NIELSEN REALISATION FOR TWO-DIMENSIONAL RIGHT-ANGLED ARTIN GROUPS

SEBASTIAN HENSEL AND DAWID KIELAK

1. Introduction

A right-angled Artin group (RAAG) A_Γ is a group given by a very simple presentation, which is defined by a graph Γ: A_Γ has one generator for each vertex of Γ, and two generators commute if and only if the corresponding vertices are joined by an edge in Γ.

RAAGs have been an object of intense study over the last years, and indeed they seem to be ubiquitous in geometry and topology. The most striking example is possibly the role they played in the recent solution of the virtual Haken conjecture. But also as objects on their own, they show a rich structure, which can be seen for example by looking at the variety of surprising properties their subgroups can exhibit (see e.g. [BB]).

A general RAAG A_Γ can be seen as interpolating between a non-abelian free group F_n (corresponding to the graph with n vertices and no edges) and a free Abelian group $\mathbb{Z}^n$ (defined by the complete graph on n vertices). If a property holds for both F_n and $\mathbb{Z}^n$, it is then natural to find an analogue that works for all RAAGs.

In this article we investigate *Nielsen realisation* from this point of view. For free groups this takes the following form: suppose one is given a finite subgroup $H < \mathrm{Out}(F_n)$. Is there a graph X with $\pi_1(X) = F_n$ on which the group H acts by isometries, inducing the given action on the fundamental group? The answer turns out to be yes (as shown independently by Culler [Cul], Khramtsov [Khr], and Zimmermann [Zim1]; see also [HOP] for a more recent, topological proof).

Let us note here that Nielsen realisation for free groups is equivalent to the statement that finite subgroups of $\mathrm{Out}(F_n)$ have fixed points when acting on the Culler–Vogtmann Outer Space. The result is also an essential tool in the work of Bridson–Vogtmann [BV] and the second-named author [Kie1, Kie2], and is used to prove certain rigidity phenomena for $\mathrm{Out}(F_n)$.

The corresponding statement for free abelian groups follows from the (classical) fact that any finite (in fact compact) subgroup of $\mathrm{GL}_n(\mathbb{R})$ can be conjugated to be a subgroup of the orthogonal group. This implies that any finite $H < \mathrm{Out}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ acts isometrically on an n-torus, and the induced action on the fundamental group is the given one.

For RAAGs the natural analogue is as follows: suppose one is given a finite subgroup $H < \mathrm{Out}(A_\Gamma)$ in the outer automorphism group of a RAAG. Is there a non-positively curved metric space on which H acts by isometries, realising the action on the fundamental group?

The close relationship between RAAGs and cube complexes tempts one to ask the above question with cube complexes in place of metric spaces. This is however bound to lead to a negative answer, since already for general finite subgroups of $GL_n(\mathbb{Z})$ the action on the torus described above cannot be made cubical (at least not in dimension n).

The main result of this article proves Nielsen Realisation for a large class of RAAGs and finite groups. The restrictions are chosen in a way allowing us to use cube complexes, and we obtain

Theorem. *Suppose Γ is a connected graph without leaves and triangles, and let $H < \mathrm{Out}^0(A_\Gamma)$ be finite. Then there is a non-positively curved square complex realising the action of H.*

The group $\mathrm{Out}^0(A_\Gamma) < \mathrm{Out}(A_\Gamma)$ is a finite-index subgroup which is usually studied in the literature.

In fact, we show a more general result (Theorem 9.6) whose statement is more technical, and not suited for this informal discussion.

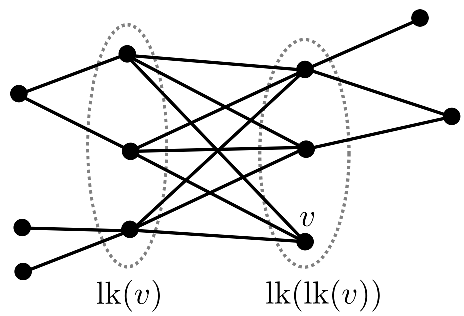
However, it is likely that the methods developed in this article can be generalized to apply to more general settings – both the restriction to subgroups of $\mathrm{Out}^0(A_\Gamma)$ and to graphs without triangles are most likely not necessary. This will be investigated in forthcoming articles. Of course when all conditions on Γ are dropped, we will no longer be able to use cube complexes.

1.1. **Outline of the proof.** Since this article is rather substantial in length, let us offer here a somewhat informal outline of the proof of the main theorem.

The proof (Section 10) is inductive on the dimension of Γ, that is the maximal size of a maximal clique in Γ. We look at Γ without triangles, and so the dimension is at most 2.

Take any vertex $v \in \Gamma$. Since Γ is connected and triangle-free, its link $\mathrm{lk}(v)$ is a non-empty discrete graph. It is also *preserved*, which means that we have an induced action of H on $A_{\mathrm{lk}(v)}$, the subgroup of A_Γ generated by vertices of $\mathrm{lk}(v)$. Since $\mathrm{lk}(v)$ is discrete, $A_{\mathrm{lk}(v)}$ is a free group. By the inductive assumption realisation this is possible for this action and yields a graph with an H-action (this at first simply seems to be Nielsen realisation for free groups, but see below for a caveat!). Now the link $\mathrm{lk}(\mathrm{lk}(v))$ (see the figure) is again preserved, and again of dimension smaller than that of Γ, and hence we can again find a graph with an H-action realising this.

We take the product of these two graphs, and in this way we can construct a cube complex realising the action on $A_{\widehat{\mathrm{st}}(v)}$, where the graph $\widehat{\mathrm{st}}(v)$ (the extended star of v) is just the union of $\mathrm{lk}(v)$ and $\mathrm{lk}(\mathrm{lk}(v))$.



Observing that $v \in \widehat{\mathrm{st}}(v)$, we see that we can cover Γ with subgraphs which are invariant, and for each we have constructed a cube complex realising the induced action. To form a complex realising the full action ϕ we now need to glue the smaller complexes. The second half of the paper (starting with Section 7) is devoted to formalising the gluing and showing that it can in fact be done.

Let us look at two complexes $X_{\widehat{\mathrm{st}}(v)}$ (realising the action on $A_{\widehat{\mathrm{st}}(v)}$) and $X_{\widehat{\mathrm{st}}(w)}$ (realising the analogous action), such that $\widehat{\mathrm{st}}(v) \cap \widehat{\mathrm{st}}(w) \neq \emptyset$. For us to be able to glue them they need to contain an H-invariant subcomplex realising the action on $A_{\widehat{\mathrm{st}}(v) \cap \widehat{\mathrm{st}}(w)}$.

This requirement motivates our main definition, namely that of cubical systems (Section 7). These are exactly cube complexes with invariant subcomplexes realising actions on relevant subgroups.

Once $X_{\widehat{\mathrm{st}}(v)}$ and $X_{\widehat{\mathrm{st}}(w)}$ contain appropriate subcomplexes, we may glue one to the other, provided that the complexes are H-equivariantly isomorphic. This requirement introduces (yet another) complication to our proof. For the complex $X_{\widehat{\mathrm{st}}(v)}$ it is enough to contain any subcomplex realising the action on $A_{\widehat{\mathrm{st}}(v) \cap \widehat{\mathrm{st}}(w)}$. In order to be able to glue $X_{\widehat{\mathrm{st}}(v)}$ and $X_{\widehat{\mathrm{st}}(w)}$, we need the latter not only to contain some subcomplex realising the action on $A_{\widehat{\mathrm{st}}(v) \cap \widehat{\mathrm{st}}(w)}$, but the exact same one as $X_{\widehat{\mathrm{st}}(v)}$ contains.

Having constructed the complexes $X_{\widehat{\mathrm{st}}(v)}$ and $X_{\widehat{\mathrm{st}}(w)}$ as above, we may now glue them along the isomorphic subcomplexes. It is by no means obvious (in fact in many cases not true), that the glued up object realises the given action of H. It is however true that there exists a correct way of gluing the objects and obtaining the desired action (Section 8). We still need the complex to have a structure of a cubical system, since we might have to glue another complex to it. Verifying that it can indeed be given the desired structure is the chief technical difficulty of the main proof (Section 10).

To start the induction we need to focus on Γ of dimension 1, that is on a finite discrete graph.

As observed above, the usual formulation of Nielsen realisation is not sufficient for our purposes, as we need to make sure that the realising graphs contain invariant subgraphs so we can perform the gluings.

The main tool which allows us to construct such graph systems is Adapted Realisation (Section 3) which should be of independent interest. It is a refined version of the classical Nielsen realisation for free groups, with the additional property that when our finite action preserves a free product decomposition of F_n, and each of the induced actions is already realised by a graph, we can build a graph realising the whole action in such a way that it contains all the given smaller graphs as invariant subgraphs.

The proof of the Adapted Realisation is topological in flavour; it exploits the action of $\mathrm{Out}(F_n)$ on isotopy classes of spheres in a doubled-up handlebody of genus n, which can be thought of as a topological model for F_n (Subsection 2.5). The main tool here is the notion of dismantlability of graphs, introduced an studied by Osajda, Przytycki and the first-named author in [HOP].

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2. Preliminaries

2.1. **Graphs and RAAGs.** Throughout the paper Γ will denote a fixed simplicial graph. We define the associated RAAG A_Γ to be the group generated by the vertices of Γ, and with a presentation in which the only relations occurring are commutators of vertices adjacent in Γ.

The only subgraphs of Γ we will encounter will be induced subgraphs; such a subgraph is uniquely determined by its vertex set (since Γ is fixed). Hence we will use $\cap, \cup, \smallsetminus$ etc. of two graphs to denote the induced subgraph spanned by the corresponding operation applied to the vertices of the two graphs.

Definition 2.1. Let Δ, Σ be two induced subgraphs of Γ. We say that they form a *join* if and only if each vertex in Δ is connected by an edge (in Γ) to each vertex of Σ. The induced subgraph spanned by all vertices of Δ and Σ will be denoted by $\Delta * \Sigma$.

An induced subgraph Θ is a *join* if and only if we have $\Theta = \Delta * \Sigma$ for some non-empty induced subgraphs Δ and Σ; furthermore Θ is a cone if Δ can be taken to be a singleton.

Note that, in accordance with our convention, we have $\Delta * \Sigma = \Delta \cup \Sigma$; the join notation indicated the presence of the relevant edges.

Note that induced subgraphs (their vertices to be more specific) generate subgroups of A_Γ; given such a subgraph Δ we will call the corresponding subgroup A_Δ. Note that it is abstractly isomorphic to the RAAG defined by Δ. We adopt the convention $A_\emptyset = \{1\}$.

Throughout the paper we use the (standard) convention of denoting the normaliser, centraliser, and centre of a subgroup $H \leqslant A_\Gamma$ by, respectively,

$N(H), C(H)$ and $Z(H)$. We will also use $c(x) \in \mathrm{Aut}(A_\Gamma)$ to denote conjugation by $x \in A_\Gamma$.

We will need the following definitions throughout the paper. Some of them are new; others may be non-standard.

Definition 2.2. Suppose $\Delta \subseteq \Gamma$ is an induced subgraph.

i) The *link* of Δ is
$$\mathrm{lk}(\Delta) = \bigcap_{v \in \Delta} \mathrm{lk}(v)$$
ii) The *star* of Δ is
$$\mathrm{st}(\Delta) = \mathrm{lk}(\Delta) * \Delta$$
iii) The *extended star* of Δ is
$$\widehat{\mathrm{st}}(\Delta) = \mathrm{lk}(\Delta) * \mathrm{lk}(\mathrm{lk}(\Delta)) = \mathrm{st}(\mathrm{lk}(\Delta))$$
iv) Given a second full subgraph $\Theta \subseteq \Gamma$ with $\Delta \subseteq \Theta$ we define the *restricted link* and *restricted star* of Δ in Θ to be respectively
$$\mathrm{lk}_\Theta(\Delta) = \mathrm{lk}(\Delta) \cap \Theta \text{ and } \mathrm{st}_\Theta(\Delta) = \mathrm{st}(\Delta) \cap \Theta$$

Let us observe the following direct consequences of the definition.

Lemma 2.3. *Let Δ and Θ be two induced subgraphs of Γ, and let v be a vertex of Γ. Then*

(1) $\Delta \subseteq \mathrm{lk}(\Theta) \Leftrightarrow \Theta \subseteq \mathrm{lk}(\Delta)$
(2) $\Delta \subseteq \mathrm{st}(v) \Rightarrow v \in \mathrm{st}(\Delta)$
(3) $\mathrm{lk}(\Delta) \subseteq \mathrm{st}(v) \Rightarrow v \in \widehat{\mathrm{st}}(\Delta)$

Proof.

(1) Both statements are equivalent to saying that each vertex in Δ is connected to each vertex in Θ.
(2) If $v \in \Delta$ then the result follows trivially. If not, then $\Delta \subseteq \mathrm{lk}(v)$ and the result follows from the previous one.
(3) $\mathrm{lk}(\Delta) \subseteq \mathrm{st}(v) \Rightarrow v \in \mathrm{st}(\mathrm{lk}(\Delta)) = \widehat{\mathrm{st}}(\Delta)$. □

Definition 2.4 (Join decomposition)**.** Let $\Delta \subseteq \Gamma$ be an induced subgraph. We say that
$$\Delta = \Delta_1 * \cdots * \Delta_k$$
is a *join decomposition* of Δ if and only if each Δ_i is an induced subgraph of Γ which is not a join.

We define $Z(\Delta)$ to be the union all subgraphs Δ_i which are singletons.

Such a decomposition is unique up to reordering the factors.

Proposition 2.5 ([CCV, Proposition 2.2])**.** *Given $\Delta \subseteq \Gamma$ we have the following identifications*

- $N(A_\Delta) = A_{\mathrm{st}(\Delta)} = A_\Delta \times A_{\mathrm{lk}(\Delta)}$*;*
- $Z(A_\Delta) = A_{Z(\Delta)}$*;*
- $C(A_\Delta) = A_{Z(\Delta)} \times A_{\mathrm{lk}(\Delta)}$.

Given another induced subgraph $\Sigma \subseteq \Gamma$ we also have

$$x^{-1}A_\Delta x \leqslant A_\Sigma \iff x \in N(A_\Delta)N(A_\Sigma) \text{ and } \Delta \subseteq \Sigma$$

Definition 2.6. Given a simplicial graph Γ we define its *dimension* $\dim \Gamma$ to be the number of vertices in a largest clique in Γ.

The dimension of Γ coincides with the dimension of the Salvetti complex of A_Γ.

2.2. **Words in RAAGs.** Since A_Γ is given in terms of a presentation, its elements are equivalence classes of words in the alphabet formed by vertices of Γ (which we will refer to simply as the alphabet Γ). Since the presentation is very simple, there is a robust notion of normal form based on reduced and cyclically reduced words. We will only mention the results necessary for our arguments; for further details see the work of Servatius [Ser].

Definition 2.7. Given a word $w = v_1 \cdots v_n$, where each v_i is a letter, i.e. a vertex of Γ or its inverse, we define two *basic moves*:

- *reduction*, which consists of removing v_i and v_j from w (with $i < j$), provided that $v_i = v_j^{-1}$, and that v_k commutes with v_i in A_Γ for each $i < k < j$.
- *cyclic reduction*, which consists of removing v_i and v_j from w (with $i < j$), provided that $v_i = v_j^{-1}$, and that v_k commutes with v_i in A_Γ for each $k < i$ and $j < k$.

A word w which does not allow for any reduction is called *reduced*; if in addition it does not allow for any cyclic reduction, it is called *cyclically reduced*.

Servatius shows that, starting with a word w, there is a unique reduced word obtainable from w by reductions, and a unique cyclically reduced word obtainable from w by basic moves. It is clear that the former gives the same element of A_Γ as w did, and the latter gives the same conjugacy class.

He also shows that two reduced words give the same element in A_Γ if and only if they differ by a sequence of moves replacing a subword vv' by $v'v$ with v, v' being commuting letters; let us call those *swaps*.

Lemma 2.8. *Let $\Sigma \subseteq \Gamma$ be an induced subgraph, and suppose that an element $x \in A_\Gamma$ satisfies $x \in y^{-1}A_\Sigma y$ for some $y \in A_\Gamma$. Then the elements of A_Γ given by cyclically reduced words representing the conjugacy class of x lie in A_Σ.*

Proof. Take a reduced word w in Σ representing yxy^{-1}; let w_y be a word in Γ representing y. Then $w_y w w_y^{-1}$ represents x, and it is clear that there is a series of cyclic reductions taking this word to w. Further reductions and cyclic reduction will yield another word in Σ representing the conjugacy class of x. Hence there exists a cyclically reduced word representing the conjugacy class of x as required.

Now suppose that we have two cyclically reduced words, w and w', representing the same conjugacy class in A_Γ, and such that w is a word in Σ. In A_Γ we have the equation

$$w = z^{-1} w' z$$

where z is some reduced word in Γ.

Suppose that the word $z^{-1}w'z$ is not reduced. Then there is a reduction allowed, and it cannot happen within w', since w' is reduced. Thus there exists a letter v such that, without loss of generality, it occurs in z, its inverse occurs in w', and the two can be removed. Thus we can perform a number of swaps to w' and obtain a reduced word $w''v^{-1}$; we can do the same for z and obtain a reduced word vz'. We now have the following equality in A_Γ

$$w = z'^{-1} v^{-1} w'' v^{-1} v z' = z'^{-1} v^{-1} w'' z'$$

with $v^{-1}w''$ consisting of exactly the same letters as w' (it is its cyclic conjugate), and z' shorter than z. We repeat this procedure until we obtain a reduced word. But then we know that it differs from w by a sequence of swaps, and hence is a word in Σ. Thus w' must have been a word in Σ as well. $\square$

Now we can prove the following proposition.

Proposition 2.9. *Let $\phi \in \mathrm{Aut}(A_\Gamma)$. Let $\Sigma \subseteq \Gamma$ be such that for all $x \in A_\Sigma$, the element $\phi(x)$ is conjugate to some element of A_Σ. Then there exists $y \in A_\Gamma$ such that*

$$\phi(A_\Sigma) \leqslant y^{-1} A_\Sigma y$$

Proof. Let the vertex set of Σ be $\{v_1, \ldots, v_m\}$; these letters are then generators of A_Σ. Let $Z_\Gamma = H_1(A_\Gamma; \mathbb{Z})$ denote the abelianisation of A_Γ, and let $Z_\Sigma \leqslant Z_\Gamma$ denote the image of A_Σ in the abelianisation. Note that Z_Σ is also generated by $\{v_1, \ldots, v_m\}$ in a natural way.

Let $\phi_* \colon Z_\Gamma \to Z_\Gamma$, be the induced isomorphism on abelianisations. Now, by assumption, ϕ_* induces a surjection $Z_\Gamma / Z_\Sigma \to Z_\Gamma / Z_\Sigma$; the group in question is however isomorphic to $\mathbb{Z}^n$ for some n, and such groups are Hopfian, so this induced morphism is an isomorphism. Hence $\phi_*|_{Z_\Sigma}$ is an isomorphism (since ϕ_* is), and so there exists an element $w \in A_\Sigma$, such that its image in Z_Σ is mapped by ϕ_* to the element $v_1 \cdots v_m$. This implies that w is mapped by ϕ to a conjugate (by some element y^{-1}) of a cyclically reduced word x, which contains each letter v_i. Crucially, an element $z \in A_\Gamma$ commutes with x if and only if $z \in C(A_\Sigma)$ (as two reduced words define the same element if and only if one can be obtained from the other by a sequence of swaps described above).

Consider $\psi = c(y)\phi$, so that $\psi(w) = x$. We now aim to show that $\psi(A_\Sigma) \leqslant A_\Sigma$.

Suppose for a contradiction that there exists $u \in A_\Sigma$ such that $\psi(u) \notin A_\Sigma$.

It could be possible that $\psi(u) \in A_{\mathrm{st}(\Sigma)} = A_\Sigma \times A_{\mathrm{lk}(\Sigma)}$. But the only elements in $A_\Sigma \times A_{\mathrm{lk}(\Sigma)}$ conjugate to elements in A_Σ are in fact the elements of A_Σ. Hence we can assume that $\psi(u) \notin A_{\mathrm{st}(\Sigma)}$.

Since $\psi(u)$ is conjugate to an element in A_Σ, yet is not in $A_{\mathrm{st}(\Sigma)}$, we can write

$$\psi(u) = a^{-1}b^{-1}v^{-1}x'vba$$

where the word is reduced, a is a subword containing only letters in

$$Z(\Sigma) * \mathrm{lk}(\Sigma)$$

the subword b contains only letters in $\Sigma \smallsetminus Z(\Sigma)$, the letter v does not lie in $\mathrm{st}(\Sigma)$, and x' is any subword. We will obtain a contradiction from this form of the word.

By assumption $\psi(wu) = xa^{-1}b^{-1}v^{-1}x'vba$ is conjugate to an element of A_Σ. In particular it lies in A_Σ after a sequence of reductions and cyclic reductions, by Lemma 2.8. We are going to visualise the situation as follows. We take a polygon with the number of vertices matching the length of the word describing $\psi(wu)$; now we label the vertices by letters so that going around the polygon clockwise and reading the labels gives us a cyclic conjugate of our word.

In this picture, a reduction or cyclic reduction consists of a deletion of two vertices V_1, V_2 labeled by the same letter but with opposite signs, and such that there is a path between V_1 and V_2 whose vertices are only labeled by letters commuting with the letter labeling V_1.

Recall that we have a finite sequence of basic moves which takes the word

$$xa^{-1}b^{-1}v^{-1}x'vba$$

to a word in A_Σ. Note that after every successive move we can still identify which part of our new word came from x, and which from $a^{-1}b^{-1}v^{-1}x'vba$.

We claim that we can never remove two occurrences of letters in

$$a^{-1}b^{-1}v^{-1}x'vba$$

along a path lying in this subword (and the same is true for the subword x)

Consider the first time we use a move which deletes the occurrences of a letter and its inverse in (what remains of) the subword $a^{-1}b^{-1}v^{-1}x'vba$ along a path lying in this subword. Let q denote the letter we are removing. Since the subword is reduced, such a move was not possible until a prior removal of a letter q' lying between the two letters we are removing, and such that q and q' do not commute. But now q' must have been removed by a path not contained in our subword, since the removal of q is the first move of this type. Therefore the path used to remove q' must contain q, and so q and q' must commute. This is a contradiction which shows the claim.

It is clear that we can remove all letters in a and a^{-1} with a path going over x; let us perform these moves first.

Since our finite sequence of moves takes us to a word in A_Σ, it must at some point remove the letters v and v^{-1}. Consider the first move removing occurrences of this letter. Note that both these occurrences must lie in the subword $v^{-1}x'v$. Hence our move removes this occurrences along a path containing all of x. We have however assumed that v does not commute with x, and so we must have first removed an occurrence of a letter q from x, where q does not commute with v. Let us denote this instance of q by q_1. The occurrence of q^{-1} which we remove together with q_1, denoted by q_1^{-1}, cannot lie in x; it must therefore lie in b or b^{-1}, since otherwise q and v would commute. Without loss of generality suppose that q_1^{-1} lies in b^{-1}. Then b contains another instance of the letter q, say q_2. This letter must again be removed before we can remove any letters v, therefore the subword bxb^{-1} contains q_2^{-1} (with notation as before). This occurrence cannot lie in b; it cannot lie in b^{-1} either since otherwise it would commute with x and so lie in a^{-1} and not in b^{-1}. Thus it must lie in x. So x contains two instances of the letter q, namely q_1 and q_2^{-1}, and one of them can be taken to be the first letter of x, while the other can be taken to be the last letter of x (by swaps). Hence x is not cyclically reduced. This is a contradiction. □

2.3. **Automorphisms of a RAAG.** Let us here briefly discuss a generating set for the group $\mathrm{Aut}(A_\Gamma)$.

By work of Servatius [Ser] and Laurence [Lau2], $\mathrm{Aut}(A_\Gamma)$ is generated by the following classes of automorphisms:

i) Inversions
ii) Partial conjugations
iii) Transvections
iv) Graph symmetries

Here, an *inversion* maps one generator of A_Γ to its inverse, fixing all other generators.

A *partial conjugation* requires a vertex v in Γ whose star disconnects Γ. For such a v, a partial conjugation is an automorphism which conjugates all generators in one of the complementary components of $\mathrm{st}(v)$ by v and fixes all other generators.

A *transvection* requires vertices v, w with $\mathrm{st}(v) \supseteq \mathrm{lk}(w)$. For such v, w, a transvection is the automorphism which maps w to wv, and fixes all other generators. Transvections come in two types: *type I* occurs when

$$\mathrm{lk}(v) \supseteq \mathrm{lk}(w)$$

and *type II* when $v \in \mathrm{lk}(w)$.

The group $\widetilde{\mathrm{Aut}}(A_\Gamma)$ is defined to be the subgroup generated by all generators from our list except the transvections of type II (which are sometimes also called *adjacent transvections*).

A *graph symmetry* is an automorphism of A_Γ which permutes the generators according to a combinatorial automorphism of Γ.

The group $\mathrm{Aut}^0(A_\Gamma)$ is defined to be the subgroup generated by generators of the first four types, i.e. without graph symmetries.

