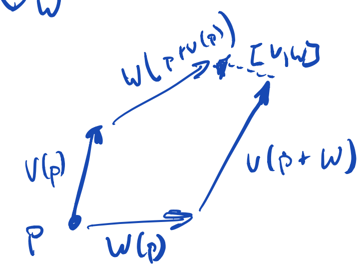
$$f\,\Omega(V,W)\psi \qquad \square$$

So $\left(\Omega(V,W)\psi\right)(p)$ only depends on $V(p), W(p)$ and $\psi(p)$. Thus

$$\Omega \in \Gamma\left(\Omega^2 \otimes \mathrm{End}(E)\right).$$

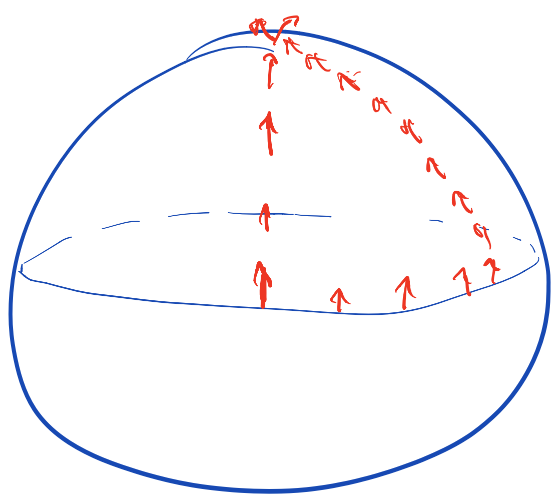
Geometric interpretation:

$D_V D_W$



$\Omega(V,W)\psi$: compare the change of ψ between infinitesimally path to (and close the parallelogram via $D_{[V,W]}$.

$\Omega(V,W)$ is the change of ψ when going around an infinitesimal parallelogram (plus close) spanned by vectors V and W.



<u>remark</u> In a chart, we can use basis vectors ∂_i and compute $\Omega(\partial_i, \partial_j)\psi$. These have the advantage $[\partial_i, \partial_j] = 0$ so there is

no last term.

<u>example</u> Consider again the hermitian $\mathbb{C}$-bundle describing electrodynamics. Here

$$\Omega(\partial_i,\partial_j)\,\psi = \nabla_{\partial_i}\nabla_{\partial_j}\psi - \nabla_{\partial_j}\nabla_{\partial_i}\psi$$

$$= (\partial_i + iA_i)(\partial_j + iA_j)\psi$$

$$-(\partial_j + iA_j)(\partial_i + iA_i)\psi$$

$$= \cancel{\partial_i\partial_j\psi} + iA_i\cancel{\partial_j\psi}$$

$$- A_i\cancel{A_j}\psi \quad + i(\partial_i A_j)\psi + iA_j\cancel{\partial_i\psi}$$

$$- \cancel{\partial_j\partial_i\psi} - i(\partial_j A_i)\psi$$

$$- iA_i\cancel{\partial_j\psi} - iA_j\cancel{\partial_i\psi} + A_j\cancel{A_i}\psi$$

$$= i(\partial_i A_j - \partial_j A_i)\psi$$

$$= i F_{ij}\psi$$

We find the electromagnetic field strength as the curvature of the $\mathbb{C}$-bundle.