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Exam preparation notes, and other assorted sundries

Important Homework problems

This list of exercises should give you an idea of some of the most important things to consider when preparing for the exam. The list is not binding – just because something is not covered here does not mean that it cannot appear in the exam. Rather, this is meant to indicate the type (difficulty, and length) of problems you might encounter. Also, if you have problems with any of the exercises on this list, discussing them with your tutor is a good idea.

- Sheet 2, Problems 2 and 3.
- Sheet 3, Problems 1-3. For problem 1: you only need to give the reasons why something fails to be a manifold if you want to show that it is not a manifold. For problem 2: discard the part about constructing an example. For problem 3: in showing that the product is a manifold, it is important to say what one needs to check, but it is (by now!) perfectly fine to assert that certain things are clear (e.g. that the product of two charts is smooth). The second part should be easy and quick.
- Sheet 4, Problem 1 (again, discard the construction of an example)
- Sheet 5, Problem 2b (try to find a solution which is not very long), Problem 3
- Sheet 6, Problem 2, 4
- Sheet 7, Problem 4
- Sheet 9, Problem 1, 2a.
- Sheet 10, Problem 1, 2c (see solution below; the local setup could be much briefer in the exam)
- Sheet 11, Problem 2
- Sheet 12, Problem 3

1. Solving 10.2c

Since a number of people asked me about the required detail to solve things like 2c on sheet 10, here is a solution I consider complete. Since the notation apparently caused some confusion, this is changed here a bit.

First, we discuss the local setup. In the homework solution (or the exam) this could be much briefer, simply asserting the identifications of sections with the corresponding local objects – the details are here to add formal clarity. (For example: it is perfectly acceptable to suppress the map f below from the notation, as was done in the formulation of the exercise)

Suppose that U is a neighbourhood over which E is trivialised, i.e. there is a bundle isomorphism
$$f : U \times \mathbb{R}^k \to \pi^{-1}(U).$$
A section of $E|_U$ is a map of the form
$$\beta(u) = f(u, b(u)),$$
for a smooth map $b : U \to \mathbb{R}^k$. Similarly, a section $\sigma \in \Gamma(\mathrm{End}(E|_U))$ has the form
$$\sigma(u)(f(u,x)) = f(u, S(u)x)$$
for a smooth map $S : U \to \mathcal{M}_{k\times k}(\mathbb{R})$. Note that this equation completely determines σ, if such an S is given.
If we assume that U is in addition a chart neighbourhood, then we obtain a connection ∇^0 on $E|_U$ from the usual derivative of vector valued functions in $\mathbb{R}^n$. More precisely, suppose that $\varphi : U \to V$ is a (fixed) chart, and suppose that β is a section, with corresponding function $b : U \to V$. We then let
$$\nabla^0_X \beta(u) = f(u, D_{\varphi(u)}(b \circ \varphi^{-1})(d\varphi_u(X(u)))).$$
(As this connection is simply the usual derivative under our identifications, it was denote by d on the problem set) Since the difference between any two connections is a End-valued 1–form, we thus find
$$\nabla^E - \nabla^0 = \alpha$$
where we have
$$\alpha_u(X_u)(f(u,x)) = f(u, A(X_u)x)$$
Here, A is a smooth map $A : TU \to \mathcal{M}_{k\times k}(\mathbb{R})$.
Similarly, we can also use the usual derivative (of matrix valued functions) to define a connection $\nabla^{0,\mathrm{End}}$ on $\mathrm{End}(E|_U)$:
$$\nabla^{0,\mathrm{End}}_X \sigma(f(u,x)) = f(u, D_{\varphi(u)} S \circ \varphi^{-1}(d\varphi_u(X(u)))).$$
(again, on the problem set, this was simply called d). Note that the product rule for matrix multiplication
$$D_x(Ab)(v) = (D_x A)(v) b(x) + A(x) D_x b(v)$$
yields the product rule
$$\nabla^0_X \sigma \circ \beta = (\nabla^{0,\mathrm{End}}_X \sigma) \circ \beta + \sigma \circ \nabla^0_X \beta.$$
showing that $\nabla^{0,\mathrm{End}}$ is the connection induced by ∇^0 in the sense of part b).

With the setup in place, we are now ready for the actual exercise. Let α be as above, with corresponding function A as above. Further, let σ be a section of $\mathrm{End}(E)$, with corresponding function S.
Let β be a section of $E|_U$ and X be a vector field. By the defining property of ∇^{End} we then have
$$\nabla^E_X(\sigma \circ \beta) = (\nabla^{\mathrm{End}}_X \sigma) \circ \beta + \sigma \circ \nabla^E_X \beta.$$

To compute the left-hand side, note first that

$$\sigma \circ \beta(u) = f(u, S(u)b(u))$$

Thus,

$$\nabla^E_X(\sigma \circ \beta)(u) = \nabla^0_X(\sigma \circ \beta)(u) + f(u, A(X(u))S(u)b(u)).$$

Similarly,

$$\sigma \circ \nabla^E \beta(u) = \sigma \circ \nabla^0 \beta(u) + f(u, S(u)A(X(u))b)$$

Together this means (suppressing dependence on u to clean up notation)

$$(\nabla^{\mathrm{End}}_X \sigma) \circ \beta = \nabla^E_X(\sigma \circ \beta) - \sigma \circ \nabla^E_X \beta = \nabla^0_X(\sigma \circ \beta) + f(u, A(X)Sb) - \sigma \circ \nabla^0 \beta - f(u, SA(X)b)$$
$$= (\nabla^{0,\mathrm{End}}_X \sigma) \circ \beta + f(u, A(X)Sb - SA(X)b)$$

where we have used that $\nabla^{\mathrm{End},0}$ is the connection induced by ∇^0, and $f(u, \cdot)$ is linear. Since an endomorphism is determined by its values, this shows the claim.