The group $\widetilde{\mathrm{Aut}}^0(A_\Gamma)$ is defined to be the subgroup generated by all generators from our list except the transvections of type II and graph symmetries.

For each of the groups $\mathrm{Aut}^0(A_\Gamma)$, $\widetilde{\mathrm{Aut}}(A_\Gamma)$ and $\widetilde{\mathrm{Aut}}^0(A_\Gamma)$ we denote their respective images in $\mathrm{Out}(A_\Gamma)$ by $\mathrm{Out}^0(A_\Gamma)$, $\widetilde{\mathrm{Out}}(A_\Gamma)$ and $\widetilde{\mathrm{Out}}^0(A_\Gamma)$. We are following the notation of Charney–Vogtmann [CV] here.

Lemma 2.10. *Suppose that* $\mathrm{lk}(v)$ *is not a cone for all vertices* v *of* Γ. *Then* $\widetilde{\mathrm{Aut}}(A_\Gamma) = \mathrm{Aut}(A_\Gamma)$ *and* $\widetilde{\mathrm{Aut}}^0(A_\Gamma) = \mathrm{Aut}^0(A_\Gamma)$.

Proof. It is enough to show that the assumption prohibits the existence of adjacent transvections. Let us suppose (for a contradiction) that such a transvection exists; this is equivalent to assuming that there exist vertices v and w such that $v \in \mathrm{lk}(w) \subseteq \mathrm{st}(v)$. But in this case we have

$$\mathrm{lk}(w) = (v \cap \mathrm{lk}(w)) * (\mathrm{lk}(v) \cap \mathrm{lk}(w)) = v * (\mathrm{lk}(w) \smallsetminus v)$$

which is a cone. □

2.4. Markings.

Definition 2.11. We say that a path-connected topological space X with a universal covering $u\colon \widetilde{X} \to X$ is *marked* by a group A if and only if it comes equipped with an isomorphism

$$m\colon A \to \mathrm{Deck}(u)$$

which we call a *marking*.

Remark 2.12. Given a space X marked by a group A, we obtain an isomorphism $A \cong \pi_1(X, p)$ by choosing a basepoint $\widetilde{p} \in \widetilde{X}$ (where p denotes its projection in X). We adopt the notation that the image of a point or set under the universal covering map will be denoted as its *projection*. To keep the notation uniform, we will also call X the projection of $\widetilde{X}$.

Conversely, an isomorphism $A \cong \pi_1(X, p)$ together with a choice of a lift $\widetilde{p} \in \widetilde{X}$ of p determines the marking in the sense of the previous definition.

Definition 2.13. Suppose that we are given an embedding $\pi_1(X) \hookrightarrow \pi_1(Y)$ of fundamental groups of two path-connected spaces X and Y, both marked. A map $\iota\colon X \to Y$ is said to *respect the markings via the map* $\widetilde{\iota}$ if and only if $\widetilde{\iota}\colon \widetilde{X} \to \widetilde{Y}$ is $\pi_1(X)$-equivariant (with respect to the given embedding $\pi_1(X) \hookrightarrow \pi_1(Y)$), and satisfies the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\iota}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & Y \end{array}$$

We say that ι *respects the markings* if and only if such an $\widetilde{\iota}$ exists.

To keep then notation (slightly) more clean, given a space X_Δ we will denote its universal cover by $\widetilde{X}_\Delta$ (rather than $\widetilde{X_\Delta}$).

Next we describe a construction that allows to glue two marked spaces.

Lemma 2.14. *Suppose that for each $i \in \{0,1,2\}$ we are given a group A_i and a space X_i marked by this group. Suppose further that for each $i \in \{1,2\}$ we are given a monomorphism $\phi_i \colon A_0 \hookrightarrow A_i$, and a continuous embedding $\iota_i \colon X_0 \hookrightarrow X_i$ which respect the markings via a map $\widetilde{\iota}_i$.*

Then there exists a group A, a space X marked by A, and maps $\phi'_i \colon A_i \to A$ and $\iota'_i \colon X_i \to X$, the latter respecting the markings via maps $\widetilde{\iota}'_i$, such that the following diagrams commute

$$\begin{array}{ccc} A_0 & \xrightarrow{\phi_1} & A_1 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi'_1} \\ A_2 & \xrightarrow{\phi'_2} & A \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\iota_1} & X_1 \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota'_1} \\ X_2 & \xrightarrow{\iota'_2} & X \end{array} \qquad \begin{array}{ccc} \widetilde{X}_0 & \xrightarrow{\widetilde{\iota}_1} & \widetilde{X}_1 \\ \downarrow{\scriptstyle \widetilde{\iota}_2} & & \downarrow{\scriptstyle \widetilde{\iota}'_1} \\ \widetilde{X}_2 & \xrightarrow{\widetilde{\iota}'_2} & \widetilde{X} \end{array}$$

We will refer to the construction in this lemma as *obtaining $\widetilde{X}$ from $\widetilde{X}_1$ and $\widetilde{X}_2$ by gluing* $\mathrm{im}(\widetilde{\iota}_1)$ *to* $\mathrm{im}(\widetilde{\iota}_2)$. Note that the projection X of $\widetilde{X}$ is in fact obtained by an honest gluing of the projections X_1 and X_2 along the respective subspaces.

Proof. We define A and X to be the push-outs of the appropriate diagrams. Take a point $p \in X_0$, and its lift $\widetilde{p} \in \widetilde{X}_0$. The point p has its copies in X_1, X_2 and X; let $\widetilde{q}$ denote some lift of p in $\widetilde{X}$. We also have copies of the point $\widetilde{p}$ in $\widetilde{X}_1$ and $\widetilde{X}_2$. We define the maps $\widetilde{\iota}'_i$ to be the unique maps satisfying the following commutative diagrams of pointed spaces

$$\begin{array}{ccc} \widetilde{X}_i, \widetilde{p} & \xrightarrow{\widetilde{\iota}'_i} & \widetilde{X}, \widetilde{q} \\ \downarrow & & \downarrow \\ X_i, p & \xrightarrow{\iota_i} & X, p \end{array}$$

where the vertical maps are the coverings. The verification that these maps satisfy the third commutative diagram above is easy. □

2.5. **Free groups and 3-manifolds.** Let $W = W_n$ be a doubled handlebody of genus n (in other words, the connected sum of n copies of $S^1 \times S^2$). We fix a marking of W and a base point $\widetilde{p}$ in the universal cover of W, and obtain the corresponding identification of $\pi_1(W,p)$ with the free group F_n where $p \in W$ is the projection of $\widetilde{p}$. In this way elements of F_n correspond to homotopy classes of loops in W (based at p), and conjugacy classes of elements in F_n correspond to free homotopy classes of closed curves in W.

An *essential sphere* σ in W is an embedded sphere in W which does not bound a ball in W. Usually we will not distinguish between a sphere and its isotopy class. A *sphere system* is a finite collection of disjoint spheres which are pairwise non-isotopic. A sphere system is called *simple* if each of its complementary components is simply connected.

The sphere graph $\mathcal{S}(W)$ is the graph whose vertex set is the set of isotopy classes of essential spheres in W. Two spheres are connected by an edge if and only if they can be realised disjointly.

By a theorem of Laudenbach [Lau1] there is a short exact sequence

$$1 \to K \to \mathrm{Mcg}(W) \to \mathrm{Out}(F_n)$$

where $\mathrm{Mcg}(W)$ is the mapping class group of W. The map $\mathrm{Mcg}(W) \to \mathrm{Out}(F_n)$ is induced by the action of diffeomorphisms on the fundamental group. The kernel K is finite and generated by sphere Dehn twists. The exact definition of sphere Dehn twists is irrelevant here, but we note that these mapping classes preserve the homotopy class of each essential sphere and each essential closed curve in W. As a consequence, outer automorphisms of F_n act on the sets of essential spheres and essential curves in W, preserving topological properties like disjointness. By abuse of notation we will in the sequel simply speak about outer automorphisms acting on spheres and curves. Readers who dislike this shortcut notation should instead replace groups $H < \mathrm{Out}(F_n)$ by their preimage in $\mathrm{Mcg}(W)$, and work with groups that genuinely act on W (up to isotopy).

We will use the notion of *outermost surgery.* We refer to Section 7.2 of [HOP] for a detailed description of the version we use. We recall the necessary notation here for convenience. Let σ and σ' be two spheres which intersect non-trivially. We say that σ and σ' are in *normal position* if the following holds for every pair $\widetilde{\sigma}, \widetilde{\sigma}'$ of lifts of σ and σ' to the universal cover of W: either $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are disjoint, or they intersect in a single circle. In the latter case, we also require that none of the disks in $\widetilde{\sigma} \smallsetminus \widetilde{\sigma}'$ is homotopic into $\widetilde{\sigma}'$ relative to its boundary. This notion of normal position for spheres extends Hatcher's notion of normal position for sphere systems (see [Hat] and Section 7.1 of [HOP]).

Suppose now that σ and σ' are in normal position. A disk $D \subset \sigma$ is *outermost* if $D \cap \sigma' = \partial D$. Denote by D'_+ and D'_- the two components of $\sigma' \smallsetminus \partial D$. Then we say that the spheres $\{D \cup D'_+, D \cup D'_-\}$ are an *outermost surgery pair of σ' in direction of σ*. By the uniqueness of normal position, the notion of outermost disks and outermost surgery pairs is well-defined for isotopy classes of spheres (see [HOP, Lemma 7.3]).

Definition 2.15. Let $\mathbf{P}$ be a property that isotopy classes of spheres can exhibit. We say that $\mathbf{P}$ is *surgery invariant* if the following holds. Suppose that σ, σ' are spheres with $\mathbf{P}$. Then any member of a surgery pair of σ in direction of σ' has $\mathbf{P}$.

In addition to surgery, we need a second way to modify sphere systems. Namely, let Σ be a sphere system. Consider the 3-manifold $W_0 = W \smallsetminus \Sigma$ which is obtained from W by removing Σ from W and taking the closure with respect to some path metric. The manifold W_0 has a boundary component for each side of each sphere in W. There is a natural map from W_0 to W, and we will identify subsets of W_0 with subsets of W using this map. In addition, we consider the manifold W' obtained by gluing a ball to each boundary component of W_0. Again, there is a natural map $W_0 \to W'$ and we will often identify sets in W_0 with their images in W' under this map. At this point, it is helpful to contrast isotopies in the different manifolds we consider. Take two spheres σ, σ' in the complement of Σ. These are isotopic in W if and only if they are isotopic in W_0. However, their images in W' may be isotopic, even if they are not isotopic in W_0 (there may be a boundary component "between them"). To avoid confusion, we will usually indicate in what manifold the spheres we consider are supposed to be isotopic.

Now let $\Sigma' = \{\sigma'_1, \dots, \sigma'_k\}$ be any sphere system in W'. We define a graph $\mathcal{F}(\Sigma')$ in the following way. A vertex of $\mathcal{F}(\Sigma')$ corresponds to the isotopy class (in W_0) of a sphere system $\{\sigma_1, \dots, \sigma_k\}$ in W_0 such that σ_i is isotopic to σ'_i in W' for each i. Note in particular that each sphere system corresponding to a vertex of $\mathcal{F}(\Sigma')$ corresponds to a sphere system with the same number of spheres as Σ'. We join two vertices with an edge if the corresponding sphere systems can be realised disjointly. Thus, cliques in $\mathcal{F}(\Sigma')$ correspond to sphere systems, which are homotopic in W' to Σ', but where several spheres are allowed to be homotopic in W'.

Now let $\Sigma \cup \Sigma_0$ be a sphere system, so that Σ_0 defines a sphere system Σ' in W' (i.e. not all spheres in Σ_0 become inessential in W'). Let Σ_1 be a sphere system corresponding to a vertex in $\mathcal{F}(\Sigma')$. We then say that the sphere system $\Sigma \cup \Sigma_1$ (in W) is obtained from $\Sigma \cup \Sigma_0$ by a *slide move*.

Definition 2.16. Let **P** be a property that spheres can exhibit. We say that a sphere system *satisfies* **P** if and only if each of its members exhibits **P**.

We say that the property **P** is *preserved under slides* if and only if every sphere system obtained by a slide move from a sphere system satisfying **P** also satisfies **P**.

The following theorem is a central tool of this section. For the formulation, given a finite subgroup $H < \mathrm{Out}(F_n)$ and a property **P** of spheres invariant under isotopy, we say that H *preserves* **P** if for every sphere system Σ with **P** and every element $h \in H$, any sphere system representing $h\Sigma$ has **P**.

Theorem 2.17. *Let* **P** *be a property of spheres which is preserved under isotopy, outermost surgery and slides. Let H be a finite subgroup of* $\mathrm{Out}(F_n)$ *which preserves* **P**.

i) If there exists a sphere with **P***, then there exists a sphere system which is preserved by H satisfying* **P**.

ii) Let Σ be a sphere system, satisfying **P***, which is preserved by H.*

Suppose there is a system Σ_0 so that $\Sigma \cup \Sigma_0$ satisfies **P** *and Σ_0 defines a nontrivial sphere system in W'. Then there is a strictly bigger sphere system $\Sigma' \supset \Sigma$, satisfying* **P***, which is preserved by H.*

The proof uses techniques developed in [HOP]. In particular, we use the notion of a *dismantling projection*. Intuitively speaking, a dismantling projection for a graph Θ is a procedure that yields an explicit way of contracting the flag complex defined by Θ to a point. Formally, we have

Definition 2.18. Let Θ be a (possibly infinite) graph, and denote by V its vertex set. Fix a vertex $\sigma \in V$.

A *dismantling projection* Π (or *σ-projection*) is a map that assigns to each vertex $\rho \in V \smallsetminus \{\sigma\}$ a nonempty finite set $\Pi(\rho)$ consisting of pairs of elements of V, which satisfies the following axioms:

(i) For each finite set of vertices $R \subseteq V$ with nonempty $R \smallsetminus \{\sigma\}$ there is an *exposed* vertex $\rho \in R \smallsetminus \{\sigma\}$, that is a vertex with $\mathrm{st}(\rho) \cap R \subseteq \mathrm{st}(\pi)$ for both π from some pair of $\Pi(\rho)$.

(ii) There is no cycle of vertices $\rho_0, \ldots, \rho_{m-1} \in V$ with $\rho_{i+1} \in \Pi^*(\rho_i)$, where i is considered modulo m. Here, $\Pi^*(\rho) \subseteq V$ denotes the set of all vertices appearing in pairs from $\Pi(\rho)$.

Note that in [HOP] the notation $N(\rho)$ is used to denote the star of a vertex ρ, but in this article we use the notation $\mathrm{st}(\rho)$ to stay consistent with the remaining sections.

In the case where Θ is the sphere graph of W_n, there is a dismantling projection given by surgery: for a base sphere σ and any other sphere ρ, we can define $\Pi_\sigma(\rho)$ to be the set of all possible outermost surgery pairs of ρ in direction of σ. This dismantling projection is defined and studied in detail in Section 7.2 of [HOP].

The utility of dismantling projections stems from the following theorem, which follows from Lemma 2.7 of [HOP] and Polat's theorem (see e.g. Theorem 2.4 of [HOP]).

Theorem 2.19. *Let Θ be a finite graph with a dismantling projection (for some σ). Then any finite group of automorphisms of Θ fixes a clique in Θ.*

Even though the sphere graph is far from being finite (in fact, it is not even locally finite), one can use this theorem to find fixed spheres under the action of finite subgroups of $\mathrm{Out}(F_n)$. Namely, a version of Theorem 2.19 also holds for infinite graphs and finite group actions, provided one can find a finite "hull", which is "convex with respect to Π". Formally, we need

Definition 2.20. Let Θ be a graph with vertex set V and let Π be a σ-projection. A subset $R \subseteq V$ is called Π-*convex* if for every vertex $\rho \in R \smallsetminus \{\sigma\}$ each pair in $\Pi(\rho)$ intersects R.

The following is a rephrasing of Proposition 2.11 part ii) of [HOP].

Proposition 2.21. *Let Θ be a graph with dismantling projections Π_σ for every vertex σ. Let R be a finite subset of vertices, and let $\sigma \in R$ be one of them. Suppose that a finite group H acts on Θ such that for all $\sigma' \in H\sigma$ the set R is $\Pi_{\sigma'}$-convex. Then H fixes a clique in Θ.*

Note that the fixed clique need not be completely contained in R (in fact, R need not be invariant under the group H). The finite hulls R as required by Proposition 2.21 are constructed carefully for sphere graphs in Section 7.3 of [HOP]. Here, we just give a brief overview.

Fix once and for all a maximal (with respect to inclusion) sphere system S_0. One can then define the *width* of a sphere with respect to S_0. The formal definition can be found in the beginning of Section 7.3 of [HOP]; intuitively, the width of σ is the number of connected components of $\sigma \setminus S_0$ if σ is made to intersect S_0 minimally.

Now, let $\sigma_1, \ldots, \sigma_k$ be any collection of essential spheres (e.g. the set $H\sigma$ required in Proposition 2.21), and let w be the maximal width of any of the σ_i. We define

$$\mathcal{H}(\sigma_1, \ldots, \sigma_k)$$

to be the set of all spheres of width at most $2w$. This is a finite set of spheres which will serve as the set R in Proposition 2.21. One can show that this set is $\Pi_{\sigma'}$-convex for any $\sigma' = \sigma_i$ [HOP, Lemma 7.9].

Proof of Theorem 2.17. i) Let $\mathcal{S}^{\mathbf{P}}(W)$ be the subgraph of $\mathcal{S}(W)$ spanned by spheres with $\mathbf{P}$. By assumption this subgraph is nonempty. Since $\mathbf{P}$ is preserved by H, the action of H on the sphere graph restricts to an action of H on $\mathcal{S}^{\mathbf{P}}(W)$.

By surgery invariance of $\mathbf{P}$, the dismantling projection Π_σ defined above (and in Section 7.2 of [HOP]) restricts to a dismantling projection on $\mathcal{S}^{\mathbf{P}}(W)$ for every base vertex $\sigma \in \mathcal{S}^{\mathbf{P}}(W)$. Similarly, define a finite hull

$$\mathcal{H}^{\mathbf{P}}(\sigma_1, \ldots, \sigma_k) = \mathcal{H}(\sigma_1, \ldots, \sigma_k) \cap \mathcal{S}^{\mathbf{P}}(W)$$

for the finite hulls $\mathcal{H}$ defined above (and in Section 7.3 of [HOP]). Surgery invariance of $\mathbf{P}$ and the fact that the hulls $\mathcal{H}(\sigma_1, \ldots, \sigma_k)$ are Π_σ–convex for all $\sigma = \sigma_i$ implies the same convexity for the hulls $\mathcal{H}^{\mathbf{P}}$.

Now assertion i) follows from Proposition 2.21: given any finite group $H < \mathrm{Out}(F_n)$, let σ be a sphere with $\mathbf{P}$ and put $\{\sigma_1, \ldots, \sigma_k\} = H\sigma$. Then $\mathcal{H}^{\mathbf{P}}(\sigma_1, \ldots, \sigma_k)$ satisfies the requirements of Proposition 2.21. Therefore, the there is an H-invariant clique in $\mathcal{S}^{\mathbf{P}}(W)$, which corresponds to the desired sphere system.

ii) We follow the strategy of Section 8 of [HOP] to find an H-invariant extension of Σ. Namely, let W_0 be the disjoint union of the complementary components of Σ. Let W' be the manifold obtained from W_0 by gluing a ball to each boundary sphere of W_0.

Say that a sphere σ' in W' has $\mathbf{P}$, if there is a sphere σ in $W_0 \subset W$ which has $\mathbf{P}$ (as a sphere in W), and which is homotopic (thus isotopic) to σ' in W_0. By assumption, there is a sphere in W' which has $\mathbf{P}$. Also, H acts on these spheres, and this notion of $\mathbf{P}$ in W' is also surgery invariant.

Since H preserves Σ, it acts on the sphere graph of W_0 and W'. Namely, the sphere graph of W_0 can be identified with the link of Σ

in the sphere graph of W. The assumption tells us that there exists a sphere in $W_0 = W \smallsetminus \Sigma$, which has $\mathbf{P}$; also by assumption it is still an essential sphere with $\mathbf{P}$ in W'. Hence, by statement i) applied to W' and this sphere, there is a sphere system $\Sigma' = \{\sigma'_1, \dots, \sigma'_k\}$ in W' which is fixed by the action of H on the sphere graph of W'.

Let $\mathcal{F}(\Sigma')$ be the graph defined above. Since Σ' has $\mathbf{P}$ and $\mathbf{P}$ is preserved under slides, each vertex of $\mathcal{F}(\Sigma')$ represents a sphere system in W_0 with $\mathbf{P}$.

Arguing as in Section 8.2 of [HOP], one can define a dismantling projection of $\mathcal{F}(\Sigma')$. This dismantling projection exactly uses the slide moves described above. Then, one can define finite hulls using intersection numbers with arc systems. Details are found in Section 8.2 of [HOP]. In our case, we can simply use Lemma 8.4 of [HOP] which states that there is a dismantlable finite subgraph of $\mathcal{F}(\Sigma')$, and therefore an H–invariant clique. By slide invariance of $\mathbf{P}$ this then corresponds to the desired H–invariant extension of Σ with $\mathbf{P}$. □

3. Adapted Realisation

Definition 3.1. Let A and H be groups, and let $\phi\colon H \to \mathrm{Out}(A)$ be a homomorphism. We say that a metric space X, on which H acts by isometries, *realises* the action ϕ if and only if X is marked by A, and the action of H on conjugacy classes in $\pi_1(X) \cong A$ is equal to the one induced by ϕ.

If X is a (metric) graph or a cube complex, we require the action to respect the combinatorial structure as well.

Before stating a result fundamental for the entire article, we need to observe that whenever we have an action $H \to \mathrm{Out}(F_n)$, where F_n denotes the free group of rank n, such that the action preserves the conjugacy class of some free factor $A \leqslant F_n$, then in fact we get an induced action $H \to \mathrm{Out}(A)$. This follows from the fact that non-trivial free factors are their own normalisers in F_n.

Throughout, we consider only graphs without vertices of valence 1, that is without leaves.

Theorem 3.2 (Adapted Realisation)**.** *Let H be finite and $\phi\colon H \to \mathrm{Out}(F_n)$ be a homomorphism. Suppose*

$$F_n = A_1 * \cdots * A_k * B$$

is a free splitting such that $\phi(H)$ preserves the conjugacy class of A_i for each $1 \leqslant i \leqslant k$. Let $\phi_i : H \to \mathrm{Out}(A_i)$ denote the induced actions. for each i let X_i be a marked metric graph realising ϕ_i.

Then there is a marked metric graph X with the following properties.

i) H acts on X by combinatorial isometries, realising ϕ.
ii) There are isometric embeddings $\iota_i\colon X_i \to X$ which respect the markings via maps $\widetilde{\iota}_i$.

iii) The embedded subgraphs X_i are preserved by H, and the restricted action induces the action ϕ_i on X_i up to homotopy.

When $\pi_1(X_i) = A_i \ncong \mathbb{Z}$, the map ι_i is actually H-equivariant (and not just H-equivariant up to homotopy).

Note that since we only require the embeddings ι_i to be isometric, the presence of vertices of valence 2 is completely irrelevant. They might appear in the graphs X_i as well as X to make the action of H combinatorial, but we do not require these appearances to agree under ι_i.

Corollary 3.3. *Suppose that in the above theorem we have*

$$F_n = A_1 * A_2$$

Then, in addition to the properties above, we can take the graph X so that $X = \mathrm{im}(\iota_1) \cup \mathrm{im}(\iota_2)$, and $\mathrm{im}(\widetilde{\iota}_1) \cap \mathrm{im}(\widetilde{\iota}_2)$ is a single point.

Lemma 3.4. *The projection of the intersection point*

$$\mathrm{im}(\widetilde{\iota}_1) \cap \mathrm{im}(\widetilde{\iota}_2)$$

in X is H-fixed.

Proof. The Seifert–van Kampen Theorem tells us that the intersection

$$\mathrm{im}(\iota_1) \cap \mathrm{im}(\iota_2)$$

is simply-connected, and so in particular path connected. But this means that it can be lifted to the universal cover in such a way that the lift lies within

$$\mathrm{im}(\widetilde{\iota}_1) \cap \mathrm{im}(\widetilde{\iota}_2)$$

which is just a singleton. Hence so is $\mathrm{im}(\iota_1) \cap \mathrm{im}(\iota_2)$. Now this point is the intersection of two H-invariant subspaces, and so is itself H-invariant, and thus H-fixed. □

The following subsections contain the proof of the Adapted Realisation (Theorem 3.2) and Corollary 3.3.

3.1. **Finding isolating sphere systems.** For the rest of this section, we let $\mathcal{A}$ be a set of free homotopy classes of closed curves in W. The guiding example to have in mind in light of Theorem 3.2 is the set of all closed curves representing conjugacy classes contained in some free factor A_i.

Definition 3.5. Let $\mathcal{A}$ be a set of free homotopy classes of closed curves in W. A sphere system Σ is $\mathcal{A}$-*adapted* if each simple closed curve $c \in \mathcal{A}$ can be made disjoint from Σ by a free homotopy.

The goal of this section is to show that if a finite group H of outer automorphisms of the free group preserves $\mathcal{A}$ and if there is any $\mathcal{A}$-adapted sphere system, then there is also an $\mathcal{A}$-adapted sphere system which is preserved by H.

One core tool to these arguments is a normal position for arcs and curves with respect to sphere systems. This is done for arcs in detail e.g. in

Section 8.1 of [HOP]. The proof of Lemma 8.2 in [HOP] verbatim also applies to curves, and yields

Lemma 3.6. *Let Σ, Σ' be two sphere systems which are in normal position with respect to each other. Let α be an embedded curve in W (or let a be a properly embedded arc). Then α (or a) can be homotoped (relative to its endpoints) to be in minimal position with respect to Σ and Σ' at the same time.*

Without going into detail about normal position, we note the following corollary, which is most important for our applications. For curves, this was also shown in Lemma 2.2 of [HV].

Corollary 3.7. *Suppose Σ, Σ' are sphere systems with the property that a curve α (or an arc a) can be made disjoint by homotopies from Σ and Σ' individually.*

Then α (or a) can be homotoped to be disjoint from $\Sigma \cup \Sigma'$.

Lemma 3.8.
i) Suppose that σ, σ' are two $\mathcal{A}$-adapted spheres, and let σ_+ be obtained from σ' by an outermost surgery in direction of σ. Then σ_+ is $\mathcal{A}$-adapted.
ii) Suppose that $\Sigma \cup \Sigma_0$ is an $\mathcal{A}$-adapted sphere sphere system, and suppose that $\Sigma \cup \Sigma_1$ is obtained from $\Sigma \cup \Sigma_0$ by a slide move. Then $\Sigma \cup \Sigma_1$ is $\mathcal{A}$-adapted.

Proof. i) Let α be a curve which is disjoint up to homotopy from σ and σ'. Then α can be made disjoint from $\sigma \cup \sigma'$ by Corollary 3.7.

This implies that α is also disjoint from any surgery of σ in direction of σ', which implies i).

ii) Let α be an element of $\mathcal{A}$ which is disjoint from $\Sigma \cup \Sigma_0$. If $\Sigma \cup \Sigma_1$ is obtained from Σ by a slide move, then Σ_1 and Σ_0 are homotopic after filling in the boundary of $W \setminus \Sigma$. In particular, α is disjoint from Σ_1 up to homotopy in the manifold W'. Now put α in minimal position with respect to $\Sigma \cup \Sigma_1$. If α would intersect Σ_1 in this minimal position, then α would also have an essential intersection with Σ_1 in W' which is impossible. □

Suppose now that $\mathcal{A}$ is the disjoint union of sets $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Let Σ be an $\mathcal{A}$-adapted sphere system. We say that Σ is *isolating*, if the following holds. Suppose that α and β are closed curves representing free homotopy classes contained in different sets $\mathcal{A}_i$. Then α and β can be chosen so that they are contained in different complementary components of Σ.

Theorem 3.9. *Let $\phi : H \to \mathrm{Out}(F_n)$ be a finite action which permutes the sets $\mathcal{A}_i$. Suppose that there is an isolating sphere system Σ_0. Let Σ be an adapted sphere system preserved by H. If Σ is not isolating, then there is an adapted sphere system Σ' which is preserved by H and which strictly contains Σ.*

Proof. Let Σ be a sphere system as above. We use the manifolds W_0 and W' as in the proof of Theorem 2.17. Recall that W_0 is simply the union of the complementary components of Σ. It is a manifold with boundary, which consists of 2-spheres (corresponding to the sides of the spheres in Σ). The manifold W' is obtained form W_0 by gluing a ball to each boundary component of W_0.

Suppose that Σ is not isolating. We first claim that there is an adapted sphere system Σ' in W'. Since Σ is not isolating, there are curves α, β contained in different sets $\mathcal{A}_i$, which are contained in the same connected component W^1 of W_0. Put Σ_0 in normal position with respect to Σ. Then the intersection of Σ_0 with W^1 is a collection of spheres with boundary, say $S_1, \ldots, S_N$. Since Σ_0 is isolating, the union of the S_i disconnects W_0 so that α and β lie in different complementary components of the spheres S_i.

By Lemma 3.7 of [HH] one can glue disks, which are contained in Σ, to each boundary component of each S_i to obtain a sphere system Σ_1 in the complement of Σ. Since both Σ and Σ_0 are adapted, every curve α corresponding to an element of $\mathcal{A}$ can be made disjoint from both Σ and Σ_0 at the same time. Since each sphere in Σ_1 is obtained by gluing subsets of Σ and Σ_0, α is then also disjoint from Σ_1. Hence, Σ_1 is adapted.

The sphere system Σ_1 has the property that α and β lie in different complementary components. Namely, suppose not. Then there would be an arc joining α to β in the complement of Σ which is disjoint from Σ_1. By Lemma 3.8 of [HH] this arc could then also be made disjoint from all of the S_i (the statement of Lemma 3.8 in [HH] is for curves, not arcs, but the proof works for arcs in the same way). This however violates the assumption that Σ_0 is isolating.

In particular, Σ_1 also defines a sphere system in W'. Namely, if not, then each sphere in Σ_1 would bound a ball in W', and hence all essential curves (thus α and β) would lie on the same side of Σ_1.

In particular, the prerequisites of Theorem 2.17 ii) are met, and the theorem follows. □

An immediate consequence is the following.

Corollary 3.10. *Let $\phi : H \to \mathrm{Out}(F_n)$ be a finite action which permutes the sets $\mathcal{A}_i$. Suppose that there is an isolating sphere system Σ^0_{iso}. Then a maximal H-invariant adapted sphere system is isolating.*

3.2. **Proof of adapted realisation.** In this section we prove Theorem 3.2.

In order to do so, we need to recall the duality between marked metric graphs (without leaves) and weighted sphere systems in doubled handlebodies. A *weighted sphere system* is a sphere system Σ together with a positive real number r_i for each sphere $\sigma_i \in \Sigma$.

The following is standard and folklore, see e.g. [Vog, Section 1.3] for a discussion.

Lemma 3.11. *Suppose that $n > 1$. Let W be a marked doubled handlebody of rank n, and let Σ be a weighted sphere system in W. Then the dual graph $X(\Sigma)$ to Σ inherits a marking from W and a metric from the weights of the spheres. Furthermore, every marked metric graph without leaves and vertices of valence 2 of rank n arises in this way.*

If $H < \mathrm{Out}(F_n)$ is a finite subgroup which fixes Σ, then the action of H is realised by the dual graph $X(\Sigma)$. Every marked metric graph realising the action of H is dual to a weighted sphere system fixed by H, provided that the graph has no vertices of valence 1 or 2.

Note that when $n = 1$ the situation is slightly more complicated: the metric graphs without leaves realising the action $\phi\colon H \to \mathrm{Out}(\mathbb{Z})$ can have any (positive) number of vertices (necessarily of valence 2), whereas the handlebody allows only one isotopy class of essential 2-spheres. This is however not a problem since if one of the free factors in

$$F_n = A_1 * \cdots * A_k * B$$

is cyclic, then the given graph realising this action is only supposed to be embedded in the graph X isometrically, and the action is only supposed to be H-equivariant up to homotopy. This essentially means that we can treat the given graph as if it only has a single vertex, and the action is either trivial or a flip fixing this vertex. These two situations have a direct analogue in the handlebody picture.

Proof of Theorem 3.2. Let

$$F_n = A_1 * \cdots * A_k * B$$

be a splitting of a free group. Let $\phi\colon H \to \mathrm{Out}(F_n)$ be a finite action which preserves the conjugacy class of A_i for each $1 \leqslant i \leqslant k$.

Recall that $\phi_i\colon H \to \mathrm{Out}(A_i)$ denotes the induced finite actions and X_i are given marked metric graphs on which ϕ_i is realised for each $1 \leqslant i \leqslant k$.

As in Section 3.1, denote by $\mathcal{A}_i$ the set of all homotopy classes of closed curves in W (the doubled-up handlebody of rank n) that represent elements in A_i. Then there exists an isolating sphere system Σ^0_{iso} – choose it to be a sphere system such that the fundamental groups of the complementary regions are $A_1, \ldots, A_k$ and B. Hence, by Theorem 2.17 there is some maximal $\mathcal{A}$-adapted sphere system Σ_{iso} which is preserved by H. Corollary 3.10 implies that Σ_{iso} is also isolating.

Denote the complementary components of Σ_{iso} by $W_1, \ldots, W_l$. By reordering the indices, we may assume that the fundamental group of W_i contains A_i, for $1 \leqslant i \leqslant k$. In fact, we may assume that the fundamental group of W_i equals A_i by maximality of Σ_{iso}, since if the fundamental group is strictly larger than A_i then there is an extension of the isolating sphere system Σ_{iso} (as then A_i defines a free factor of the fundamental group of W_i). By Theorem 2.17 ii) we can then find an extension which is preserved by H.

Consider now a specific i with $1 \leqslant i \leqslant k$. The action of H on spheres in W preserves spheres lying in W_i, in particular the boundary. Thus H acts on $\pi_1(W_i)$ by outer automorphisms, realising the action of ϕ_i on A_i.

Let $W(A_i)$ be a doubled handlebody marked by the group A_i. First suppose that the rank of A_i is at least 2. In this case Lemma 3.11 applies and yields associated to the marked metric graph X_i a dual weighted sphere system $\Sigma(X_i)$ in $W(A_i)$ which is fixed by the action of H_i.

The marking m of W induces a marking $m|_{A_i}$ of W_i. Let W_i' be the manifold obtained by gluing balls to the boundary components of W_i. The marking $m|_{A_i}$ of W_i defines a marking of W_i', which by slight abuse of notation we denote by the same symbol.

The manifold W_i' is diffeomorphic to $W(A_i)$, and we can choose a diffeomorphism such that the marking of $W(A_i)$ agrees with the restriction marking $m|_{A_i}$. Namely, any two markings of $W(A_i)$ differ by precomposition with an automorphism of the free group A_i. By Laudenbach's theorem, there is a self-diffeomorphism which realises this automorphism. Furthermore, this diffeomorphism is unique up to a map which preserves the homotopy class of every sphere system and every closed curve.

Using this diffeomorphism, the image of $\Sigma(A_i)$ is a sphere system Σ_i' in W_i', such that the dual graph with the marking induced by W is the marked metric graph X_i. By construction, Σ_i' is preserved by the action of H on W_i'.

If the rank of A_i is 1, then we let $\Sigma(A_i)$ be the (unique) essential sphere in W_i' (which is then homeomorphic to $S^1 \times S^2$), and we assign the weight according to the length of the loop in X_i.

Let $\Sigma_i' = \{\sigma_1', \dots, \sigma_r'\}$. We now consider the graph $\mathcal{F}(\Sigma_i')$ as defined above. Recall that vertices of $\mathcal{F}(\Sigma_i')$ correspond to sphere systems $\{\sigma_1, \dots, \sigma_r\}$ in W_i, such that σ_j and σ_j' are isotopic as spheres in W_i'.

Since H preserves Σ_{iso} and Σ_i', it acts on $\mathcal{F}(\Sigma_i')$. Arguing as in the proof of part ii) of Theorem 2.17, one uses dismantlability to show that H fixes a clique in $\mathcal{F}(\Sigma_i')$. Such a clique corresponds to a sphere system Σ_i in W_i (and hence in W) each sphere of which is homotopic (in W_i') to one of the σ_i'.

However, in the sphere system Σ_i several spheres may be homotopic to the same sphere σ_j'. This happens exactly when Σ_i contains two spheres which cobound a simply-connected region containing boundary components of W_0. Since W_0 may have many boundary components, many spheres of Σ_i can be homotopic to the same sphere σ_j'. In other words, the graphs dual to Σ_i have additional edges which do not appear in the graphs dual to Σ_i'. To guarantee that the dual graphs will nevertheless be isometric (as claimed in the theorem) we need to make a final modification.

The sphere system Σ_i naturally inherits the structure of a weighted sphere system from Σ_i'. Namely, let $\sigma_1, \ldots, \sigma_l \in \Sigma$ be the spheres which are homotopic to σ_j'. Then we assign each of them the weight $\frac{1}{l}$ times the weight of σ_j'.

Since H_i preserves the weighted sphere system Σ_i', the action of H preserves Σ_i respecting the weights.

The union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k \cup \Sigma_{\mathrm{iso}}$ is then a weighted (assigning weight one to each sphere in Σ_{iso}) sphere system which is preserved by H.

If Σ_{iso} has a complementary component which does not correspond to one of the A_i, then extend Σ in any way to a sphere system. Again by Theorem 2.17 ii) we may extend Σ to a weighted simple sphere system which is preserved by H and contains Σ_i and Σ_{iso}.

As Σ is preserved by H, it acts as isometries on the metric graph X dual to the weighted sphere system Σ. As H preserves Σ_{iso}, the graph X has subgraphs Y_i, which are dual to the sub-sphere systems Σ_i and which are invariant under the action of H. By construction the graphs Y_i are isometric to X_i. In particular there is an isometric embedding $\iota_i \colon X_i \to X$ as claimed.

Since the action of H on spheres disjoint from Σ_{iso} induces the action of H on spheres and curves in W_i', the isometry ι_i is ϕ_i-equivariant up to homotopy. If the graph X_i has rank at least 2, this implies that ι_i is actually ϕ_i-equivariant as claimed, since the only action on a leafless finite graph of rank $n \geqslant 2$ which is trivial on the conjugacy classes in the fundamental group is the trivial action.

Hence, X is the metric graph whose existence is claimed in Theorem 3.2.

To show the corollary to the theorem, it suffices to remark that if the splitting $F_n = A_1 * A_2$ has only two free factors, then there is a unique separating sphere $\sigma \subset W$ with the property that the fundamental groups of the complementary components are A_i. Thus, ϕ automatically preserves this sphere σ, which is a maximal separating sphere system. Applying the construction as before, we get a graph X consisting of X_1, X_2 joined by a single edge (corresponding to σ). Collapsing that edge yields the graph as claimed in the corollary. □

4. Systems of subgraphs and invariance

Definition 4.1. Given a homomorphism $\phi \colon H \to \mathrm{Out}(A_\Gamma)$ we define *the system of invariant subgraphs* $\mathcal{L}^\phi$ to be the set of induced subgraphs $\Delta \subseteq \Gamma$ (including the empty one) such that $\phi(H)$ preserves the conjugacy class of $A_\Delta \leqslant A_\Gamma$. Note that $\mathcal{L}^\phi$ is partially ordered by inclusion.

We next show that $\mathcal{L}^\phi$ is closed under taking intersections and certain unions.

Lemma 4.2. *Let* $\Delta, \Sigma \in \mathcal{L}^\phi$. *Then*

i) $\Delta \cap \Sigma \in \mathcal{L}^\phi$.

ii) If $\mathrm{lk}(\Delta \cap \Sigma) \subseteq \mathrm{st}(\Delta)$ *then* $\Delta \cup \Sigma \in \mathcal{L}^\phi$

Proof. i) Pick $h \in H$. Since $\Delta \in \mathcal{L}^\phi$, there exists a representative

$$h_1 \in \mathrm{Aut}(A_\Gamma)$$

of $\phi(h)$ such that $h_1(A_\Delta) = A_\Delta$. Analogously, there exists $h_2 \in \mathrm{Aut}(A_\Gamma)$ representing $\phi(h)$ such that $h_2(A_\Sigma) = A_\Sigma$.

Since h_1 and h_2 represent the same element in $\mathrm{Out}(A_\Gamma)$, we have

$$h_1^{-1}h_2 = c(r)$$

with $r \in A_\Gamma$.

Let $\Theta = \Delta \cap \Sigma$, and take $x \in A_\Theta$. Then $h_1(x) \in A_\Delta$, and so any cyclically reduced word in the alphabet Γ representing the conjugacy class of $h_1(x)$ lies in A_Δ (by Lemma 2.8). Now

$$c(r)h_1(x) = h_2(x) \in A_\Sigma$$

and thus any reduced word representing the conjugacy class of $h_1(x)$ lies in A_Σ. Hence such a word lies in A_Θ. But this implies that $h_1(x)$ is a conjugate of an element of A_Θ for each $x \in A_\Theta$, and so we apply Proposition 2.9 and conclude that

$$h_1(A_\Theta) \leqslant y^{-1}A_\Theta y$$

for some $y \in A_\Gamma$.

We repeat the argument for h_1^{-1}, which is a representative of $\phi(h^{-1})$, and obtain

$$A_\Theta = h_1^{-1}h_1(A_\Theta) \leqslant h_1^{-1}(y^{-1}A_\Theta y) \leqslant y'^{-1}A_\Theta y'$$

for some $y' \in A_\Gamma$. Now Proposition 2.5 tells us that $y'^{-1}A_\Theta y' = A_\Theta$ and so both inequalities in the expression above are in fact equalities. Thus

$$h_1(A_\Theta) = y^{-1}A_\Theta y$$

and so $\Theta \in \mathcal{L}^\phi$ as claimed.

ii) Now suppose that $\mathrm{lk}(\Theta) \subseteq \mathrm{st}(\Delta)$. Since $\Theta \in \mathcal{L}^\phi$, there exists a representative h_3 of $\phi(h)$ which fixes A_Θ. Since $\Delta \in \mathcal{L}^\phi$, the subgroup $h_3(A_\Delta)$ is a conjugate of A_Δ by an element $r \in A_\Gamma$. Now we have

$$rA_\Theta r^{-1} \leqslant A_\Delta$$

and so by Proposition 2.5 we know that

$$r \in N(A_\Theta)N(A_\Delta) = A_{\mathrm{st}(\Theta)}A_{\mathrm{st}(\Delta)} = A_{\mathrm{st}(\Delta)}$$

since $\mathrm{lk}(\Theta) \subseteq \mathrm{st}(\Delta)$ by assumption. But then $r^{-1}A_\Delta r = A_\Delta$, and so $h_3(A_\Delta) = A_\Delta$.

Let us apply the same argument to $h_3(A_\Sigma)$ – it must be a conjugate of A_Σ by some $s \in A_\Gamma$, and we conclude as above that

$$s \in N(A_\Theta)N(A_\Sigma) \leqslant A_{\mathrm{st}(\Delta)}A_{\mathrm{st}(\Sigma)}$$

We take a new representative h_4 of $\phi(h)$, which differs from h_3 by the conjugation by the $A_{\mathrm{st}(\Delta)}$-factor of s. This way we get $h_4(A_\Delta) = A_\Delta$, and $h_4(A_\Sigma)$ equal to a conjugate of A_Σ by an element of $A_{\mathrm{st}(\Sigma)} = N(A_\Sigma)$. Hence we have $h_4(A_\Sigma) = A_\Sigma$, and the result follows.

□

The following lemma is very much motivated by the work of [CCV].

Lemma 4.3. *Suppose that $\phi(H) \leqslant \mathrm{Out}^0(A_\Gamma)$. Then $\mathcal{L}^\phi$ contains*

(1) each connected component of Γ which contains at least one edge;
(2) the extended star of each induced subgraph;
(3) the link of each subgraph Δ, such that Δ is not a cone;
(4) the star of each subgraph in $\mathcal{L}^\phi$.

Proof. We will prove the first three points on our list for $\mathrm{Out}^0(A_\Gamma)$ (and therefore for any subgroup). It is enough to verify that each type of generator of $\mathrm{Out}^0(A_\Gamma)$ preserves the listed subgroups up to conjugacy. It is certainly true for all inversions, and thus we only need to verify it for transvections and partial conjugations.

We will make a rather liberal use of Lemma 2.3.

Transvections. Take two vertices in Γ, say v and w, such that

$$\mathrm{lk}(w) \subseteq \mathrm{st}(v)$$

In this case we have a transvection $w \mapsto wv$. To prove our assertion we need to check that whenever w belongs to the subgraph defining our subgroup, so does v.

(1) If w belongs to a connected component of Γ which is not a singleton, then $\mathrm{lk}(w) \neq \emptyset$ lies in the same component, and so our assumption $\mathrm{lk}(w) \subseteq \mathrm{st}(v)$ forces v to lie in the component as well.
(2) Take $\Delta \subseteq \Gamma$ with $w \in \widehat{\mathrm{st}}(\Delta)$. If $w \in \mathrm{lk}(\mathrm{lk}(\Delta))$ then
$$\mathrm{lk}(\Delta) \subseteq \mathrm{lk}(w) \subseteq \mathrm{st}(v)$$
and so $v \in \widehat{\mathrm{st}}(\Delta)$. If $w \in \mathrm{lk}(\Delta)$ then $\Delta \subseteq \mathrm{lk}(w) \subseteq \mathrm{st}(v)$ and so $v \in \mathrm{st}(\Delta)$.
(3) Take $\Delta \subseteq \Gamma$ which is not a cone, and such that $w \in \mathrm{lk}(\Delta)$. Then $\Delta \subseteq \mathrm{st}(v)$, and so $v \in \mathrm{st}(\Delta)$ as above. However, if $v \in \Delta$, then $\Delta \smallsetminus \{v\} \subseteq \mathrm{st}(v) \smallsetminus \{v\} = \mathrm{lk}(v)$, and so Δ is a cone over v, which is a contradiction. Thus $v \in \mathrm{lk}(\Delta)$.

Note that in the last part the assumption of Δ not being a cone is used only to guarantee that $v \notin \Delta$. If the transvection under consideration was of type I, we would now this immediately since $\Delta \subseteq \mathrm{lk}(w) \subseteq \mathrm{lk}(v)$ in this case, and so the assumption of Δ not being a cone would be unnecessary.

Partial conjugations. Take a vertex v in Γ, such that its star disconnects Γ. In this case we have a partial conjugation of the subgroup A_Θ by v, with $\Theta \subseteq \Gamma \smallsetminus \mathrm{st}(v)$ being a connected component. Let $\Sigma = \Gamma \smallsetminus (\Theta \cup \mathrm{st}(v))$. To show that this automorphism preserves the desired subgroups up to conjugation, we need to show that if the subgraphs defining the subgroups do not contain v, then they cannot intersect both Θ and Σ.

(1) If a connected component intersects (and so contains) Θ but does not contain v, then it is in fact equal to Θ, and so intersects Σ trivially.
(2) Take $\Delta \subseteq \Gamma$ with $v \notin \widehat{\mathrm{st}}(\Delta)$, and such that $\widehat{\mathrm{st}}(\Delta)$ intersects Θ and Σ non-trivially. If $\mathrm{lk}(\Delta) \subseteq \mathrm{st}(v)$, then $v \in \widehat{\mathrm{st}}(\Delta)$, which is a contradiction. Hence $\mathrm{lk}(\mathrm{lk}(\Delta))$ cannot intersect both Θ and Σ, since if it did, then we would have $\mathrm{lk}(\Delta) \subseteq \mathrm{st}(v)$. Similarly, if $\Delta \subseteq \mathrm{st}(v)$, then $v \in \mathrm{st}(\Delta)$. This is again a contradiction, and again we conclude that $\mathrm{lk}(\Delta)$ cannot intersect both Θ and Σ. Hence, without loss of generality, we have $\mathrm{lk}(\mathrm{lk}(\Delta))$ intersecting Θ and $\mathrm{lk}(\Delta)$ intersecting Σ. But then the two cannot form a join. This is a contradiction.
(3) Take $\Delta \subseteq \Gamma$, with $\mathrm{lk}(\Delta)$ intersecting both Θ and Σ non-trivially. This condition forces $\Delta \subseteq \mathrm{st}(v)$. In fact we see that $\Delta \subseteq \mathrm{lk}(v)$, since otherwise we would have $\mathrm{lk}(\Delta) \subseteq \mathrm{st}(v)$, and thus the link would intersect neither Θ nor Σ. Now $\Delta \subseteq \mathrm{lk}(v)$ gives $v \in \mathrm{lk}(\Delta)$.

Note that in the last part we did not use the assumption on Δ not being a cone.

Now we need to prove (4). Take $\Delta \in \mathcal{L}^\phi$. Pick $h \in H$ and let

$$h_1 \in \mathrm{Aut}(A_\Gamma)$$

be a representative of $\phi(h)$ such that $h_1(A_\Delta) = A_\Delta$. Then also

$$h_1(N(A_\Delta)) = N(A_\Delta) = A_{\mathrm{st}(\Delta)}$$ □

Definition 4.4. We say that $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ is *link-preserving* if and only if $\mathcal{L}^\phi$ contains links of all induced subgraphs of Γ.

Note that if $\dim \Gamma = 1$, then every action ϕ is link-preserving.

Remark 4.5. Note that when ϕ is link-preserving, then so is every induced action $H \to \mathrm{Out}(A_\Sigma)$ for each $\Sigma \in \mathcal{L}^\phi$.

Lemma 4.6. *If $\phi(H) \leqslant \widetilde{\mathrm{Out}}^0(A_\Gamma)$ then ϕ is link-preserving.*

Proof. Let $\Delta \subseteq \Gamma$ be given. Again we will show that each generator of $\widetilde{\mathrm{Out}}^0(A_\Gamma)$ sends $A_{\mathrm{lk}(\Delta)}$ to a conjugate of itself. The inversions clearly have the desired property; so do partial conjugations and transvections of type I, since the proof of (3) above did not use the assumption on Δ not being a cone (as remarked). But these are the generators of $\widetilde{\mathrm{Out}}^0(A_\Gamma)$ and so we are done. □

Corollary 4.7. *If $\dim \Gamma = 2$ and Γ has no leaves, then any $\phi\colon H \to \mathrm{Out}^0(A_\Gamma)$ is link-preserving.*

Proof. Since $\dim \Gamma = 2$, the links of vertices are discrete graphs; since Γ has no leaves, they contain at least 2 vertices. Hence such links are never cones, and Lemma 2.10 tells us that

$$\mathrm{Out}^0(A_\Gamma) = \widetilde{\mathrm{Out}}^0(A_\Gamma)$$

Lemma 4.6 completes the proof. ☐

Definition 4.8. i) Any subset $\mathcal{S}$ of $\mathcal{L}^\phi$ closed under taking intersections of its elements will be called a *subsystem of invariant subgraphs.*

ii) Given such a subsystem $\mathcal{S}$, and any induced subgraph $\Theta \in \mathcal{L}^\phi$, we define
- $\mathcal{S}_\Theta = \{\Sigma \cap \Theta \mid \Sigma \in \mathcal{S}\}$
- $\bigcup \mathcal{S} = \bigcup_{\Sigma \in \mathcal{S}} \Sigma$
- $\bigcap \mathcal{S} = \bigcap_{\Sigma \in \mathcal{S}} \Sigma$

Lemma 4.9. *Let $\mathcal{P} \subseteq \mathcal{L}^\phi$ be a subsystem of invariant graphs, and let $\Theta \in \mathcal{P}$. Then*

$$\mathcal{P}_\Theta = \{\Delta \in \mathcal{P} \mid \Delta \subseteq \Theta\}$$

Proof. This follows directly from the fact that $\mathcal{P}$ is closed under taking intersections. ☐

Note that given $\Delta \in \mathcal{L}^\phi$, we get an induced action $\psi\colon H \to \mathrm{Out}(A_\Delta)$. This follows from the fact that the normaliser $N(A_\Delta)$ satisfies

$$N(A_\Delta) = A_{\mathrm{st}(\Delta)} = A_\Delta \times A_{\mathrm{lk}(\Delta)}$$

and $A_{\mathrm{lk}(\Delta)}$ centralises A_Δ.

It is immediate that $\mathcal{L}^\psi = \mathcal{L}^\phi_\Delta$.

5. Graph systems

In this section we assume that $\dim \Gamma = 1$. In other words, the RAAG A_Γ is a free group. The construction in this section already shows the strategy employed in the general case, but has easier technical details.

Definition 5.1 (Graph systems)**.** Suppose we have a subsystem of invariant subgraphs $\mathcal{P} \subseteq \mathcal{L}^\phi$, with Γ of dimension 1. A *graph system* $\mathcal{X}$ for $\mathcal{P}$ consists of the following data.

(1) For each $\Delta \in \mathcal{P}$ a connected metric graph X_Δ without leaves, realising $H \to \mathrm{Out}(A_\Delta)$.

(2) For each pair $\Delta, \Theta \in \mathcal{P}$ with $\Delta \subseteq \Theta$ and $\Delta \neq \emptyset$, an H-equivariant isometric embedding

$$\iota_{\Delta,\Theta}\colon X_\Delta \to X_\Theta$$

which respects the markings via a map $\widetilde{\iota}_{\Delta,\Theta}$. We set $\iota_{\Delta,\Delta}$ and $\widetilde{\iota}_{\Delta,\Delta}$ to be the respective identities.

We require the data to satisfy the following conditions.

Composition Axiom: for any $\Delta, \Sigma, \Theta \in \mathcal{P} \smallsetminus \{\emptyset\}$ with $\Delta \subseteq \Sigma \subseteq \Theta$ we require

$$\mathrm{im}(\widetilde{\iota}_{\Delta,\Theta}) = \mathrm{im}(\widetilde{\iota}_{\Sigma,\Theta} \circ \widetilde{\iota}_{\Delta,\Sigma})$$

Pair Intersection Axiom: given $\Delta, \Delta' \in \mathcal{P} \smallsetminus \{\emptyset\}$ with $\Delta \cup \Delta' \in \mathcal{P}$ we require

- $\mathrm{im}(\widetilde{\iota}_{\Delta,\Delta\cup\Delta'}) \cap \mathrm{im}(\widetilde{\iota}_{\Delta',\Delta\cup\Delta'}) = \mathrm{im}(\widetilde{\iota}_{\Delta\cap\Delta',\Delta\cup\Delta'})$ when $\Delta \cap \Delta' \neq \emptyset$;
- $\mathrm{im}(\widetilde{\iota}_{\Delta,\Delta\cup\Delta'}) \cap \mathrm{im}(\widetilde{\iota}_{\Delta',\Delta\cup\Delta'})$ equal to a point when $\Delta \cap \Delta' = \emptyset$.

Lemma 5.2. *Let $\mathcal{X}$ be a graph system for $\mathcal{L}^\phi$. Suppose that $\Delta, \Sigma \in \mathcal{L}^\phi \smallsetminus \{\emptyset\}$ are such that*

$$\mathrm{im}(\widetilde{\iota}_{\Delta,\Gamma}) \cap \mathrm{im}(\widetilde{\iota}_{\Sigma,\Gamma}) \neq \emptyset$$

Then $\Delta \cup \Sigma \in \mathcal{L}^\phi$.

Proof. Let $\widetilde{p} \in \mathrm{im}(\widetilde{\iota}_{\Delta,\Gamma}) \cap \mathrm{im}(\widetilde{\iota}_{\Sigma,\Gamma})$, and let p denote its projection in X_Γ. Since $\iota_{\Delta,\Gamma}$ is H-equivariant, its image in X_Γ is preserved by H; the same is true for $\mathrm{im}(\iota_{\Sigma,\Gamma})$, and hence for their union. Now the fundamental group of this union based at p is $A_{\Sigma\cup\Delta}$ (bearing in mind that choosing a basepoint $\widetilde{p}$ fixes an isomorphism $\pi_1(X_\Gamma, p) \cong A_\Gamma$). □

Let us also define a more general System Intersection Property.

Definition 5.3 (System Intersection Property)**.** Let $\mathcal{P}$ be a subsystem of invariant graphs, and let $\mathcal{X}$ be a graph system for $\mathcal{P}$. We say that $\mathcal{X}$ satisfies the *System Intersection Property* if and only if given any subsystem $\mathcal{S} \subseteq \mathcal{P}$, which is closed under taking unions and intersections, we have

- $\bigcap\limits_{\Sigma\in\mathcal{S}\smallsetminus\{\emptyset\}} \mathrm{im}(\widetilde{\iota}_{\Sigma,\Theta}) = \mathrm{im}(\widetilde{\iota}_{\Delta,\Theta})$ when $\Delta \neq \emptyset$;
- $\bigcap\limits_{\Sigma\in\mathcal{S}\smallsetminus\{\emptyset\}} \mathrm{im}(\widetilde{\iota}_{\Sigma,\Theta})$ equal to a point when $\Delta = \emptyset$.

where $\Delta = \bigcap \mathcal{S}$ and $\Theta = \bigcup \mathcal{S}$.

Remark 5.4. Observe that when $\Delta = \emptyset$, the image of the intersection point $\bigcap_{\Sigma\in\mathcal{S}\smallsetminus\{\emptyset\}} \mathrm{im}(\widetilde{\iota}_{\Sigma,\Theta}) \in \widetilde{X_\Theta}$ in X_Θ is the only point in $\bigcap_{\Sigma\in\mathcal{S}\smallsetminus\{\emptyset\}} \mathrm{im}(\iota_{\Sigma,\Theta})$. Hence this latter point is H-fixed, since each of the images is H-invariant (compare Lemma 3.4).

Lemma 5.5. *A graph system satisfies the System Intersection Property.*

Proof. Let $\mathcal{S}$ be closed under taking unions and intersections, and let $\Theta = \bigcup \mathcal{S}$. Take non-empty subgraphs $\Sigma_1, \dots, \Sigma_n \in \mathcal{S}$. We will show by induction on n that

$$\bigcap_{i=1}^{n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) = \mathrm{im}(\widetilde{\iota}_{\bigcap_{i=1}^n \Sigma_i,\Theta})$$

when $\bigcap_{i=1}^n \Sigma_i \neq \emptyset$, and that the left-hand side is a point when $\bigcap_{i=1}^n \Sigma_i = \emptyset$.

Suppose that $n = 2$. In this case we need only to observe that the Composition Axiom allows us to replace Θ by $\Sigma_1 \cup \Sigma_2$ in the above expressions, and then the statement follows directly from the Pair Intersection Axiom.

Let us now suppose that the claim holds for all integers smaller than n.

When $\bigcap_{i<n} \Sigma_i \neq \emptyset$, we have

$$\bigcap_{i<n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) = \mathrm{im}(\widetilde{\iota}_{\bigcap_{i<n} \Sigma_i,\Theta})$$

and so

$$\bigcap_{i=1}^{n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) = \bigcap_{i<n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) \cap \mathrm{im}(\widetilde{\iota}_{\Sigma_n,\Theta}) = \mathrm{im}(\widetilde{\iota}_{\bigcap_{i<n}\Sigma_i,\Theta}) \cap \mathrm{im}(\widetilde{\iota}_{\Sigma_n,\Theta})$$

which yields the desired result by the Pair Intersection Axiom (again using the Composition Axiom).

When $\bigcap_{i<n} \Sigma_i = \emptyset$, then

$$\bigcap_{i<n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta})$$

is a point which lies in

$$\bigcap_{1<i<n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta})$$

We may also assume that $\bigcap_{1<i} \Sigma_i = \emptyset$, since otherwise we are in the previous case (after reordering the subgraphs). If $\bigcap_{1<i<n} \Sigma_i = \emptyset$, then

$$\bigcap_{i<n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) = \bigcap_{1<i} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta}) = \bigcap_{i=1}^{n} \mathrm{im}(\widetilde{\iota}_{\Sigma_i,\Theta})$$

We are left with the case in which $\bigcap\limits_{1<i<n} \Sigma_i \neq \emptyset$. If in addition $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ we use the Pair Intersection Axiom with the pair $\bigcap\limits_{1<i<n} \Sigma_i$ and $\Sigma_1 \cap \Sigma_2$. When $\Sigma_1 \cap \Sigma_2 = \emptyset$ we apply the Pair Intersection Axiom to each pair in $\{\Sigma_1, \Sigma_n, \bigcap\limits_{1<i<n} \Sigma_i\}$, and obtain three points in $\widetilde{X}_\Theta$. These three points form a triangle in $\widetilde{X}_\Theta$, with one edge in each $\widetilde{X}_{\Sigma_1}$,$\widetilde{X}_{\Sigma_n}$ and $\widetilde{X}_{\bigcap\limits_{1<i<n} \Sigma_i}$. Since this is a triangle in a tree, it contains a point common to all its edges. Thus the three points we obtained from the Pair Intersection Axiom are in fact equal. This concludes the proof. □

Definition 5.6 (Extending graph systems)**.** Suppose we are given two subsystems of invariant graphs, $\mathcal{S}$ and $\mathcal{S}'$, with $S \subseteq S'$, and two graphs systems, $\mathcal{X}$ and $\mathcal{X}'$ for $\mathcal{S}$ and $\mathcal{S}'$ respectively.

When $\mathcal{S} = \{\emptyset, \Delta\}$ or $\mathcal{S} = \{\Delta\}$, and Δ is a singleton, we say that $\mathcal{X}'$ *extends* $\mathcal{X}$ if and only if there is an isometry $X_\Delta \to X'_\Delta$ which is H-equivariant up to homotopy. We say that $\mathcal{X}'$ extends $\mathcal{X}$ *strongly* if and only if the isometry can be made H-equivariant.

For any other $\mathcal{S}$ we say that $\mathcal{X}'$ *extends* $\mathcal{X}$, or equivalently that it extends *strongly* if and only if for every $\Delta \in \mathcal{S}$ we have X_Δ and X'_Δ H-equivariantly isometric, and for all $\Delta, \Theta \in \mathcal{S}$ with $\Delta \subseteq \Theta$ the maps $\iota_{\Delta,\Theta}, \widetilde{\iota}_{\Delta,\Theta}$ and $\iota'_{\Delta,\Theta}, \widetilde{\iota}'_{\Delta,\Theta}$ form commutative diagrams with the isometries.

Lemma 5.7. *Let H be finite. Suppose we are given two subgraphs $\Sigma, \Theta \in \mathcal{L}^\phi$ with at least two vertices each, such that $\Gamma = \Sigma \cup \Theta$ and $\Sigma \cap \Theta = \{s\} \in \mathcal{L}^\phi$ is a single vertex. Suppose we have two graphs systems, $\mathcal{X}$ and $\mathcal{X}'$, for $\mathcal{L}^\phi_\Sigma$ and $\mathcal{L}^\phi_\Theta$ respectively. Then $\mathcal{X}'$ extends the subsystem $\mathcal{X}_{\{s\}}$ strongly.*

Moreover, there exists a graph Y realising ϕ, such that $\widetilde{Y}$ is obtained from $\widetilde{X}_\Sigma$ and $\widetilde{X}'_\Theta$ by gluing $\widetilde{X}_{\{s\}}$ and $\widetilde{X}'_{\{s\}}$.

The proposition above is a direct consequence of Lemma 8.3; the lemma is formulated for cubical systems, which we define later, but when $\dim \Gamma = 1$ cubical systems and graph systems (essentially) coincide – see Lemma 7.8.

Proposition 5.8. *Let H be finite. Suppose we are given two subgraphs $\Sigma, \Theta \in \mathcal{L}^\phi$ such that $\Gamma = \Sigma \cup \Theta$. Let $E = \Sigma \cap \Theta \in \mathcal{L}^\phi$, and suppose that it is non-empty. Suppose further that we have two graphs systems, $\mathcal{X}$ and $\mathcal{X}'$, for $\mathcal{L}^\phi_\Sigma$ and $\mathcal{L}^\phi_\Theta$ respectively, such that $\mathcal{X}'$ extends the subsystem $\mathcal{X}_E$ strongly. Then there exists a graph Y realising ϕ, such that $\widetilde{Y}$ is obtained from $\widetilde{X}_\Sigma$ and $\widetilde{X}'_\Theta$ by gluing $\widetilde{X}_E$ and $\widetilde{X}'_E$.*

This follows from Proposition 8.5 – we only need to remark that $C(A_E) = Z(A_E)$ since A_Γ is a free group here.

6. Realisation for free groups

In this section we assume that H is a finite group.

Theorem 6.1. *Suppose $\phi \colon H \to \mathrm{Out}(F_n)$ is a homomorphism with a finite domain. Suppose we are given $\Xi \in \mathcal{L}^\phi$ and a graph system $\mathcal{X}_\Xi$ for $\mathcal{L}^\phi_\Xi$. Then there exists a graph system $\mathcal{X}$ for $\mathcal{L}^\phi$ which extends $\mathcal{X}_\Xi$.*

Proof. Let Γ be a graph with n vertices and no edges, so that $A_\Gamma = F_n$.

First we note that when $\Xi = \Gamma$ we take $\mathcal{X} = \mathcal{X}_\Xi$ and then there is nothing to prove. We will proceed by induction on the *depth* k of Γ, defined to be the length of a maximal chain of proper inclusions

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

where $\Gamma_i \in \mathcal{L}^\phi$ for each i.

Note that $\Xi \in \mathcal{L}^\phi$. When $k = 1$ we must then have $\Xi = \emptyset$, in which case the theorem is reduced to the classical Nielsen Realisation for free groups [Cul, Khr, Zim2]. Note that we require our graphs to be leaf-free, so we might need to prune the leaves.

Now suppose the result holds whenever the depth is at most $k-1$, and suppose that Γ is of depth k. Consider the set of subgraphs $\Sigma_1, \ldots, \Sigma_r \in \mathcal{L}^\phi$ which are properly contained in Γ, and maximal with this property. Note that this implies that, without loss of generality, $\Xi \subseteq \Sigma_1$, where the equality is certainly possible.

We distinguish two cases.

Case 1: The subgraphs Σ_i are mutually disjoint.

In that case we have a free factor decomposition

$$F_n = A_\Gamma = A_{\Sigma_1} * \cdots * A_{\Sigma_r} * B \tag{1}$$

with $B = A_{\Gamma \smallsetminus \bigcup \Sigma_i}$. The inductive step gives us a graph system $\mathcal{X}^i$ for each $\mathcal{L}^\phi_{\Sigma_i}$, with the additional property that $\mathcal{X}^1$ extends $\mathcal{X}_\Xi$.

We apply Adapted Realisation (Theorem 3.2) to find a marked graph X_Γ with embeddings of graphs $X^i_{\Sigma_i} \in \mathcal{X}^i$. Note that we also apply Corollary 3.3 when the decomposition (1) has the form $A_\Gamma = A_{\Sigma_1} * A_{\Sigma_2}$. We are left with (the easy task of) checking the axioms.

Graphs. We now define the graphs in $\mathcal{X}$ as follows. Take $\Delta \in \mathcal{L}^\phi$. When $\Delta = \Gamma$ we take $X_\Delta = X_\Gamma$ defined above. When $\Delta \neq \Gamma$ then there exists j such that $\Delta \subseteq \Sigma_j$. If Σ_j is not a singleton, we define $X_\Delta = X^j_\Delta$.

We are left with defining X_{Σ_i} when Σ_i is a singleton. We take this graph to be the image of $X^i_{\Sigma_i}$ in X_Γ obtained by the Adapted Realisation, with the restricted H-action. The point here is that both X_{Σ_i} and $X^i_{\Sigma_i}$ admit an H-action, but it is the same action only up to homotopy.

When Ξ is a singleton we already see that $\mathcal{X}$ extends $\mathcal{X}_\Xi$, since $\mathcal{X}^1$ did – in this case $\mathcal{X}_\Xi$ contains no maps.

Maps. Given $\Delta \subseteq \Delta'$, both non-empty and in $\mathcal{L}^\phi$, we need to specify the map $\widetilde{\iota}_{\Delta,\Delta'}$. We define $\widetilde{\iota}_{\Delta,\Delta}$ to be the identity, and will assume henceforth that the subgraphs are distinct. If $\Delta' \neq \Gamma$ then there exists j such that $\Delta' \subseteq \Sigma_j$. We note that necessarily Σ_j is not a singleton, and we take $\widetilde{\iota}_{\Delta\Delta'} = \widetilde{\iota}^j_{\Delta,\Delta'}$. The last remaining case occurs when $\Delta' = \Gamma$. When $\Delta = \Sigma_j$, we take $\widetilde{\iota}_{\Sigma_j,\Gamma}$ to be the embedding given by the Adapted Realisation. For a general Δ we take $\widetilde{\iota}_{\Delta\Gamma} = \widetilde{\iota}_{\Sigma_l,\Gamma} \circ \widetilde{\iota}_{\Delta,\Sigma_l}$, where $\Delta \subseteq \Sigma_l$ (with l unique).

Note that $\mathcal{X}$ extends $\mathcal{X}_\Xi$ when Ξ is not a singleton, since $\mathcal{X}$ extends $\mathcal{X}^1$.

Composition Axiom. The property of map composition follows directly from the corresponding property in systems $\mathcal{X}^i$, unless we are considering $\widetilde{\iota}_{\Delta,\Gamma}$ for some $\Delta \in \mathcal{L}^\phi \smallsetminus \{\emptyset\}$, in which case the property follows directly from the construction of this map.

Pair Intersection Axiom. We are left with verifying that our system satisfies the Pair Intersection Axiom. Let $\Delta, \Delta' \in \mathcal{L}^\phi \smallsetminus \{\emptyset\}$ be such that $\Delta \cup \Delta' \in \mathcal{L}^\phi$. The property is somewhat vacuously satisfied when $\Delta \subseteq \Delta'$ or vice versa, so we may assume that neither is the case. In particular neither Δ nor Δ' is equal to Γ. If there exists j such that $\Delta \cup \Delta' \subseteq \Sigma_j$, then the property is satisfied, since it is satisfied in $\mathcal{X}^j$.

We are left with the possibility that $\Delta \subseteq \Sigma_i$ and $\Delta' \subseteq \Sigma_j$ with $i \neq j$. Suppose that $\Delta \subset \Sigma_i$ is a proper inclusion. Then $\Sigma_j \cup \Delta = \Sigma_j \cup \Sigma_i \cup \Delta$ is a superset of Σ_j, properly contained in Γ. But it is also contained in $\mathcal{L}^\phi$, by the second assertion of Lemma 4.2 (since links of non-empty subgraphs of Γ are empty in our case). This contradicts the maximality of Σ_j. Therefore $\Delta = \Sigma_i$, and by an analogous argument $\Delta' = \Sigma_j$. Also, since $\Delta \cup \Delta' \in \mathcal{L}^\phi$, we must have $\Gamma = \Sigma_i \cup \Sigma_j$, as otherwise neither Σ_i nor Σ_j would be maximal. Thus

$$F_n = A_{\Sigma_i} * A_{\Sigma_j}$$

and so we may apply Corollary 3.3. Therefore the images of $\widetilde{\iota}_{\Sigma_i,\Gamma}$ and $\widetilde{\iota}_{\Sigma_j,\Gamma}$ intersect in the desired way.

Case 2: There exists $\Sigma' \in \mathcal{L}^\phi \smallsetminus \{\Gamma\}$ such that $\Sigma_1 \cap \Sigma' \neq \emptyset$ and $\Sigma' \not\subseteq \Sigma_1$.

This is the trickier case, and the one in which we use the System Intersection Property in a crucial way.

To ease up notation, let us set $\Sigma = \Sigma_1$.

The union $\Sigma \cup \Sigma'$ lies in $\mathcal{L}^\phi$ as well by Lemma 4.2, and thus by maximality of Σ, we have $\Sigma' \cup \Sigma = \Gamma$. In particular this implies that $\Sigma' \supseteq \Gamma \smallsetminus \Sigma$, and this is true for all $\Sigma' \in \mathcal{L}^\phi \smallsetminus \{\Gamma\}$ such that $\Sigma_1 \cap \Sigma' \neq \emptyset$ and $\Sigma' \not\subseteq \Sigma_1$.

Let

$$\mathcal{P} = \{\Sigma' \in \mathcal{L}^\phi \mid \Sigma' \supseteq \Gamma \smallsetminus \Sigma\}$$

It is closed under taking unions and intersections by Lemma 4.2. Let

$$\Theta = \bigcap \mathcal{P}$$

Crucially, every element in $\mathcal{L}^\phi$ not contained in Σ nor in Θ must contain Θ by the argument above. Note that $\Sigma \not\subseteq \Theta$, since otherwise Σ would not be maximal, as then we would have $\Sigma \subset \Sigma'$. For the same reason Σ cannot be a singleton.

Induction gives us a graph system $\mathcal{X}'$ for $\mathcal{L}^\phi_\Sigma$ extending $\mathcal{X}_\Xi$. Let $E = \Theta \cap \Sigma$. The system $\mathcal{X}'$ contains a subsystem $\mathcal{X}'_E$ for $\mathcal{L}^\phi_E$. Induction gives a graph system $\mathcal{X}''$ for $\mathcal{L}^\phi_\Theta$ extending $\mathcal{X}'_E$. When E is not a singleton, the systems are strongly extending by definition. When E is a singleton, we need to observe that neither Σ nor Θ is, and hence that they contain E properly. Thus we apply Lemma 5.7 and conclude that in this case the systems are extending strongly as well.

Graphs. We will now define the graphs constituting $\mathcal{X}$. Whenever we have $\Delta \in \mathcal{L}^\phi$ with $\Delta \subseteq \Sigma$, we declare $X_\Delta = X'_\Delta$. Whenever $\Delta \subseteq \Theta$ but $\Delta \not\subseteq \Sigma$ we take $X_\Delta = X''_\Delta$.

Now let us construct X_Γ; we will then define X_Δ for remaining Δ by taking subgraphs of X_Γ. We need to consider two cases, depending on the cardinality of $E = \Sigma \cap \Theta$.

If $|E| > 0$, then the tree $\widetilde{X}_\Gamma$ is obtained from $\widetilde{X}_\Sigma$ and $\widetilde{X}_\Theta$ by applying Proposition 5.8.

Let us now suppose that $|E| = 0$.

Since $\mathcal{P}$ is closed under taking unions and intersections, so is $\mathcal{S} = \mathcal{P}_\Sigma$. We can thus apply the Intersection Axiom to $\mathcal{S} \subseteq \mathcal{L}^\phi_\Sigma$ and obtain a point of intersection $\widetilde{p} = \bigcap_{\Delta \in \mathcal{S} \smallsetminus \{\emptyset\}} \mathrm{im}(\widetilde{\iota}_{\Delta,\Sigma}) \in \widetilde{X_\Sigma}$, since $\bigcup \mathcal{S} = \bigcup \mathcal{P} \cap \Sigma = \Gamma \cap \Sigma = \Sigma$. Let p be the projection of $\widetilde{p}$ in X_Σ.

The graph system $\mathcal{X}''$ for $\mathcal{L}^\phi_\Theta$ contains a graph X''_Θ. Let us apply Adapted Realisation (together with Corollary 3.3) to $A_\Gamma = A_\Sigma * A_\Theta$ with graphs $X_\Sigma = X'_\Sigma$ and $X_\Theta = X''_\Theta$. We obtain a graph Y_Γ, with embeddings of $\widetilde{X}_\Sigma$ and $\widetilde{X}_\Theta$ into $\widetilde{Y}_\Gamma$ which intersect in a single point; let $\widetilde{q}$ denote this point in $\widetilde{X}_\Theta$. In particular q, the projection of $\widetilde{q}$ in the quotient Y is H-fixed. Hence, picking $\widetilde{q}$ as a basepoint of $\widetilde{Y}$, we get a representative $h_q \in \mathrm{Aut}(A_\Gamma) = \mathrm{Aut}(\pi_1(Y, q))$

of $\phi(h)$ for each $h \in H$. This representative satisfies

$$h_q(A_\Sigma) = A_\Sigma \text{ and } h_q(A_\Theta) = A_\Theta$$

We have already defined the point $\widetilde{p} \in \widetilde{X}_\Sigma$; it naturally can also be viewed as a point in $\widetilde{Y}$. Similarly p becomes a point in Y which is also H-fixed. Thus we can use $\widetilde{p}$ as a basepoint, and for each $\phi(h)$ obtain a representative $h_p \in \mathrm{Aut}(A_\Gamma)$ such that $h_q(A_\Sigma) = A_\Sigma$.

Now $h_q h_p^{-1} = c(x)$ since they are representatives of the same outer automorphism, with $x \in A_\Gamma$.

Take $\Delta \in \mathcal{P}$. Now

$$A_\Theta = h_q(A_\Theta) \leqslant h_q(A_\Delta) = A_\Delta^y$$

for some $y \in A_\Gamma$, since $\Delta \in \mathcal{L}^\phi$. Proposition 2.5 implies that

$$y \in N(A_\Delta)N(A_\Theta) = A_\Delta$$

and so $h_q(A_\Delta) = A_\Delta$.

Since $h_q(A_\Delta) = A_\Delta$ and $h_q(A_\Sigma) = A_\Sigma$, we have $h_q(A_{\Delta\cap\Sigma}) = A_{\Delta\cap\Sigma}$. But we also have $h_p(A_{\Delta\cap\Sigma}) = A_{\Delta\cap\Sigma}$, since the basepoint $\widetilde{p}$ lies in the image of $\widetilde{\iota}_{\Delta\cap\Sigma,\Sigma}$ for each $\Delta \in \mathcal{P}$. Thus

$$A_{\Delta\cap\Sigma} = h_q h_p^{-1}(A_{\Delta\cap\Sigma}) = {A_{\Delta\cap\Sigma}}^x$$

and thus $x \in N(A_{\Delta\cap\Sigma}) = A_{\Delta\cap\Sigma}$ for each $\Delta \in \mathcal{S}$. Hence $x \in A_E = \{1\}$, and so $h_q = h_p$. This equality means that the inherited action of H on X_Γ, the quotient of the tree $\widetilde{X}_\Gamma$ obtained from $\widetilde{X}_\Sigma$ and $\widetilde{X}_\Theta$ by gluing $\widetilde{p}$ and $\widetilde{q}$, induces ϕ.

Our construction gives us maps $\widetilde{\iota}_{\Sigma,\Gamma}$ and $\widetilde{\iota}_{\Theta,\Gamma}$ which satisfy the requirements.

We finish the construction by taking $\Delta \subset \Gamma$ not contained in Σ nor Θ. Thus $\Delta \in \mathcal{P}$. We define X_Δ to be the subgraph of X_Γ obtained by taking the union of $\mathrm{im}(\iota_{\Delta\cap\Sigma,\Gamma})$ and $\mathrm{im}(\iota_{\Theta,\Gamma})$; we define $\widetilde{X}_\Delta$ to be the subtree of $\widetilde{X}_\Gamma$ which becomes the universal covering after taking the restriction of the covering $\widetilde{X}_\Gamma \to X_\Gamma$, and which contains $\mathrm{im}(\widetilde{\iota}_{\Delta\cap\Sigma,\Gamma})$ and $\mathrm{im}(\widetilde{\iota}_{\Theta,\Gamma})$.

Again, our construction gives us maps $\widetilde{\iota}_{\Sigma\cap\Delta,\Delta}$ and $\widetilde{\iota}_{\Theta,\Delta}$.

As before, when Ξ is a singleton, we immediately see that $\mathcal{X}$ extends $\mathcal{X}_\Xi$.

Maps. We now need to specify the maps $\widetilde{\iota}$. Let $\Delta, \Delta' \in \mathcal{L}^\phi$ with $\Delta \subseteq \Delta'$. When $\Delta' \subseteq \Sigma$ we take $\widetilde{\iota}_{\Delta\Delta'} = \widetilde{\iota'}_{\Delta\Delta'}$. When $\Delta' \subseteq \Theta$ we take $\widetilde{\iota}_{\Delta\Delta'} = \widetilde{\iota''}_{\Delta\Delta'}$.

The remaining case occurs when $\Theta \subset \Delta'$. When additionally $\Theta \subseteq \Delta$, we define $\widetilde{\iota}_{\Delta\Delta'}$ to be the inclusion map coming from viewing $\widetilde{X}_\Delta$ and $\widetilde{X}_{\Delta'}$ as embedded in $\widetilde{X}_\Gamma$; this map is as required since the Composition Axiom for $\mathcal{X}'$ implies that $\mathrm{im}(\widetilde{\iota}_{\Delta\cap\Sigma,\Sigma}) \subseteq \mathrm{im}(\widetilde{\iota}_{\Delta'\cap\Sigma,\Sigma})$, and thus

$$\mathrm{im}(\widetilde{\iota}_{\Delta\cap\Sigma,\Gamma}) \subseteq \mathrm{im}(\widetilde{\iota}_{\Delta'\cap\Sigma,\Gamma})$$

by construction.

When $\Delta \subseteq \Sigma$, we define

$$\widetilde{\iota}_{\Delta\Delta'} = \widetilde{\iota}_{\Delta'\cap\Sigma,\Delta'} \circ \widetilde{\iota}_{\Delta,\Delta'\cap\Sigma}$$

Similarly we define

$$\widetilde{\iota}_{\Delta\Delta'} = \widetilde{\iota}_{\Theta\Delta'} \circ \widetilde{\iota}_{\Delta\Theta}$$

when $\Delta \subseteq \Theta$ but $\Delta \not\subseteq \Sigma$.

Composition Axiom. Suppose we are given $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{L}^\phi \smallsetminus \emptyset$ such that $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$. When $\Delta_2 \subseteq \Sigma$ then either $\Delta_3 \subseteq \Sigma$ as well, or we have

$$\widetilde{\iota}_{\Delta_2\Delta_3} = \widetilde{\iota}_{\Delta_3\cap\Sigma,\Delta_3} \circ \widetilde{\iota}_{\Delta_2,\Delta_3\cap\Sigma} \text{ and } \widetilde{\iota}_{\Delta_1\Delta_3} = \widetilde{\iota}_{\Delta_3\cap\Sigma,\Delta_3} \circ \widetilde{\iota}_{\Delta_1,\Delta_3\cap\Sigma}$$

by construction. In either case the Composition Axiom follows immediately from the same axiom in $\mathcal{X}'$.

The situation is analogous when $\Delta_2 \subseteq \Theta$.

We are left with the case in which $\Delta_2 \supset \Theta$. Then also $\Delta_3 \supset \Theta$, and the image $\operatorname{im}(\widetilde{\iota}_{\Delta_2\Delta_3})$ is obtained from the union of $\operatorname{im}(\widetilde{\iota}_{\Delta_2\cap\Sigma,\Delta_3\cap\Sigma})$ and $\widetilde{X}_\Theta$. If $\Delta_1 \supseteq \Theta$ then the composition axiom follows from the same axiom for $\mathcal{X}'$ applied to the triple $\Delta_1\cap\Sigma, \Delta_2\cap\Sigma, \Delta_3\cap\Sigma$. The very last possibility occurs when $\Delta_1 \subset \Theta$. In this case we only need to observe that $\widetilde{\iota}_{\Delta_2,\Delta_3}$ restricts to the identity on $\widetilde{X}_\Theta$.

Pair Intersection Axiom. Let $\Delta, \Delta' \in \mathcal{L}^\phi$ be non-empty and such that

$$\Delta \cup \Delta' \in \mathcal{L}^\phi$$

If $\Delta \cup \Delta' \subseteq \Sigma$ or $\Delta \cup \Delta' \subseteq \Theta$ then we are done. Without loss of generality let us suppose that $\Delta \cap (\Sigma \smallsetminus E) \neq \emptyset$ and $\Delta' \cap (\Theta \smallsetminus E) \neq \emptyset$. Then the union intersects both Θ and $\Sigma \smallsetminus \Theta$, and so it must contain Θ.

If Δ intersects $\Theta \smallsetminus E$, then it must contain Θ. If Δ does not intersect $\Theta \smallsetminus E$, then the latter is a subgraph of Δ', and hence also $\Theta \subseteq \Delta'$. Thus at least one of the two graphs contains Θ.

If $\Theta \not\subset \Delta'$, then $\Delta' \subseteq \Theta \subseteq \Delta$, since Δ' is not allowed to intersect $\Sigma \smallsetminus E$. In this case the Intersection Axiom is satisfied trivially.

We remain with the possibility that $\Theta \subset \Delta'$. We apply the Intersection Axiom (satisfied by $\mathcal{X}'$) to $\Delta \cap \Sigma$ and $\Delta' \cap \Sigma$; when $\Delta \subseteq \Sigma$ the intersection we obtained this way is the desired intersection. Otherwise, $\Theta \subseteq \Delta$, and so we need to extend this intersection by $\widetilde{X}_\Theta$. $\square$

7. Cubical systems

In this section we give the definition of the most fundamental object in the paper. We then begin proving some central properties that will be used throughout.

Definition 7.1 (Metric cube complex). i) A *metric cube complex* is a (realisation of a) combinatorial cube complex, which comes equipped with a metric such that every n-cube in X (for each n) is isometric to a Cartesian product of n closed intervals in $\mathbb{R}$.

ii) Given two such complexes X and Y, we say that Y is a *subdivision* of X if and only if the combinatorial cube complex underlying Y is a subdivision of X, and the induced map (on the realisations) $Y \to X$ is an isometry.
iii) We say that $Z \subseteq X$ is a *subcomplex* of X if and only if there exists a subdivision Y of X such that, under the identification $X = Y$, the subspace $Z \subseteq Y$ is a subcomplex in the combinatorial sense.
iv) A metric cube complex is called *non-positively curved*, or NPC, whenever its universal cover is CAT(0).

At this point we want to warn the reader that our notion of subcomplex is more relaxed than the usual definition.

Note further that we will make no distinction between a metric cube complex X and its subdivisions; as metric spaces they are isomorphic; moreover given any group action (by isometries) $H \curvearrowright X$ which respects the combinatorial structure of X and any subdivision Y of X, there exists a further subdivision Z of Y such that the inherited action of H on Z respects the combinatorial structure of Z.

Definition 7.2 (Cubical systems)**.** Suppose we have a subsystem of invariant subgraphs $\mathcal{P}$, such that $\mathcal{P}$ is closed under taking restricted links, that is for all $\Delta, E \in \mathcal{P}$ with $\Delta \subseteq E$ we have $\mathrm{lk}_E(\Delta) \in \mathcal{P}$. A *cubical system* $\mathcal{X}$ (for $\mathcal{P}$) consists of the following data.

(1) For each $\Delta \in \mathcal{P}$ a marked metric NPC cube complex X_Δ, of the same dimension as A_Δ, realising $H \to \mathrm{Out}(A_\Delta)$. We additionally require X_Δ not to have leaves when Δ is 1-dimensional.
(2) For each pair $\Delta, \Theta \in \mathcal{P}$ with $\Delta \subseteq \Theta$, an H-equivariant isometric embedding

$$\iota_{\Delta,\Theta} \colon X_\Delta \times X_{\mathrm{lk}_\Theta(\Delta)} \to X_\Theta$$

whose image is a subcomplex, and which respects the markings via a map $\widetilde{\iota}_{\Delta,\Theta}$, where the product is given the product marking. We set $\iota_{\Delta,\Delta}$ and $\widetilde{\iota}_{\Delta,\Delta}$ to be the respective identity maps.

Given $\Delta, \Theta \in \mathcal{P}$ we will refer to the map $\widetilde{\iota}_{\Delta,\Theta}|_{\widetilde{X}_\Delta \times \{x\}}$ for any $x \in \widetilde{X}_{\mathrm{lk}_\Theta(\Delta)}$ as the *standard copy of* $\widetilde{X}_\Delta$ *in* $\widetilde{X}_\Theta$ *determined by* x, or simply *a standard copy of* $\widetilde{X}_\Delta$ *in* $\widetilde{X}_\Theta$.

We say that such a standard copy is *fixed* if and only if its projection in X_Θ is H-invariant.

We require our maps to satisfy four conditions.

Product Axiom. Given $\Delta, \Theta \in \mathcal{P}$ such that $\Theta = \Delta * \mathrm{lk}_\Theta(\Delta)$, we require $\widetilde{\iota}_{\Delta,\Theta}$ to be surjective.

Orthogonal Axiom. Given $\Delta \subseteq \Theta$, both in $\mathcal{P}$, for each $\widetilde{x} \in \widetilde{X}_\Delta$ we require $\widetilde{\iota}_{\Delta,\Theta}(\{\widetilde{x}\} \times \widetilde{X}_{\mathrm{lk}_\Theta(\Delta)})$ to be equal to the image of some standard copy of $\widetilde{X}_{\mathrm{lk}_\Theta(\Delta)}$ in $\widetilde{X}_\Theta$.

Intersection Axiom. Let $\Sigma_1, \Sigma_2, \Theta \in \mathcal{P}$ be such that $\Sigma_i \subseteq \Theta$ for both values of i. Suppose that we are given standard copies of $\widetilde{X}_{\Sigma_i}$ in $\widetilde{X}_\Theta$ whose images intersect non-trivially. Then the intersection of the images is equal to the image of a standard copy of $\widetilde{X}_{\Sigma_1 \cap \Sigma_2}$ in $\widetilde{X}_\Theta$. Moreover, this intersection is also the image of a standard copy of $\widetilde{X}_{\Sigma_1 \cap \Sigma_2}$ in $\widetilde{X}_{\Sigma_i}$, under the given standard copy $\widetilde{X}_{\Sigma_i} \to \widetilde{X}_\Theta$, for both values of i

System Intersection Axiom. Let $\mathcal{S} \subseteq \mathcal{P}$ be a subsystem of invariant graphs closed under taking unions of its elements. Then for each $\Sigma \in \mathcal{S}$ there exists a standard copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\bigcup \mathcal{S}}$ such that the images of all of these copies intersect non-trivially.

Remark 7.3. We will often make no distinction between a standard copy and its image. Let us remark here that any standard copy is a subcomplex (with our non-standard definition of a subcomplex; see above).

Now we will list some implications of the definition.

Remark 7.4. Suppose we are given a cubical system $\mathcal{X}$ for $\mathcal{L}^\phi$, and let $\Delta \in \mathcal{L}^\phi$. Then the subsystem $\mathcal{X}_\Delta$, consisting of all complexes $X_\Sigma \in \mathcal{X}$ with $\Sigma \subseteq \Delta$ together with all relevant maps, is a cubical system for $\mathcal{L}^\phi_\Delta$.

Lemma 7.5. *Let $\mathcal{X}$ be a cubical system for $\mathcal{L}^\phi$, and let $\mathcal{P} \subseteq \mathcal{L}^\phi$ be a subsystem of invariant graphs which is closed under taking unions. Suppose that $\mathcal{P}$ contains another subsystem $\mathcal{P}'$, also closed under taking unions, such that $\Sigma \cup \bigcap \mathcal{P}' \in \mathcal{P}'$ for each $\Sigma \in \mathcal{P}$. Suppose further that for each $\Sigma' \in \mathcal{P}'$ we are given a standard copy $\widetilde{Y}_{\Sigma'}$ of $\widetilde{X}_{\Sigma'}$ in $\widetilde{X}_{\bigcup \mathcal{P}}$ such that*

$$\bigcap_{\Sigma' \in \mathcal{P}'} \widetilde{Y}_{\Sigma'} \neq \emptyset$$

Then for each $\Sigma \in \mathcal{P} \smallsetminus \mathcal{P}'$ there exists a standard copy $\widetilde{Y}_\Sigma$ of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\bigcup \mathcal{P}}$ such that

$$\bigcap_{\Sigma \in \mathcal{P}} \widetilde{Y}_\Sigma \neq \emptyset$$

Proof. Let us first set some notation: we let $\Delta = \bigcap \mathcal{P}'$, and for each $\Sigma \in \mathcal{P}$ we set $\Sigma' = \Sigma \cup \Delta \in \mathcal{P}'$.

The System Intersection Axiom gives us for each $\Sigma \in \mathcal{P}$ a standard copy $\widetilde{Z}_\Sigma$ of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\bigcup \mathcal{P}}$ such that there is a point $\widetilde{b} \in \widetilde{X}_{\bigcup \mathcal{P}}$ with

$$\widetilde{b} \in \bigcap_{\Sigma \in \mathcal{P}} \widetilde{Z}_\Sigma.$$

In particular $\widetilde{b} \in \widetilde{Z}_\Delta$, and thus there exists a point $\widetilde{b}' \in \widetilde{Y}_\Delta$ such that both $\widetilde{b}$ and $\widetilde{b}'$ lie in the same standard copy $\widetilde{W}$ of $\widetilde{X}_{\mathrm{lk}_{\bigcup \mathcal{P}}(\Delta)}$ (due to the Orthogonal Axiom). Note that the Intersection Axiom guarantees that $\widetilde{b}' \in \widetilde{Y}_{\Sigma'}$ for each $\Sigma' \in \mathcal{P}'$.

We want to construct standard copies $\widetilde{Y}_\Sigma$ for each $\Sigma \in \mathcal{P}$ which contain $\widetilde{b}'$.

Let δ denote the geodesic in the complete CAT(0) space $\widetilde{X}_{\bigcup \mathcal{P}}$ connecting $\widetilde{b}$ to $\widetilde{b}'$. Since all maps $\widetilde{\iota}$ are isometric embeddings, the geodesic δ lies in $\widetilde{W}$.

Take $\Sigma \in \mathcal{P}$. Then δ connects two standard copies of $\widetilde{X}_{\Sigma'}$, namely $\widetilde{Y}_\Sigma$ and $\widetilde{Z}_\Sigma$, and hence it must lie in a standard copy of $\widetilde{X}_{\mathrm{st}_{\bigcup \mathcal{P}}(\Sigma')}$; this copy is unique since the link of $\mathrm{st}_{\bigcup \mathcal{P}}(\Sigma')$ in $\bigcup \mathcal{P}$ is trivial. Thus the geodesic lies in the intersection of this copy and $\widetilde{W}$, which itself is a standard copy of

$$\widetilde{X}_{\mathrm{lk}_{\bigcup \mathcal{P}}(\Delta) \cap \mathrm{st}_{\bigcup \mathcal{P}}(\Sigma')} = \widetilde{X}_{\mathrm{lk}_{\Sigma'}(\Delta) * \mathrm{lk}_{\bigcup \mathcal{P}}(\Sigma')}$$

by the Intersection Axiom.

Crucially, $\Sigma' = \Sigma \cup \Delta$, and so $\mathrm{lk}_{\Sigma'}(\Delta) = \mathrm{lk}_\Sigma(\Delta \cap \Sigma)$. Thus δ lies in a standard copy of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta \cap \Sigma) * \mathrm{lk}_{\bigcup \mathcal{P}}(\Sigma')}$. In particular so does $\widetilde{b}$, which also lies in $\widetilde{Z}_\Sigma$, and hence in the unique standard copy of $\widetilde{X}_{\mathrm{st}_{\bigcup \mathcal{P}}(\Sigma)}$. Therefore $\widetilde{b}$ lies in the intersection of this copy and the copy of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta \cap \Sigma) * \mathrm{lk}_{\bigcup \mathcal{P}}(\Sigma')}$. But

$$\mathrm{lk}_\Sigma(\Delta \cap \Sigma) * \mathrm{lk}_{\bigcup \mathcal{P}}(\Sigma') \subseteq \mathrm{st}_{\bigcup \mathcal{P}}(\Sigma)$$

(since $\Sigma \subseteq \Sigma'$), and thus the Intersection Axiom implies that the copy of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta\Sigma) * \mathrm{lk}_{\bigcup \mathcal{P}}(\Sigma')}$, which contains δ, lies within the unique copy of $\widetilde{X}_{\mathrm{st}_{\bigcup \mathcal{P}}(\Sigma)}$. This must be also true for δ itself, and so there exists a standard copy $\widetilde{Y}_\Sigma$ of $\widetilde{X}_\Sigma$ which contains $\widetilde{b}'$, the other endpoint of δ. ☐

Lemma 7.6 (Matching Property)**.** *Let $\mathcal{X}$ be a cubical system for $\mathcal{L}^\phi$, and let $\Sigma_1, \Sigma_2 \in \mathcal{L}^\phi$ be such that $\Theta = \Sigma_1 \cup \Sigma_2 \in \mathcal{L}^\phi$ as well. Let $\widetilde{Y}_i$ be a given standard copy of $\widetilde{X}_{\Sigma_i}$ in $\widetilde{X}_\Theta$. Then their images intersect in a standard copy of $\widetilde{X}_\Delta$, with $\Delta = \Sigma_1 \cap \Sigma_2$.*

Proof. Let

$$\mathcal{P} = \{\Delta, \Sigma_1, \Sigma_2, \Theta\}$$

and

$$\mathcal{P}' = \{\Sigma_1, \Theta\} \subseteq \mathcal{P}$$

Note that $\mathcal{P}'$ satisfies the assumptions of the previous lemma, and so there exists a standard copy $\widetilde{Z}_{\Sigma_2}$ of $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_\Theta$ such that $\widetilde{Y}_{\Sigma_1}$ and $\widetilde{Z}_{\Sigma_2}$ intersect.

Now there exists a geodesic Δ connecting standard copies $\widetilde{Z}_{\Sigma_2}$ and $\widetilde{Y}_{\Sigma_2}$, such that δ lies in a standard copy of $\widetilde{X}_{\mathrm{lk}_\Theta(\Sigma_2)}$, and one of the endpoints of δ lies in $\widetilde{Y}_{\Sigma_1} \cap \widetilde{Z}_{\Sigma_2}$. But

$$\mathrm{lk}_\Theta(\Sigma_2) \subseteq \Sigma_1$$

and so δ lies in $\widetilde{Y}_{\Sigma_1}$ entirely (by the Intersection Axiom). Thus $\widetilde{Y}_{\Sigma_1}$ and $\widetilde{Y}_{\Sigma_2}$ intersect non-trivially (at the other endpoint of δ). ☐

Lemma 7.7 (Composition Property)**.** *Let $\mathcal{X}$ be a cubical system for $\mathcal{P}$, and let $\Delta, \Sigma, \Theta \in \mathcal{P}$ satisfy $\Delta \subseteq \Sigma \subseteq \Theta$. Suppose that we are given a standard copy $\widetilde{Y}_\Delta$ of $\widetilde{X}_\Delta$ in $\widetilde{X}_\Sigma$, and a standard copy $\widetilde{Z}_\Sigma$ of $\widetilde{X}_\Sigma$ in $\widetilde{X}_\Theta$. Then there*

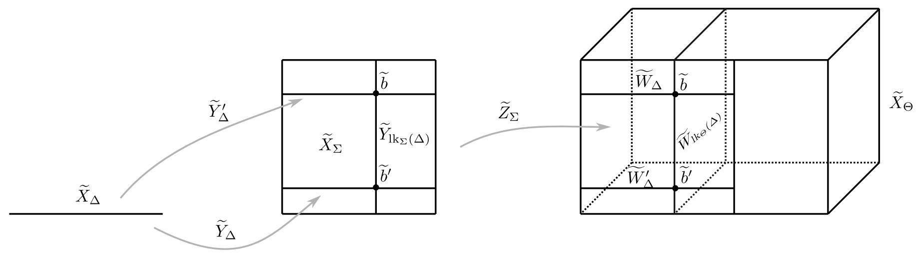

FIGURE 1. A schematic for the proof of Lemma 7.7

exists a standard copy of $\widetilde{X}_\Delta$ *in* $\widetilde{X}_\Theta$*, whose image is equal to the image of* $\widetilde{Y}_\Delta$ *in* $\widetilde{Z}_\Sigma$*.*

Proof. The System Intersection Axiom, Lemma 7.5, and the Intersection Axiom give us a standard copy $\widetilde{W}_\Delta$ of $\widetilde{X}_\Delta$ in $\widetilde{X}_\Theta$ contained in $\widetilde{Z}_\Sigma$, which is an image of a standard copy $\widetilde{Y}'_\Delta$ of $\widetilde{X}_\Delta$ in $\widetilde{X}_\Sigma$. Now the Orthogonal Axiom gives us a standard copy $\widetilde{Y}_{\mathrm{lk}_\Sigma(\Delta)}$ of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta)}$ in $\widetilde{X}_\Sigma$ which intersects both $\widetilde{Y}_\Delta$ and $\widetilde{Y}'_\Delta$, the latter in a point $\widetilde{b}$.

Now the Orthogonal Axiom gives us a standard copy $\widetilde{W}_{\mathrm{lk}_\Theta(\Delta)}$ of $\widetilde{X}_{\mathrm{lk}_\Theta(\Delta)}$ in $\widetilde{X}_\Theta$ which intersects $\widetilde{W}_\Delta$ at the image of point $\widetilde{b}$. The Intersection Axiom implies that $\widetilde{W}_{\mathrm{lk}_\Theta(\Delta)}$ intersects $\widetilde{Z}_\Sigma$ in a copy of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta)}$, which is also the image of a copy of $\widetilde{X}_{\mathrm{lk}_\Sigma(\Delta)}$ in $\widetilde{X}_\Sigma$. But this copy contains $\widetilde{b}$, and two standard copies of a given complex intersect if and only if they coincide. Therefore it is equal to $\widetilde{Y}_{\mathrm{lk}_\Sigma(\Delta)}$. Thus it intersects $\widetilde{Y}'_\Delta$ in a point $\widetilde{b}'$, and the image of this point in $\widetilde{X}_\Theta$ lies in the image of $\widetilde{Y}_\Delta$ in $\widetilde{Z}_\Sigma$ and $\widetilde{W}_{\mathrm{lk}_\Theta(\Delta)}$.

The Orthogonal Axiom gives us a copy $\widetilde{W}'_\Delta$ of $\widetilde{X}_\Delta$ in $\widetilde{X}_\Theta$ which contains $\widetilde{b}'$. Hence $\widetilde{W}'_\Delta$ intersects $\widetilde{Z}_\Sigma$, and the Intersection Axiom implies that this intersection is a standard copy equal to the image of a standard copy of $\widetilde{X}_\Delta$ in $\widetilde{X}_\Sigma$. But this standard copy intersects $\widetilde{Y}_\Delta$ in $\widetilde{b}'$, and so coincides with $\widetilde{Y}_\Delta$. This finishes the proof. ☐

The following lemma will allow us to ignore the (formal) differences between graph systems and cubical systems when A_Γ is a free group.

Lemma 7.8. *Suppose that* $\dim\Gamma = 1$*, let* $\mathcal{P}$ *be a subsystem of invariant graphs, and let* $\mathcal{X}$ *be a graph system for* $\mathcal{P}$*. There exists a cubical system* $\mathcal{X}'$ *for* $\mathcal{P}$ *such that* $X'_\Sigma = X_\Sigma$ *for all* $\Sigma \in \mathcal{P}$*, and* $\widetilde{\iota}'_{\Delta\Sigma} = \widetilde{\iota}_{\Delta,\Sigma}$ *for all* $\Delta, \Sigma \in \mathcal{P} \smallsetminus \{\emptyset\}$ *with* $\Delta \subseteq \Sigma$*.*

Proof. We define $X'_\Sigma = X_\Sigma$ for all $\Sigma \in \mathcal{P}$, and $\widetilde{\iota}'_{\Delta\Sigma} = \widetilde{\iota}_{\Delta,\Sigma}$ for all

$$\Delta, \Sigma \in \mathcal{P} \smallsetminus \{\emptyset\}$$

with $\Delta \subseteq \Sigma$. We also set $\widetilde{\iota}'_{\emptyset\Delta}$ to be the identity map for all $\Delta \in \mathcal{P}$.

We have thus defined all the data, and now we need to verify the four axioms.

Product and Orthogonal Axioms. These are trivially satisfied.

Intersection Axiom. Let $\Sigma_1, \Sigma_2, \Theta \in \mathcal{L}^\phi$ be such that $\Sigma_i \subseteq \Theta$ for both values of i. Suppose that we are given standard copies of $\widetilde{X}_{\Sigma_i}$ in $\widetilde{X}_\Theta$ whose images intersect non-trivially. If $\Sigma_1, \Sigma_2 \neq \emptyset$ then Lemma 5.2 implies that $\Sigma_1 \cup \Sigma_2 \in \mathcal{L}^\phi$. Thus the Intersection Axiom for pairs (in a graph system) implies that the images of $\widetilde{X}_{\Sigma_1}$ and $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_{\Sigma_1 \cup \Sigma_2}$ intersect in the image of $\widetilde{X}_{\Sigma_1 \cap \Sigma_2}$. Then the Composition Axiom for graph systems informs us that the corresponding statement is true in $\widetilde{X}_\Theta$ (since $\Sigma_1 \cup \Sigma_2 \subseteq \Theta$).

Now suppose that, without loss of generality, $\Sigma_2 = \emptyset$. Then the standard copy of $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_\Theta$ we are given is a point, and so it is a standard copy of $\widetilde{X}_{\Sigma_1 \cap \Sigma_2}$ in $\widetilde{X}_\Theta$.

System Intersection Axiom. This is equivalent to the System Intersection Property for graph systems. □

Definition 7.9. Let $\Delta \subseteq \Sigma$ be two elements of $\mathcal{L}^\phi$, and suppose that $\mathcal{X}'$ is a cubical system for $\mathcal{L}^\phi_\Delta$. Let $\Delta = \Delta_1 * \cdots * \Delta_k$ be a join decomposition. We say that a cubical system $\mathcal{X}$ for $\mathcal{L}^\phi_\Sigma$ *extends* $\mathcal{X}'$ if and only if

- when $|\Delta_i| \geqslant 2$, for every $E \in \mathcal{L}^\phi_{\Delta_i}$ we have an H-equivariant isometry $j_E \colon X'_E \to X_E$ which preserves the markings via a map $\widetilde{j}_E$;
- when Δ_i is a singleton we have an isometry $j_{\Delta_i} \colon X'_{\Delta_i} \to X_{\Delta_i}$, which preserves the markings via a map $\widetilde{j}_{\Delta_i}$, and is H-equivariant up to homotopy;
- the maps $\widetilde{\iota}', \widetilde{\iota}$ and $\widetilde{j}$ make the obvious diagrams commute.

We say that $\mathcal{X}$ *strongly extends* $\mathcal{X}'$ if and only if all the maps j are H-equivariant.

Note that this notion generalises the notion of extending and strongly extending for graph systems.

Proposition 7.10. *Suppose that $\Gamma = \Gamma_1 * \Gamma_2$ with $\Gamma_i \in \mathcal{L}^\phi$ for both values of i. Let $\mathcal{X}^i$ be a cubical system for $\mathcal{L}^\phi_{\Gamma_i}$. Then there exists a cubical system $\mathcal{X}$ for $\mathcal{L}^\phi$ which extends both $\mathcal{X}^1$ and $\mathcal{X}^2$ strongly.*

Proof. This construction is fairly straightforward. Given $\Sigma \in \mathcal{L}^\phi$ we define $\Sigma_i = \Sigma \cap \Gamma_i$. Note that we have $\Sigma = \Sigma_1 * \Sigma_2$.

Take such a Σ. We define $\widetilde{X}_\Sigma = \widetilde{X}^1_{\Sigma_1} \times \widetilde{X}^2_{\Sigma_2}$. We mark it with the product marking, and immediately see that $\widetilde{X}_\Sigma$ and its quotient X_Σ are of the form required in the definition of a cubical system.

Now let us also take $\Theta \in \mathcal{L}^\phi$ such that $\Sigma \subseteq \Theta$. Crucially,

$$\mathrm{lk}_\Theta(\Sigma) = \mathrm{lk}_{\Theta_1}(\Sigma_1) * \mathrm{lk}_{\Theta_2}(\Sigma_2)$$

We define $\widetilde{\iota}_{\Sigma,\Theta} = \widetilde{\iota}_{\Sigma_1,\Theta_1} \times \widetilde{\iota}_{\Sigma_2,\Theta_2}$. Again, it is clear that these maps have the form required in the definition.

The four axioms are immediate; they all follow from the observation that any standard copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_\Theta$ for any $\Sigma, \Theta \in \mathcal{L}^\phi$ with $\Sigma \subseteq \Theta$ is equal to a product of standard copies of $\widetilde{X}_{\Sigma_i}$ in $\widetilde{X}_{\Theta_i}$ for both values of i, and vice-versa – taking a product of two such copies yields a copy of the former kind. □

We will say that the cubical system $\mathcal{X}$ obtained above is the *product* of systems $\mathcal{X}^1$ and $\mathcal{X}^2$.

Lemma 7.11. *Suppose that Γ is a cone over $s \in \Gamma$, with $\{s\}, \Gamma \smallsetminus \{s\} \in \mathcal{L}^\phi$. Let $\mathcal{X}'$ be a cubical system for $\mathcal{L}^\phi_{\{s\}}$, and let $\mathcal{X}$ be a cubical system for $\mathcal{L}^\phi$ which extends $\mathcal{X}'$. Then there exists a cubical system $\mathcal{X}''$ for $\mathcal{L}^\phi$ which extends $\mathcal{X}$ and extends $\mathcal{X}'$ strongly.*

Proof. We define $\mathcal{X}''$ to be the product of $\mathcal{X}_{\Gamma \smallsetminus \{s\}}$ and $\mathcal{X}'$. It is then clear that $\mathcal{X}''$ extends $\mathcal{X}'$ strongly. To verify that $\mathcal{X}''$ extends $\mathcal{X}$ we only need to observe that $X''_{\{s\}} = X'_{\{s\}}$, which in turn is isometric to $X_{\{s\}}$ in a way H-equivariant up to homotopy since $\mathcal{X}$ extends $\mathcal{X}'$. □

8. Gluing

Suppose that $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ is link-preserving, and that H is finite. Let $\Sigma, \Theta \in \mathcal{L}^\phi$ be such that $\Sigma \cup \Theta = \Gamma$. Take $E = \Sigma \cap \Theta$. Let us set $E' = E \smallsetminus Z(E)$, and $Z(E) = \{s_1, \ldots, s_k\}$. Since ϕ is link-preserving, we have $E' \in \mathcal{L}^\phi$ and $\{s_i\} \in \mathcal{L}^\phi$ for each i.

Suppose that we have cubical systems $\mathcal{X}$ and $\mathcal{X}'$ for $\mathcal{L}^\phi_\Sigma$ and $\mathcal{L}^\phi_\Theta$ respectively, such that $\mathcal{X}'$ extends $\mathcal{X}_E$.

The main goal of this section is to show that (under mild assumptions) one can equivariantly glue X_Σ to X'_Θ so that the result realises the correct action $\phi : H \to \mathrm{Out}(A_\Gamma)$. This will be done in two steps; first showing that the loops X_{s_i} and X'_{s_i} are actually equal (and hence that $\mathcal{X}'$ extends $\mathcal{X}_E$ strongly), and then constructing a gluing.

Throughout the paper we will repeatedly use the following construction.

Definition 8.1. Given a cube complex X_Γ realising an action $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ we say that an element $h_p \in \mathrm{Aut}(A_\Gamma)$ is a *geometric representative* of h if and only if it is obtained by the following procedure: take a basepoint $\widetilde{p} \in \widetilde{X}_\Gamma$ with a projection $p \in X_\Gamma$, and a path γ from p to $h.p$. The choice of $\widetilde{p}$ induces an identification $\pi(X_\Gamma, p) = A_\Gamma$. We now take $h_p \in \mathrm{Aut}(\pi_1(X_\Gamma, p))$ to be the automorphism induced on the fundamental group by first applying h to X_Γ, and then pushing the basepoint back to p via γ.

Suppose that H acts on a graph (without leaves) of rank 1 (i.e. on a subdivided circle). Let us fix an orientation on the circle.

Definition 8.2. Given an element $h \in H$ we say that it *flips* the circle if and only if it reverses the circle's orientation; otherwise we say that it *rotates* the circle. In the latter case we say that it rotates by k if and only if the simple path from some vertex to its image under h, going along the orientation of the circle, has combinatorial length k.

Lemma 8.3. *Suppose that $\Sigma \neq \mathrm{st}_\Sigma(s_i)$ and $\Theta \neq \mathrm{st}_\Theta(s_i)$ for some i. Then $X_{\{s_i\}}$ and $X'_{\{s_i\}}$ are H-equivariantly isometric.*

Proof. To simplify notation set $s = s_i$, $Y = X_{s_i}$ and $Z = X'_{s_i}$. Note that Y and Z are isometric. Let us subdivide the edges of Y and Z so that both actions of H are combinatorial, and so that the two loops can be made combinatorially isomorphic.

Let m denote the number of vertices in the subdivided loop Y. Fix an orientation on both Y and Z so that going around the loops once in the positive direction yields $s \in A_\Gamma$.

We first focus on those $h \in H$ which map the conjugacy class of $s \in A_\Gamma$ to itself. Then h acts on Y and Z as a rotation. We claim that the two actions of h rotate by the same number of vertices.

Consider a representative $h_0 \in \mathrm{Aut}(A_\Gamma)$ of $\phi(h)$ which preserves s. For any such representative we have $h_0^{\mathrm{ord}(h)}$ equal to a conjugation which fixes s. Hence

$$h_0^{\mathrm{ord}(h)} = c(s^{K(h_0)} t_0)$$

where

$$t_0 \in A_{\mathrm{lk}(s)}$$

We know that $s \notin Z(\Gamma)$ (otherwise $\Sigma = \mathrm{st}_\Sigma(s)$ which contradicts our assumption). Thus, the integer $K(h_0)$ is unique.

Since $\mathrm{lk}(s) \in \mathcal{L}^\phi$, the subgroup $h_0(A_{\mathrm{lk}(s)})$ is conjugate to $A_{\mathrm{lk}(s)}$. But the former subgroup must centralise $h_0(s) = s$, and so

$$h_0(A_{\mathrm{lk}(s)}) \leqslant A_{\mathrm{st}(s)}$$

The only subgroup of $A_{\mathrm{st}(s)} = A_s \times A_{\mathrm{lk}(s)}$ conjugate to $A_{\mathrm{lk}(s)}$ is $A_{\mathrm{lk}(s)}$ itself, and therefore $h_0(A_{\mathrm{lk}(s)}) = A_{\mathrm{lk}(s)}$.

Let h_1 and h_2 be representatives as above. There exists a unique integer l such that

$$h_2 = c(s^l t) \circ h_1$$

with $t \in A_{\mathrm{lk}(s)}$. Note that

$$c(x) \circ h_1 = h_1 \circ c(h_1^{-1}(x))$$

and so in particular, using $h_1(s) = s$ and $h_1(A_{\mathrm{lk}(s)}) = A_{\mathrm{lk}(s)}$, we get

$$(c(s^l t) \circ h_1)^{\mathrm{ord}(h)} = c(s^{\mathrm{ord}(h) l} t') \circ h_1^{\mathrm{ord}(h)}$$

where $t' \in A_{\mathrm{lk}(s)}$. Thus

$$\begin{aligned} c(s^{K(h_2)} t_2) &= h_2^{\mathrm{ord}(h)} \\ &= (c(s^l t) \circ h_1)^{\mathrm{ord}(h)} \\ &= c(s^{\mathrm{ord}(h) l} t') \circ h_1^{\mathrm{ord}(h)} \\ &= c(s^{\mathrm{ord}(h) l + K(h_1)} t' t_1) \end{aligned}$$

This shows that $K(h_1)$ mod $\operatorname{ord}(h)$ is independent of the representative, and so we can define $K(h) \in \mathbb{Z}/\operatorname{ord}(h)\mathbb{Z}$ in the obvious way. This algebraic invariant will be the main tool in showing that Y and Z are H-equivariantly isometric.

Fix a basepoint p in X_Σ lying in $\operatorname{im}(\iota_{E,\Sigma})$, and a basepoint q in X_Θ lying in $\operatorname{im}(\iota_{E,\Theta})$.

Let $h_p \in \operatorname{Aut}(A_\Sigma)$ be the geometric representative of h (using the action of H on X_Σ), obtained by taking the basepoint p and a path γ inside

$$\operatorname{im}(\iota_{E,\Sigma}) \cong \prod_{i=1}^{k} X_{\{s_i\}} \times X_{E'}$$

which first travels orthogonally to $Y = X_{\{s_i\}}$, and then along the copy of Y containing p (in the negative direction).

If h rotates Y by μ vertices (in the positive direction), then $(h_p)^{\operatorname{ord}(h)}$ is equal to the conjugation by $s^{\operatorname{ord}(h)\mu/m}t$ for some $t \in A_{\operatorname{lk}_\Sigma(s)}$. Since $\Sigma \neq \operatorname{st}_\Sigma(s)$, the number $\operatorname{ord}(h)\mu/m$ is unique. Hence, by taking any representative of $\phi(h)$ in $\operatorname{Aut}(A_\Gamma)$ which restricts to h_p on A_Σ, we see that

$$K(h) = \operatorname{ord}(h)\mu/m \bmod \operatorname{ord}(h)$$

Now we define h_q in the analogous manner using X_Θ instead of X_Σ. Since h_p and h_q represent the same element h, the computation above shows that they rotate by the same number of vertices. We have thus dealt with elements $h \in H$ which fix the conjugacy class of s.

If h maps the conjugacy class of s to the conjugacy class of s^{-1}, then h must flip both Y and Z, and therefore must have two fixed points on each loop. If another element $g \in H$ flips Y, then hg rotates Y, and by the above rotates Z by the same number of vertices. This implies that the fixed points of h and g on Y differ by the same number of vertices as the respective fixed points on Z.

Hence there exists an identification between Y and Z which is H-equivariant. □

Now suppose that $\mathcal{X}'$ extends $\mathcal{X}_E$ strongly. Suppose further that we have fixed standard copies of $\widetilde{X}_E$: one in $\widetilde{X}_\Sigma$, called $\widetilde{P}$, and one in $\widetilde{X}'_\Theta$, called $\widetilde{Q}$. We can from a cube complex marked by A_Γ from $\widetilde{X}_\Sigma$ and $\widetilde{X}'_\Theta$ by gluing $\widetilde{P}$ and $\widetilde{Q}$. Note that this is in general not unique, as there might be more than one marking-respecting isometry of $\widetilde{P}$ and $\widetilde{Q}$ such that the projections P and Q become H-equivariantly isomorphic.

Let $\widetilde{Y}$ denote the glued-up complex, and let Y denote its projection. Our gluing gives us an action of H on the projection Y. This induces an action $H \to \operatorname{Out}(A_\Gamma)$ in the obvious way; but this action is in general not equal to ϕ. We are now going to measure the difference of these two actions.

Let us choose a point $\widetilde{p} \in \widetilde{P}$ as a basepoint. For each $h \in H$ we choose a path $\gamma(h)$ in P connecting p to $h.p$. Since P and Q are standard copies of the same complex, they are isomorphic via a fixed isomorphism. This gives us a copy of $\widetilde{p}$ and $\gamma(h)$ in $\widetilde{Q}$ and Q respectively; let us denote the former by $\widetilde{q}$ and the latter by $\gamma'(h)$. The points p and q are naturally points in Y.

Now let h_p and h_q denote the respective geometric representatives of h. The former restricts to the same automorphism as $\phi(h)$ on A_Σ, the latter on A_Θ. They represent the same outer automorphism, and agree on A_E. Thus we have

$$h_p h_q^{-1} = c(x(h))$$

for some $x(h) \in C(A_E)$.

Definition 8.4. We say that the gluing above is *faulty within* $G \leqslant C(A_E)$ if and only if $x(h) \in G$ for all h (with our choices of $\widetilde{p}$ and $\gamma(h)$).

Proposition 8.5 (Gluing Lemma)**.** *Suppose that $\widetilde{Y}$ is faulty within $Z(A_E)$. Then there exists another gluing as above, $\widetilde{X}$, such that its projection X realises ϕ.*

Proof. First let us note that when $\Gamma = Z(E)$ then all the conjugations $c(x(h)$ are trivial, and $\widetilde{Y}$ is already as desired. We will henceforth assume that $\Gamma \neq Z(E)$.

By assumption the gluing $\widetilde{Y}$ gives us geometric representatives h_p and h_q such that

$$h_p h_q^{-1} = c(x(h))$$

with $x(h) \in Z(A_E) = A_{Z(E)}$.

We assume that $x(h) \in A_{Z(E) \smallsetminus Z(\Gamma)}$ unless $x(h)$ is the identity; we can always do this since conjugating by elements in $A_{Z(\Gamma)} = Z(A_\Gamma)$ is trivial. Now we define $h'_p \in \mathrm{Aut}(A_\Gamma)$ so that $h'_p h_p^{-1}$ is a conjugation by an element of $Z(A_\Sigma)$ and that $h'_p h_q^{-1}$ is equal to a conjugation by an element in $A_{Z(E) \smallsetminus (Z(\Gamma) \cup Z(\Sigma))}$; we further define $h'_q \in \mathrm{Aut}(A_\Gamma)$ so that $h'_q h_q^{-1}$ is a conjugation by an element of $Z(A_\Theta)$ and that

$$h'_p h'^{-1}_q = c(x'(h))$$

with $x'(h) \in A_{Z(E) \smallsetminus (Z(\Gamma) \cup Z(\Sigma) \cup Z(\Theta))}$. Since $Z(\Sigma) \cap Z(\Theta) = Z(\Gamma)$, the elements h'_p and h'_q are unique.

Consider now ${h'_p}^{\mathrm{ord}(h)}$. It is equal to a conjugation $c(y_p)$ where $y_p \in N(A_\Sigma)$ by construction. Let $y_q \in N(A_\Theta)$ be the corresponding element for h'_q. Now

$$c(y_p) = {h'_p}^{\mathrm{ord}(h)} = \big(c(x'(h)) h'_q\big)^{\mathrm{ord}(h)} = c(x'') {h'_q}^{\mathrm{ord}(h)} = c(x'') c(y_q)$$

where

$$x'' = \prod_{i=0}^{\mathrm{ord}(h)} (h'_q)^i (x'(h)) \in A_{Z(E) \smallsetminus (Z(\Sigma) \cup Z(\Theta))}$$

since $Z(E) \smallsetminus (Z(\Sigma) \cup Z(\Theta)) \in \mathcal{L}^\phi$ and $h'_q(A_E) = A_E$ by construction.

The element y_p is determined up to $C(A_\Sigma)$ by its action by conjugation on A_Σ. Here however we immediately see that $c(y_p)$ is equal to conjugation by the element given by the loop obtained from concatenating images of our path $\gamma(h)$ under successive iterations of h.

We repeat the argument for y_q and conclude that $y_p = y_q$ up to $C(A_\Sigma)C(A_\Theta)$. But we have already shown that they differ by $x'' \in A_{Z(E)\smallsetminus(Z(\Sigma)\cup Z(\Theta))}$ up to $Z(A_\Gamma)$. Hence $c(x'') = 1$ as $Z(E)$ intersects $\mathrm{lk}(\Sigma)$ and $\mathrm{lk}(\Theta)$ trivially. Thus $x'' \in Z(A_\Gamma) \cap A_{Z(E)\smallsetminus(Z(\Sigma)\cup Z(\Theta))} = \{1\}$. We obtain

$$1 = x'' = \prod_{i=0}^{\mathrm{ord}(h)} (h'_q)^i(x'(h))$$

and so $x'(h)$ must lie in the subgroup of $A_{Z(E)}$ generated by all vertices $s_i \in Z(E)$ such that h flips the loop P_i, since any other generator s_j satisfies

$$\prod_{i=0}^{\mathrm{ord}(h)} (h'_q)^i(s_j) = {s_j}^{\mathrm{ord}(h)} \neq 1$$

The purpose of the proof so far was exactly to establish that $x'(h)$ lies in the subgroup of $A_{Z(E)}$ generated by all vertices $s_i \in Z(E)$ such that h flips the loop P_i.

Now we are going to construct a whole family of gluings, and show that one of them is as desired.

To analyse the situation we need to look more closely at $\widetilde{P} = \widetilde{Q}$ (with the equality coming from the fact that $\mathcal{X}'$ extends $\mathcal{X}_E$ strongly). By the Product Axiom, we have $\widetilde{P} = \widetilde{P}_0 \times \widetilde{P_1} \times \cdots \times \widetilde{P_k}$, with $\widetilde{P}_0 = \widetilde{X}_{E'}$, and $\widetilde{P}_i = \widetilde{X}_{s_i}$ for $i \geqslant 1$. Our basepoint $\widetilde{p}$ satisfies

$$\widetilde{p} = (\widetilde{p}_0, \dots, \widetilde{p}_k) \in \widetilde{P}_0 \times \cdots \times \widetilde{P}_k$$

Let $\widetilde{q} = (\widetilde{q}_0, \dots, \widetilde{q}_k)$ be the corresponding expression for $\widetilde{q}$.

We construct complexes from $\widetilde{X}_\Sigma$ and $\widetilde{X}'_\Theta$ by gluing $\widetilde{P}$ and $\widetilde{Q}$ in a way respecting the markings, and so that the projections P and Q are glued in an H-equivariant fashion. The resulting space is determined by the relative position of $\widetilde{p}$ and $\widetilde{q}$, now both seen as points in the glued-up complex (so in particular they do not need to coincide). We glue so that the images of $\widetilde{p}$ and $\widetilde{q}$ coincide if we project $\widetilde{X}_E$ onto $\widetilde{X}_{E'}$ – this is in fact forced on us since $\widetilde{X}_{E'}$ can be glued to itself only in one way. Hence any such gluing will give us a geodesic from $\widetilde{p}$ to $\widetilde{q}$, which will lie in a subcomplex of $\widetilde{X}_E$ isomorphic to $\widetilde{P}$, and hence isometric to a Euclidean space.

We start by taking $x \in A_{Z(E)}$; we form a complex $\widetilde{X}^x$ (with projection X^x as usual) by gluing as above, so that the geodesic $\widetilde{\delta}$ we just discussed is such that if we extend it to twice its length (which is possible in a Euclidean space), still starting at $\widetilde{p}$, the projection in X^x becomes a closed loop equal to $x \in \pi_1(X^x, p)$.

Note that given $x \in Z(E)$ such a geodesic (and hence gluing) always exist: the Euclidean space we discussed is marked by $A_{Z(E)}$, and so the point $x(\widetilde{p})$ lies therein. We take the unique geodesic from $\widetilde{p}$ to $x(\widetilde{p})$, cut it in half, and declare the first half to be $\widetilde{\delta}$.

Let us now calculate the action on (conjugacy classes in) A_Γ induced from the action of h on X^x. The element h maps the local geodesic δ (the projection of $\widetilde{\delta}$) to a local geodesic $h.\delta$ connecting $h.p$ to $h.q$ such that the loop obtained by following $\gamma(h)$ (starting at p), $h.\delta$, the inverse of $\gamma'(h)$, and the inverse of δ, gives an element $\overline{x}^h \in \pi_1(P,p)$ which is the projection of x onto the subgroup of $Z(E)$ generated by all the generators s_i such that $h'_p(s_i) = s_i^{-1}$. Hence the action of h on X^x, followed by pushing the basepoint p via $\gamma(h)$ as before, gives us an automorphism equal to h'_p on the subgroup $\pi_1(X_\Sigma, p) = A_\Sigma$, and to $c(\overline{x})h'_q$ on the subgroup $\pi_1(X_\Theta, p) = A_\Theta$.

It follows that the action of h on $X^{x'(h)}$ is the correct one: by the observation above, $x'(h)$ lies in the subgroup of $A_{Z(E)}$ generated by all the vertices in $Z(E)$ which are mapped to their own inverse under h'_p. Hence $\overline{x'(h)}^h = x'(h)$, and it is enough to observe that $c(x'(h))h'_q = h_p$.

We now need to specify a single x that will work for all elements $h \in H$; we will denote such an x by $x'(H)$. If there is a vertex in $Z(E)$ which is preserved by all elements h'_p (for all $h \in H$), then we set the corresponding coordinate of $x'(H)$ to 0. If a generator is mapped to its inverse by some h'_p, then we set the corresponding coordinate of $x'(H)$ to be equal to the relevant coordinate in $x'(h)$.

Since this definition involves making choices (of elements h that flip a generator), we need to make sure that we indeed obtain the desired action. Let $g \in H$ be any element. We need to confirm that $\overline{x'(g)}^g = \overline{x'(H)}^g$.

Suppose that this is not the case; then there exists a generator s_i such that $g'(s_i) = {s_i}^{-1}$, and $x'(g)$ and $x'(H)$ disagree on the s_i-coordinate. This means that the geometric representative g'_p obtained from the action on $X^{x'(H)}$ is not a representative of $\phi(g)$. To make it such a representative we need to postcompose it with a partial conjugation of A_Θ by $\overline{x'(g)}^g \big(\overline{x'(H)}^g\big)^{-1}$, which has a non-trivial s_i-coordinate.

The construction of $x'(H)$ required an element $h \in H$ such that h'_p flips the loop P_i. We have $g = fh$ with f acting trivially on the conjugacy class of s_i. Hence, even though f might not act correctly on $X^{x'(H)}$, the geometric representative f'_p can be made into a representative of $\phi(f)$ by postcomposing it with a partial conjugation of A_Θ by an element of $Z(E)$ with a trivial s_i-coordinate. The exact same statement is true for h. Using the fact that f'_p maps each s_j either to itself or its inverse, we deduce that $f'_p h'_p$ can be made into a representative of $\phi(fh) = \phi(g)$ by postcomposing it with a partial conjugation of A_Θ by an element of $Z(E)$ with a trivial s_i-coordinate. But $f'_p h'_p$ differs from $(fh)'_p = g'_p$ by a conjugation, and hence $\overline{x'(g)}^g \big(\overline{x'(H)}^g\big)^{-1}$ cannot have a non-trivial s_i-coordinate (as $\Gamma \neq Z(E)$). $\square$

In particular we have

Corollary 8.6. *If* $\mathrm{lk}(E) = \emptyset$ *then there exists a complex obtained from* $\widetilde{X}_\Sigma$ *and* $\widetilde{X}'_\Theta$ *by gluing* $\widetilde{P}$ *to* $\widetilde{Q}$ *such that its projection realises* ϕ.

Proof. When $\mathrm{lk}(E) = \emptyset$ we have $C(A_E) = Z(A_E)$. Hence any gluing will be faulty within $Z(A_E)$, and then the existence of a desired gluing follows from the previous proposition. □

9. Realisation for right-angled Artin groups

To emphasize the inductive character of our proof (and to simplify statements) let us introduce the following definitions.

Definition 9.1. We say that *Relative Nielsen Realisation holds* for an action $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ if and only if given any $\Delta \in \mathcal{L}^\phi$ and any cubical system $\mathcal{X}'$ for $\mathcal{L}^\phi_\Delta$, there exists a cubical system $\mathcal{X}$ for $\mathcal{L}^\phi$ extending $\mathcal{X}'$.

Corollary 9.2 (of Theorem 6.1)**.** *Relative Nielsen Realisation holds for all actions* $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ *with* Γ *of dimension 1.*

Lemma 9.3. *Assume that* ϕ *is link-preserving, and that Relative Nielsen Realisation holds for all link-preserving actions* $\psi\colon H \to \mathrm{Out}(A_\Sigma)$ *with* $\dim \Sigma < \dim \Gamma$. *Suppose that* $\Gamma = \Delta * (E \cup \Theta)$, *where* Δ *and* Θ *are non-empty. Suppose that* $\Delta * E \in \mathcal{L}^\phi$, *and that we are given a cubical system* $\mathcal{X}'$ *for* $\mathcal{L}^\phi_{\Delta * E}$. *Then there exists a cubical system* $\mathcal{X}$ *for* $\mathcal{L}^\phi$ *extending* $\mathcal{X}'$, *which furthermore extends* $\mathcal{X}'_\Delta$ *strongly.*

Proof. Since $\mathcal{L}^\phi$ contains all links, we have $E \cup \Theta \in \mathcal{L}^\phi$. Note that $E \cup \Theta$ has lower dimension than Γ. Since

$$E = (E \cup \Theta) \cap (\Delta * E) \in \mathcal{L}^\phi$$

we can apply the assumption to the induced action on $A_{E\cup\Theta}$, and obtain a cubical system $\mathcal{X}_{E\cup\Theta}$ extending $\mathcal{X}'_E$, the subsystem of $\mathcal{X}'$ corresponding to $\mathcal{L}^\phi_E$. This last system also contains a subsystem $\mathcal{X}'_\Delta$ corresponding to $\mathcal{L}_\Delta$.

We now define $\mathcal{X}$ to be the product of $\mathcal{X}_{E\cup\Theta}$ and $\mathcal{X}'_\Delta$ (compare Proposition 7.10). □

Definition 9.4. Let $\Sigma \subseteq \Gamma$ be a subgraph. We define its *boundary* $\partial\Sigma$ to be the set of all vertices of Σ whose link is not contained in Σ.

.

Lemma 9.5. *Suppose that* Γ *is connected and is not a join, and let* $\Sigma \in \mathcal{L}^\phi$ *be a proper, non-empty, connected subgraph which is maximal with respect to these four properties. Then for every* $w \in \partial\Sigma$ *we have* $\Gamma \smallsetminus \Sigma \subseteq \mathrm{lk}(w)$.

Proof. Let $\Theta = \Gamma \smallsetminus \Sigma$, and let $w \in \partial\Sigma$. We have $\widehat{\mathrm{st}}(w) \in \mathcal{L}^\phi$, which is connected and properly contained in Γ, since Γ is not a join. Now

$$\mathrm{lk}(\widehat{\mathrm{st}}(w) \cap \Sigma) \subseteq \mathrm{lk}(w) \subseteq \widehat{\mathrm{st}}(w)$$

and so $\Sigma \cup \widehat{\mathrm{st}}(w) \in \mathcal{L}^\phi$ by Lemma 4.2. It is also connected and non-empty, and thus the maximality of Σ implies that $\Sigma \cup \widehat{\mathrm{st}}(w) = \Gamma$. Thus $\Theta \subseteq \widehat{\mathrm{st}}(w)$. Let $\Theta_1 = \Theta \cap \mathrm{lk}(w)$ and $\Theta_2 = \Theta \cap \mathrm{lk}(\mathrm{lk}(w))$. If the latter is empty then we are done, so let us assume that it is not empty.

If $\widehat{\mathrm{st}}(w) = \mathrm{st}(w)$ then $\Theta_2 = \Theta \cap \{w\} = \emptyset$ and we are done. Otherwise $\mathrm{lk}(\mathrm{lk}(w))$ is not a cone (since w is isolated in it), and thus the link of $\mathrm{lk}(\mathrm{lk}(w))$ is contained in $\mathcal{L}^\phi$. But this triple link is in fact equal to $\mathrm{lk}(w)$, and so we have $\mathrm{lk}(w) \in \mathcal{L}^\phi$. Since $\mathcal{L}^\phi$ is closed under taking intersections, we also have $\Sigma \cap \mathrm{lk}(w) \in \mathcal{L}^\phi$ and thus also $\mathrm{st}(\Sigma \cap \mathrm{lk}(w)) \in \mathcal{L}^\phi$. We have

$$\Theta_2 = \Theta \cap \mathrm{lk}(\mathrm{lk}(w)) \subseteq \mathrm{lk}(\mathrm{lk}(w)) \subseteq \mathrm{lk}(\Sigma \cap \mathrm{lk}(w)) \subseteq \mathrm{st}(\Sigma \cap \mathrm{lk}(w))$$

and so $\mathrm{st}(\Sigma \cap \mathrm{lk}(w))$ is not contained in Σ. We also have

$$\Sigma \cup \mathrm{st}(\Sigma \cap \mathrm{lk}(w)) \in \mathcal{L}^\phi$$

since

$$\mathrm{lk}(\Sigma \cap \mathrm{st}(\Sigma \cap \mathrm{lk}(w))) \subseteq \mathrm{lk}(\Sigma \cap \mathrm{lk}(w)) \subseteq \mathrm{st}(\Sigma \cap \mathrm{lk}(w))$$

as before. The graph $\Sigma \cup \mathrm{st}(\Sigma \cap \mathrm{lk}(w))$ is also connected and non-empty, and so it must contain Θ_1 as well. Thus $\Theta_1 \subseteq \mathrm{st}(\Sigma \cap \mathrm{lk}(w))$, and so

$$\Theta_1 \subseteq \mathrm{lk}(\mathrm{lk}_\Sigma(w))$$

We have $\Theta_1 \subseteq \mathrm{lk}(w)$ and so $\mathrm{lk}(\mathrm{lk}(w)) \subseteq \mathrm{lk}(\Theta_1)$, and thus $\Theta_1 \subseteq \mathrm{lk}(\mathrm{lk}(\mathrm{lk}(w)))$. Combining this with the previous observation we get

$$\begin{aligned} \Theta_1 &\subseteq \mathrm{lk}\big(\mathrm{lk}_\Sigma(w)\big) \cap \mathrm{lk}\big(\mathrm{lk}(\mathrm{lk}(w))\big) \\ &= \mathrm{lk}\big(\mathrm{lk}(\mathrm{lk}(w)) \cup (\mathrm{lk}(w) \cap \Sigma)\big) \\ &\subseteq \mathrm{lk}\big((\mathrm{lk}(\mathrm{lk}(w)) \cup \mathrm{lk}(w)) \cap \Sigma\big) \\ &= \mathrm{lk}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big) \\ &\subseteq \mathrm{st}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big) \end{aligned}$$

The last subgraph is a star of a subgraph in $\mathcal{L}^\phi$, and so is itself in $\mathcal{L}^\phi$. As before we have

$$\mathrm{lk}(\mathrm{st}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big) \cap \Sigma) \subseteq \mathrm{lk}(w) = \mathrm{lk}_\Sigma(w) \cup \Theta_1 \subseteq \mathrm{st}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big)$$

and so $\mathrm{st}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big) \cup \Sigma \in \mathcal{L}^\phi$. It is connected, and contains Θ_1, and therefore it must contain Θ. But then $\Theta \subseteq \mathrm{st}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big)$, which is only possible when

$$\Theta \subseteq \mathrm{lk}\big(\widehat{\mathrm{st}}(w) \cap \Sigma\big) \subseteq \mathrm{lk}(w)$$

which is the desired statement. □

Theorem 9.6. *Assume that* Γ *is connected,* ϕ *is link-preserving, and that Relative Nielsen Realisation holds for for all link-preserving actions* $\psi \colon H \to \mathrm{Out}(A_\Sigma)$ *with* $\dim \Sigma < \dim \Gamma$. *Then there exists a cubical system* $\mathcal{X}$ *for* $\mathcal{L}^\phi$.

The proof of this theorem is a long inductive procedure and will occupy the next section. Here, we record the following corollary.

Corollary 9.7. *Let* $\phi\colon H \to \mathrm{Out}^0(A_\Gamma)$ *be a homomorphism with a finite domain. Then there exists a metric NPC cube complex* X *realising* ϕ*, provided that* Γ *is connected, and triangle- and leaf-free.*

Proof. By Corollary 4.7 ϕ is link-preserving. We take $X = X_\Gamma \in \mathcal{X}$ obtained by an application of the previous theorem. □

Note that in particular the statement above holds for all $\phi\colon H \to \mathrm{Out}(A_\Gamma)$, provided that in addition Γ has no symmetries.

10. Proof of the main theorem

The rest of the paper is devoted to proving Theorem 9.6.

We proceed by induction on the dimension of the defining graph Γ. If this dimension is 1, then Γ is a single point (as it is connected) and the statement of the theorem is immediate. Let us assume for the rest of the proof that Γ contains at least one edge, and therefore the dimension is at least 2.

First we consider the case of Γ being a join $\Gamma_1 * \Gamma_2$ for some non-empty subgraphs Γ_1 and Γ_2. The dimension of Γ_1 is strictly smaller than that of Γ, and $\Gamma_1 = \mathrm{lk}(\Gamma_2) \in \mathcal{L}^\phi$ since ϕ is link-preserving . Thus, by the inductive assumption, Relative Nielsen Realisation holds for Γ_1 and yields a cubical system $\mathcal{X}_{\Gamma_1}$ for $\mathcal{L}^\phi_{\Gamma_1}$. The same applies to Γ_2 and yields a cubical system $\mathcal{L}^\phi_{\Gamma_2}$. In this setting Proposition 7.10 yields a cubical system $\mathcal{X}$ as required.

For the rest of the proof we assume that Γ is not a join. Similar to the proof of Theorem 6.1 we now proceed by induction on the *depth* k of Γ. The depth is the length k of a maximal chain of proper inclusions

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

where each $\Gamma_i \in \mathcal{L}^\phi$ is connected. Since Γ is not a join we have $k > 1$; if $k = 1$ then Γ is equal to the extended star of any of its vertices, and any such graph is a join of two non-empty subgraphs. In what follows, we fix once and for all such a maximal chain of Γ_i, and denote the penultimate term by

$$\Gamma' = \Gamma_{k-1}$$

We let Θ be the complement

$$\Theta = \Gamma \smallsetminus \Gamma'$$

Note that at this point we do not claim that Θ is contained in $\mathcal{L}^\phi$. Lemma 9.5 implies that for all $w \in \partial\Gamma'$ we have $\Theta \subseteq \mathrm{lk}(w)$. Thus we have

$$\partial\Gamma' = \mathrm{lk}(\Theta)$$

Since Γ' has strictly smaller depth than Γ, by our inductive assumption there is a cubical system

$$\mathcal{X}'$$

for the action restricted to $A_{\Gamma'}$ (compare Remark 4.5).

10.1. **Constructing the complexes.** We are now ready to define the cube complexes forming $\mathcal{X}$. The basic idea is to glue the desired complexes from pieces lying in Γ' (where, by induction, we already have them in $\mathcal{X}'$) and pieces overlapping with Θ. The details will be different depending on how the invariant graph in question intersects Γ' and Θ, and thus the construction has several steps.

To describe the cases, we need the following additional graphs. We let

$$\overline{\Theta} = \Theta * \partial\Gamma'$$

and we consider the subsystem of all invariant graphs which contain $\overline{\Theta}$:

$$\mathcal{S} = \{\Sigma \in \mathcal{L}^\phi \mid \overline{\Theta} \subseteq \Sigma\}$$

Note that $\mathcal{S}$ is closed under taking unions by Lemma 4.2, since $\mathrm{lk}(\overline{\Theta}) = \emptyset$. Hence

$$\mathcal{S}_{\Gamma'} = \{\Sigma \cap \Gamma' \mid \Sigma \in \mathcal{S}\}$$

is also closed under taking unions.

We abbreviate

$$\Delta = \bigcap \mathcal{S}_{\Gamma'}$$

For any $\Xi \in \mathcal{L}^\phi$ we let

$$\Xi_1 = \Xi \cap \Gamma' \quad \text{and} \quad \Xi_2 = \Xi \cap (\Delta \cup \Theta)$$

In other words, these are the part of Ξ which lie inside Γ' and inside $\Theta \cup \Delta$.

Step 1: Constructing X_Γ. In this step we will actually do a little more. First we will construct a suitable cubical system $\mathcal{X}''$ for $\mathcal{L}^\phi_{\mathrm{st}(\partial\Gamma')}$ extending $\mathcal{X}'_\Delta$ strongly. Then we will find a fixed standard copy $\widetilde{R}_\Delta$ of $\widetilde{X}''_\Delta$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$. We will also construct a family of standard copies $\widetilde{Z}_\Sigma$ of $\widetilde{X}'_\Sigma$ in $\widetilde{X}'_{\Gamma'}$ for each $\Sigma \in \mathcal{S}_{\Gamma'}$, such that they all intersect in the copy $\widetilde{Z}_\Delta$. We will show that this last copy is fixed, and obtain $\widetilde{X}_\Gamma$ from $\widetilde{X}'_{\Gamma'}$ and $\widetilde{X}''_{\Delta\cup\Theta}$ (which will be shown to contain $\widetilde{R}_\Delta$) by gluing $\widetilde{R}_\Delta$ and $\widetilde{Z}_\Delta$.

Note that the construction of a gluing then also gives us maps

$$\widetilde{\iota}_{\Gamma',\Gamma} \colon \widetilde{X}'_{\Gamma'} \to \widetilde{X}_\Gamma$$

and

$$\widetilde{\iota}_{\Delta\cup\Theta,\Gamma} \colon \widetilde{X}''_{\Delta\cup\Theta} \to \widetilde{X}_\Gamma$$

which will be as desired since $\mathrm{lk}(\Gamma') = \emptyset = \mathrm{lk}(\Delta \cup \Theta)$.

We need to consider two cases depending on whether $\mathrm{lk}(\partial\Gamma')$ intersects Γ' or not.

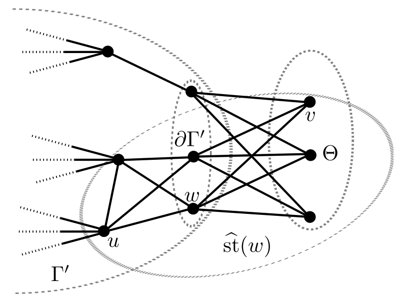

FIGURE 2. The relevant subgraphs of Γ in case 1

Case 1: $\mathrm{lk}(\partial\Gamma') \cap \Gamma' = \emptyset$. Recall that $\partial\Gamma' = \mathrm{lk}(\Theta)$. By the case assumption we therefore have

$$\Theta = \mathrm{lk}(\partial\Gamma') \in \mathcal{L}^\phi$$

Take any $v \in \Theta$. We have $\partial\Gamma' = \widehat{\mathrm{st}}(v)_1$ and thus $\partial\Gamma' \in \mathcal{L}^\phi$. Hence in particular $\Delta = \partial\Gamma'$. Since Γ is not a join, there is a vertex $w \in \partial\Gamma'$ such that $\mathrm{lk}(w) \not\subseteq \overline{\Theta}$ (otherwise $\Gamma = \overline{\Theta}$ as Γ is connected).

Considering the decomposition

$$\widehat{\mathrm{st}}(w) = (\widehat{\mathrm{st}}(w) \cap \partial\Gamma') * \big((\widehat{\mathrm{st}}(w) \cap (\Gamma' \smallsetminus \partial\Gamma')) \cup (\widehat{\mathrm{st}}(w) \cap \Theta)\big)$$

Lemma 9.3 yields an auxiliary graph system $\mathcal{X}'''$ for $\mathcal{L}^\phi_{\widehat{\mathrm{st}}(w)}$ which extends $\mathcal{X}'_{\widehat{\mathrm{st}}(w)_1}$, as

$$(\widehat{\mathrm{st}}(w) \cap \partial\Gamma') * (\widehat{\mathrm{st}}(w) \cap (\Gamma' \smallsetminus \partial\Gamma')) = \widehat{\mathrm{st}}(w)_1$$

The system $\mathcal{X}'''$ contains a complex $X'''_{\widehat{\mathrm{st}}(w)_2}$ which is isomorphic (in an H-equivariant way) to the product

$$X'''_{\widehat{\mathrm{st}}(w) \cap \partial\Gamma'} \times X'''_\Theta$$

since

$$\widehat{\mathrm{st}}(w)_2 = \widehat{\mathrm{st}}(w) \cap (\partial\Gamma' \cup \Theta) = (\widehat{\mathrm{st}}(w) \cap \partial\Gamma') \cup \Theta$$

and by the Product Axiom.

Claim. There is a point $\widetilde{r} \in \widetilde{X}'''_\Theta$ whose projection $r \in X'''_\Theta$ is fixed by the action of H.

Proof. Observe that

$$\mathrm{lk}(\widehat{\mathrm{st}}(w)_1) \subseteq \mathrm{lk}(w) \smallsetminus \widehat{\mathrm{st}}(w)_1 \subseteq \mathrm{st}(w) \smallsetminus \widehat{\mathrm{st}}(w)_1 = \mathrm{st}(w) \cap \Theta = \Theta$$

But, by our assumption on w, the subgraph $\mathrm{lk}(w)$ contains a vertex, say u, which does not lie in $\overline{\Theta}$. Hence $u \in \widehat{\mathrm{st}}(w)_1$, but it is not connected to anything in Θ, since $\mathrm{st}(\Theta) = \overline{\Theta}$. Thus

$$\mathrm{lk}(\widehat{\mathrm{st}}(w)_1) \subseteq \Theta \cap \mathrm{lk}(u) = \emptyset$$

and so $\widetilde{X}'''_{\widehat{\mathrm{st}}(w)_1}$ admits a unique (and hence fixed) standard copy in $\widetilde{X}'''_{\widehat{\mathrm{st}}(w)}$.

Similarly observe that

$$\operatorname{lk}(\widehat{\operatorname{st}}(w)_2) \subseteq \operatorname{lk}(\Theta \cup \{w\}) \smallsetminus \widehat{\operatorname{st}}(w)_2 \subseteq \operatorname{lk}_{\partial\Gamma'}(w) \smallsetminus (\operatorname{st}(w) \cap \partial\Gamma') = \emptyset$$

and therefore $\widetilde{X}'''_{\widehat{\operatorname{st}}(w)_2}$ as well admits a unique (and hence fixed) standard copy in $\widetilde{X}'''_{\widehat{\operatorname{st}}(w)}$.

The Matching Property implies that these standard copies of $\widetilde{X}'''_{\widehat{\operatorname{st}}(w)_1}$ and $\widetilde{X}'''_{\widehat{\operatorname{st}}(w)_2}$ intersect in a standard copy $\widetilde{Q}$ of $\widetilde{X}'''_{\widehat{\operatorname{st}}(w)\cap\partial\Gamma'}$; this must again be fixed, since it is an intersection of two fixed standard copies. Let $\widetilde{r} \in \widetilde{X}'''_{\Theta}$ denote the unique point such that

$$\widetilde{X}'''_{\widehat{\operatorname{st}}(w)\cap\partial\Gamma'} \times \{\widetilde{r}\}$$

is this standard copy $\widetilde{Q}$. Observe that $r \in X'''_{\Theta}$, the projection of $\widetilde{r}$, is fixed by H as claimed. □

By Proposition 7.10 there is a system $\mathcal{X}''$ for $\mathcal{L}^{\phi}_{\overline{\Theta}}$, strongly extending both the cubical subsystems $\mathcal{X}'''_{\Theta}$ of $\mathcal{X}'''$ and $\mathcal{X}'_{\partial\Gamma'}$ of $\mathcal{X}'$.

Let $\widetilde{R}$ denote the standard copy of $\widetilde{X}''_{\partial\Gamma'}$ in $\widetilde{X}''_{\overline{\Theta}}$ determined by $\widetilde{r}$. Its projection R in $X_{\overline{\Theta}}$ is preserved by H, since r is. On the other hand, let $\widetilde{Z}$ be the standard copy of $\widetilde{X}'_{\partial\Gamma'}$ in $\widetilde{X}'_{\Gamma'}$; this is unique since $\operatorname{lk}_{\Gamma'}(\partial\Gamma') = \emptyset$ by the assumption of Case 1.

The standard copies $\widetilde{R}$ and $\widetilde{Z}$ are naturally isometric since both are standard copies of the same cube complex. We now form $\widetilde{X}_{\Gamma}$ by gluing $\widetilde{X}'_{\Gamma'}$ to $\widetilde{X}''_{\overline{\Theta}}$ via the natural isomorphism $\widetilde{Z} = \widetilde{R}$.

Claim. The H-action on X_{Γ} inherited from the gluing is the correct one.

Proof. Take $h \in H$. Let us pick a basepoint $\widetilde{p} \in \widetilde{Z} = \widetilde{R}$, and let p denote its projection in $Z = R$. Note that choosing $\widetilde{p}$ fixes an isomorphism between fundamental groups of $Z, X'_{\Gamma'}, X'''_{\widehat{\operatorname{st}}(w)}$ and $X''_{\overline{\Theta}}$ (based at p), and the groups $A_{\partial\Gamma'}, A_{\Gamma'}, A_{\widehat{\operatorname{st}}(w)}$ and $A_{\overline{\Theta}}$ respectively.

Choose a path $\gamma(h)$ in Z connecting p to $h.p$. Let $h_{\Gamma'} \in \operatorname{Aut}(A_{\Gamma'})$ be the geometric representative of the restriction of h. In the same way we obtain geometric representatives $h_{\overline{\Theta}} \in \operatorname{Aut}(A_{\overline{\Theta}})$ and $h_{\widehat{\operatorname{st}}(w)} \in \operatorname{Aut}(A_{\widehat{\operatorname{st}}(w)})$ defined by the same path. Each of these representatives can be (algebraically) extended to a representative of $\phi(h)$ in $\operatorname{Aut}(A_{\Gamma})$, but this is in general not unique; two such representatives will differ by conjugation by an element of the centraliser of $A_{\Gamma'}, A_{\overline{\Theta}}$ and $A_{\widehat{\operatorname{st}}(w)}$ respectively.

Since $\operatorname{lk}(\partial\Gamma') \cap \Gamma' = \emptyset$ (which is the assumption of Case 1 we are in), $Z(\Gamma') \subseteq \partial\Gamma'$. Hence, if $Z(\Gamma')$ is non-empty, then Γ is a cone (and hence a join) over any vertex of it. As this would be a contradiction, we see that Γ' has trivial centre. It also has a trivial link, since $\Gamma' \neq \partial\Gamma'$ (as $\Gamma' = \partial\Gamma'$ would again force Γ to be a join). Therefore there is a unique way of extending

$h_{\Gamma'}$ to an automorphism of A_Γ; we will continue to denote this extension by $h_{\Gamma'}$.

Let $h_1, h_2 \in \mathrm{Aut}(A_\Gamma)$ be such extensions of $h_{\overline{\Theta}}$ and $h_{\widehat{\mathrm{st}}(w)}$ respectively. Since $h_{\Gamma'}, h_1$ and h_2 are representative of the same outer automorphism, and since each pair acts in the same way on some subgroup, we get

$$\begin{aligned} h_{\Gamma'} h_1^{-1} &= c(x_1) \quad \text{with} \quad x_1 \in C(A_{\partial\Gamma'}) = Z(A_{\partial\Gamma'}) \times A_\Theta \\ h_1 h_2^{-1} &= c(x_2) \quad \text{with} \quad x_2 \in C(A_{\widehat{\mathrm{st}}(w)_2}) \\ h_2 h_{\Gamma'}^{-1} &= c(x_3) \quad \text{with} \quad x_3 \in C(A_{\widehat{\mathrm{st}}(w)_1}) = Z(A_{\widehat{\mathrm{st}}(w)_1}) \end{aligned}$$

Observe that $x_2 x_3 = x_1^{-1}$, since Γ is not a join, and hence has no centre. Now

$$(\mathrm{lk}(\widehat{\mathrm{st}}(w)_2) * Z(\widehat{\mathrm{st}}(w)_2)) \cap \Theta = Z(\widehat{\mathrm{st}}(w)_2)) \cap \Theta = Z(\Theta)$$

since $\Theta \subseteq \widehat{\mathrm{st}}(w)_2$, and thus

$$C(A_{\widehat{\mathrm{st}}(w)_2}) \cap A_\Theta = Z(A_\Theta)$$

Similarly

$$Z(\widehat{\mathrm{st}}(w)_1) \cap \Theta \subseteq \Gamma' \cap \Theta = \emptyset$$

and so

$$Z(A_{\widehat{\mathrm{st}}(w)_1}) \cap A_\Theta = \{1\}$$

Since all groups in question are sub-RAAGs generated by vertices, this means that $x_1 = x_3^{-1} x_2^{-1}$ can only involve generators of A_Θ which are in fact contained in $Z(A_\Theta)$. This implies $x_1 \in Z(A_{\partial\Gamma'}) \times Z(A_\Theta)$.

Crucially, upon replacing h_1 by $h_1 c(x_1)$, we still have an extension as required, since $Z(A_{\overline{\Theta}}) = Z(A_{\partial\Gamma' * \Theta}) = Z(A_{\partial\Gamma'}) \times Z(A_\Theta)$. But $h_1 c(x_1) = h_{\Gamma'}$.

Let us now go back to the action of h on the glued-up complex X_Γ. Again we use $\widetilde{p}$ and $\gamma(h)$ to obtain a geometric representative in $\mathrm{Aut}(A_\Gamma)$. On the subgroup $A_{\Gamma'}$ this automorphism is equal to $h_{\Gamma'}$. On the subgroup $A_{\overline{\Theta}}$, it is equal to h_1, but also to $h_1 c(x_1) = h_{\Gamma'}$, since conjugation by x_1 is trivial here. Thus h acts as the outer automorphism $\phi(h)$, as required. □

The System Intersection Axiom gives a standard copy $\widetilde{Z}_\Sigma$ of $\widetilde{X}'_\Sigma$ in $\widetilde{X}'_{\Gamma'}$ for each $\Sigma \in \mathcal{S}_{\Gamma'}$. Note that this includes $\Delta \in \mathcal{S}_{\Gamma'}$; we have $\widetilde{Z} = \widetilde{Z}_\Delta$ since $\mathrm{lk}_{\Gamma'}(\Delta) \subseteq \mathrm{lk}_{\Gamma'}(\partial\Gamma') = \emptyset$ (by the assumption of Case 1 we are in), and so there is only one standard copy of $\widetilde{X}'_\Delta$ in $\widetilde{X}'_{\Gamma'}$.

We have thus completed the construction in Case 1.

Case 2: $\mathrm{lk}(\partial\Gamma') \cap \Gamma' \neq \emptyset$. Let

$$E = \mathrm{lk}(\partial\Gamma')_1$$

be the part of the link of $\partial\Gamma$ lying in Γ'. We define

$$\overline{E} = E * \partial\Gamma'$$

We have already defined $\overline{\Theta} = \Theta * \partial\Gamma'$. In general let us define

$$\overline{\Sigma} = \Sigma \cup \partial\Gamma'$$

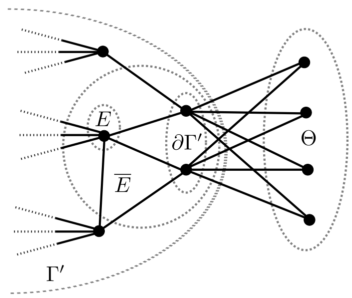

FIGURE 3. The relevant subgraphs of Γ in case 2

for any subgraph $\Sigma \subseteq \Gamma$.

Note that we may assume that $\overline{E} \neq \Gamma'$, as otherwise Γ is a join, namely $\mathrm{st}(\partial\Gamma')$. We have $\partial\Gamma' = \mathrm{lk}(\Theta \cup E)$ and thus $\partial\Gamma' \in \mathcal{L}^\phi$, since it is a link of a non-cone. Thus $\overline{E} = \mathrm{st}(\partial\Gamma')_1 \in \mathcal{L}^\phi$, by part (4) of Lemma 4.3 and by Lemma 4.2.

Claim. There exists a cubical system $\mathcal{X}''$ for $\mathcal{L}^\phi_{\mathrm{st}(\partial\Gamma')}$ which extends the subsystem $\mathcal{X}'_{\overline{E}}$ of $\mathcal{X}'$ strongly.

Proof. We begin by applying Lemma 9.3, using the decomposition

$$\mathrm{st}(\partial\Gamma') = \partial\Gamma' * (E \cup \Theta)$$

to obtain a cubical system $\mathcal{X}''$ for $\mathcal{L}^\phi_{\mathrm{st}(\partial\Gamma')}$ which extends the subsystem $\mathcal{X}'_{\overline{E}}$ of $\mathcal{X}'$, in such a way that it extends $\mathcal{X}'_{\partial\Gamma'}$ strongly. If the join decomposition (compare Definition 2.4) of E does not contain singletons, then in fact $\mathcal{X}''$ extends $\mathcal{X}'_{\overline{E}}$ strongly by definition.

Suppose that the decomposition does contain singletons; note that each such singleton belongs to $\mathcal{L}^\phi$, since ϕ is link-preserving. Moreover, such a singleton is also a singleton in the join decomposition of $\overline{E} = E * \partial\Gamma'$.

Let s be such a singleton which is not connected to every other vertex in Γ'. Then Lemma 8.3 implies that the action on the associated loop in $\mathcal{X}'$ and $\mathcal{X}''$ is the same.

The remaining case occurs when E contains a singleton s in its join decomposition, and $\mathrm{st}(s) = \Gamma'$. In this situation we will replace $\mathcal{X}'$ by the system given by Lemma 7.11, so that $\mathcal{X}'_{\{s\}}$ and $\mathcal{X}''_{\{s\}}$ are H-equivariantly isometric; we will continue to denote this slightly altered system by $\mathcal{X}'$. Repeating this operation for each singleton as described guarantees that $\mathcal{X}''$ extends $\mathcal{X}'_{\overline{E}}$ strongly. □

We need to introduce one more graph. Let

$$\Delta' = \bigcap_{\Sigma \in \mathcal{S}} \mathrm{st}(\Sigma_1)_1$$

Note that $\Delta' \subseteq \Gamma'$. The significance of Δ' will be explained shortly. For now let us observe the following.

Claim. $\Delta' \cup \Theta \in \mathcal{L}^\phi$.

Proof. We have

$$\mathrm{st}(\Sigma_1)_1 \cup \Theta = \mathrm{st}(\Sigma_1) \cup \Sigma \in \mathcal{L}^\phi$$

since $\mathrm{lk}(\mathrm{st}(\Sigma_1) \cap \Sigma) \subseteq \mathrm{lk}(\Sigma_1) \subseteq \mathrm{st}(\Sigma_1) = \mathrm{st}(\mathrm{st}(\Sigma_1))$, and thanks to part ii) of Lemma 4.2. Thus

$$\Delta' \cup \Theta = \bigcap_{\Sigma \in \mathcal{S}} (\mathrm{st}(\Sigma_1)_1 \cup \Theta) \in \mathcal{L}^\phi$$

by part i) of Lemma 4.2. □

By construction we have $\overline{\Theta} \subseteq \Delta' \cup \Theta$ and so $\Delta' \in \mathcal{S}_{\Gamma'}$. We also have $\Delta' \subseteq \overline{E}$ since $\overline{E} \in \mathcal{S}_{\Gamma'}$ satisfies $\mathrm{st}(\overline{E})_1 = \overline{E}$.

Claim. There exist fixed standard copies $\widetilde{R}_\Delta$ and $\widetilde{R}_{\Delta'}$ of, respectively, $\widetilde{X}''_\Delta$ and $\widetilde{X}''_{\Delta'}$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$ such that $\widetilde{R}_\Delta \subseteq \widetilde{R}_{\Delta'}$.

Proof. We have

$$\mathrm{lk}(\Delta' \cup \Theta) \subseteq \mathrm{lk}(\Theta) \smallsetminus \Delta' \subseteq \partial\Gamma' \smallsetminus \partial\Gamma' = \emptyset$$

Thus there is a unique (and therefore fixed) standard copy of $\widetilde{X}''_{\Delta' \cup \Theta}$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$. Since

$$\mathrm{lk}(\overline{E}) = \mathrm{lk}(\partial\Gamma') \cap \mathrm{lk}(E) \subseteq \mathrm{lk}(\partial\Gamma') \smallsetminus (E \cup \Theta) = \emptyset$$

there is a unique (and therefore again fixed) standard copy of $\widetilde{X}''_{\overline{E}}$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$. These two copies intersect in a standard copy of $\widetilde{X}''_{\Delta'}$ (by the Matching Property); let us denote it by $\widetilde{R}_{\Delta'}$. Since the two copies are fixed, so is $\widetilde{R}_{\Delta'}$.

By noting that

$$\mathrm{lk}(\Delta \cup \Theta) \subseteq \mathrm{lk}(\Theta) \smallsetminus \Delta \subseteq \partial\Gamma' \smallsetminus \partial\Gamma' = \emptyset$$

and applying the same procedure we obtain a fixed standard copy $\widetilde{R}_\Delta$ of $\widetilde{X}''_\Delta$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$.

Since the unique standard copy of $\widetilde{X}''_{\Delta \cup \Theta}$ must be contained in the unique standard copy of $\widetilde{X}''_{\Delta' \cup \Theta}$ (by the Composition Property), we have

$$\widetilde{R}_\Delta \subseteq \widetilde{R}_{\Delta'}$$

as claimed. □

The System Intersection Axiom gives a standard copy $\widetilde{Y}_\Sigma$ of $\widetilde{X}'_\Sigma$ in $\widetilde{X}'_{\Gamma'}$ for each $\Sigma \in \mathcal{S}_{\Gamma'}$ such that all these copies intersect in $\widetilde{Y}_\Delta$. For each $\Sigma \in \mathcal{S}_{\Gamma'}$ we have

$$\mathrm{lk}_{\Gamma'}(\mathrm{st}(\Sigma)_1) \subseteq \mathrm{lk}_{\Gamma'}(\Sigma) \smallsetminus \mathrm{st}(\Sigma)_1 \subseteq \mathrm{lk}(\Sigma)_1 \smallsetminus \mathrm{lk}(\Sigma)_1 = \emptyset$$

and thus the copy $\widetilde{Y}_{\mathrm{st}(\Sigma)_1}$ is fixed. Note that $\Delta' \in \mathcal{S}_{\Gamma'}$, and by the Intersection Axiom we have

$$\widetilde{Y}_{\Delta'} = \bigcap_{\Sigma \in \mathcal{S}_{\Gamma'}} \widetilde{Y}_{\mathrm{st}(\Sigma)_1}$$

Thus, the standard copy $\widetilde{Y}_{\Delta'}$ is fixed.

Let us now observe the crucial property of Δ'.

Claim. Let $\Sigma \in \mathcal{S}$, and let $\widetilde{p} \in \widetilde{Y}_{\Delta'}$. Then there exists a standard copy $\widetilde{W}_{\Sigma_1}$ of $\widetilde{X}'_{\Sigma_1}$ in $\widetilde{X}'_{\Gamma'}$ such that $\widetilde{p} \in \widetilde{W}_{\Sigma_1}$.

Proof. Note that we have a standard copy $\widetilde{Y}_{\mathrm{st}(\Sigma_1)_1}$ which contains $\widetilde{Y}_{\Delta'}$, and hence $\widetilde{p}$. We have

$$\mathrm{st}(\Sigma_1)_1 = \Sigma_1 * \mathrm{lk}_{\Gamma'}(\Sigma_1)$$

and so the Product Axiom and Composition Axiom tell us that there exists a standard copy $\widetilde{W}_{\Sigma_1}$ as required. □

We obtain a complex $\widetilde{C}$ from $\widetilde{X}'_{\Gamma'}$ and $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$ by gluing $\widetilde{Y}_{\Delta'} = \widetilde{R}_{\Delta'}$. However, the projection C of our complex $\widetilde{C}$ might not yet realise the desired action of H.

Pick $h \in H$. Take a point $\widetilde{r} \in \widetilde{R}_\Delta$ as a basepoint, let r be its projection as usual. Take a path $\gamma(h)$ in R_Δ from r to $h.r$. Note that $\widetilde{r}$ is a point in $\widetilde{R}_{\Delta'}$, and hence in $\widetilde{C}$ (after the gluing). Let $\widetilde{p} \in \widetilde{Y}_{\Delta'} = \widetilde{R}_{\Delta'}$ be the corresponding point; we view it as a point in $\widetilde{C}$ as well, and denote its projection by p as usual.

Claim. The gluing $\widetilde{C}$ is faulty within $Z(A_{\Delta'})$.

Proof. Let us first remark that $\mathrm{lk}(\Sigma) \subseteq \mathrm{lk}(\Theta) \smallsetminus \partial\Gamma' = \emptyset$ for all $\Sigma \in \mathcal{S}$.

By construction we see that

$$h_r(A_{\Delta\cup\Theta}) = A_{\Delta\cup\Theta}$$

Take $\Sigma \in \mathcal{S}$. Since Σ lies in $\mathcal{L}^\phi$, we have

$$h_r(A_\Sigma) = A_\Sigma^y$$

for some element $y \in A_\Gamma$. But then

$$A_{\Delta\cup\Theta} = h_r(A_{\Delta\cup\Theta}) \leqslant h_r(A_\Sigma) = A_\Sigma^y$$

Proposition 2.5 implies that $y \in N(A_\Sigma)N(A_{\Delta\cup\Theta}) = A_\Sigma A_{\Delta\cup\Theta} = A_\Sigma$, and thus

$$h_r(A_\Sigma) = A_\Sigma$$

Now $\Sigma_1 \in \mathcal{L}^\phi$ and so

$$h_r(A_{\Sigma_1}) = A_{\Sigma_1}^z$$

We have $A_{\Sigma_1}^z \subseteq A_\Sigma$, and so (using Proposition 2.5 again) we get, without loss of generality, $z \in N(A_\Sigma) = A_\Sigma$ since $\Sigma \in \mathcal{S}$, and so $\mathrm{lk}(\Sigma) = \emptyset$.

Let us now focus on h_p. We have shown above that there exists a standard copy of $\widetilde{X}'_{\Sigma_1}$ containing $\widetilde{p}$; it is clear that it will also contain the copy of the path $\gamma(h)$. Hence

$$h_p(A_{\Sigma_1}) = A_{\Sigma_1}$$

and so

$$A_{\Sigma_1}^z = h_r(A_{\Sigma_1}) = c(x(h)^{-1})h_p(A_{\Sigma_1}) = A_{\Sigma_1}^{x(h)^{-1}}$$

where

$$c(x(h)) = h_p h_r^{-1}$$

and $x(h) \in C(A_{\Delta'})$.

The construction also tells us that $A_{\overline{E}} = h_r(A_{\overline{E}})$, since $\widetilde{r}$ and $\gamma(h)$ lie in the fixed standard copy of $\widetilde{X}''_{\overline{E}}$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$. We have $A_{\overline{E}} = h_p(A_{\overline{E}})$ as well, since $\overline{E} \in \mathcal{S}_{\Gamma'}$. So

$$x(h) \in N(A_{\overline{E}}) \leqslant A_{\Gamma'}$$

Thus, by Proposition 2.5, $z \in N(A_{\Sigma_1})A_{\Gamma'}$.

We claim that $A_{\Sigma_1}^z = A_{\Sigma_1}$. If $\Theta \cap \mathrm{lk}(\Sigma_1) \neq \emptyset$, then $\Sigma_1 = \partial\Gamma'$, and so

$$z \in A_\Sigma \leqslant N(A_{\Sigma_1})$$

which yields the desired conclusion.

Otherwise we have $\mathrm{st}(\Sigma_1) \subseteq \Gamma'$, and so $N(A_{\Sigma_1}) \leqslant A_{\Gamma'}$, which in turn implies $z \in A_{\Gamma'}$. Now $z \in A_\Sigma \cap A_{\Gamma'} = A_{\Sigma_1}$ and so $A_{\Sigma_1}^z = A_{\Sigma_1}$ as claimed.

We have

$$A_{\Sigma_1}^{x(h)^{-1}} = A_{\Sigma_1}^z = A_{\Sigma_1}$$

for all $\Sigma \in \mathcal{S}$. Hence

$$x(h) \in \bigcap_{\Sigma \in \mathcal{S}} A_{\mathrm{st}(\Sigma_1)}$$

We have already shown that $x(h) \in A_{\Gamma'}$ and so

$$x(h) \in \bigcap_{\Sigma \in \mathcal{S}} A_{\mathrm{st}(\Sigma_1)_1} = A_{\Delta'}$$

Therefore $x(h) \in C(A_{\Delta'}) \cap A_{\Delta'} = Z(A'_\Delta)$. This statement holds for each h, and thus the fault of our gluing satisfies the claim. □

Now we are in a position to apply Proposition 8.5 and obtain a new glued up complex, which we call $\widetilde{X}_\Gamma$, which realises our action ϕ.

Recall that we have a standard copy $\widetilde{R}_\Delta$ in $\widetilde{X}''_{\mathrm{st}(\partial\Gamma')}$. The gluing sends $\widetilde{R}_\Delta$ to some standard copy of $\widetilde{X}'_\Delta$ in $\widetilde{X}'_{\Gamma'}$, which lies within $\widetilde{Y}_{\Delta'}$ (we are using the Composition Property here); let us denote this standard copy by $\widetilde{Z}_\Delta$. It is fixed since $\widetilde{R}_\Delta$ is.

By one of the claims above, we may pick a standard copy $\widetilde{Z}_\Sigma$ of $\widetilde{X}'_\Sigma$ in $\widetilde{X}'_{\Gamma'}$ for each $\Sigma \in \mathcal{S}_{\Gamma'}$ such that they will all intersect in $\widetilde{Z}_\Delta$. Let us choose such a family of standard copies.

To finish the construction in this case we need to remark that the complex $\widetilde{X}_\Gamma$ we constructed is equal to a complex obtained from $\widetilde{X}'_{\Gamma'}$ and $\widetilde{X}''_{\Delta\cup\Theta}$ by gluing $\widetilde{R}_\Delta$ and $\widetilde{Z}_\Delta$.

Step 2: Constructing X_Σ for $\Sigma \subseteq \Gamma'$ or $\Sigma \subseteq \Delta \cup \Theta$. Since $\mathcal{X}'$ and $\mathcal{X}''_{\Delta\cup\Theta}$ strongly extend $\mathcal{X}'_\Delta$, and $\Delta = \Gamma' \cap (\Delta \cup \Theta)$, we simply define the complexes in $\mathcal{X}$ for graphs $\Sigma \subseteq \Gamma'$ or $\Sigma \subseteq \Delta \cup \Theta$ to be the ones in $\mathcal{X}'$ or $\mathcal{X}''_{\Delta\cup\Theta}$, respectively.

Interlude. Before we begin with Step 3, we record the following

Claim. For any graph $\Sigma \in \mathcal{L}^\phi$ such that $\Sigma \not\subseteq \Gamma'$ and $\Sigma \not\subseteq \overline{\Theta}$ we have $\Theta \subseteq \Sigma$.

Additionally, every graph $\Sigma \in \mathcal{L}^\phi$ with $\Theta \subseteq \Sigma$ satisfies

$$(*) \qquad \mathrm{lk}(\partial\Gamma') \cap (\Theta \cup \Delta) \subseteq \Sigma$$

In particular, $\overline{\Sigma} = \Sigma \cup \Delta$ for subgraphs Σ with $(*)$, and so $\overline{\Sigma} \in \mathcal{S}$ in this case.

Proof. First assume that $\Sigma \not\subseteq \Gamma'$ and $\Sigma \not\subseteq \overline{\Theta}$. In this case $\mathrm{lk}(\Sigma_1) \subseteq \Gamma'$, since otherwise $\Sigma_1 \subseteq \partial\Gamma'$, which would force $\Sigma \subseteq \overline{\Theta}$, a contradiction. Therefore, by Lemma 4.2, $\overline{\Sigma} = \Sigma \cup \Gamma' \in \mathcal{L}^\phi$, and it is a connected subgraph. Thus

$$\Theta \subseteq \Sigma$$

since there is no connected subgraph in $\mathcal{L}^\phi$ which is properly contained in Γ and properly contains Γ'.

Now let us suppose that $\Sigma \in \mathcal{L}^\phi$ satisfies $\Theta \subseteq \Sigma$. Then

$$\mathrm{lk}(\Sigma \cap (\Delta \cup \Theta)) \subseteq \mathrm{lk}(\Theta) \subseteq \Delta \cup \Theta$$

and so $\Sigma \cup \Delta \in \mathcal{L}^\phi$ by Lemma 4.2.

We now claim that $\mathrm{lk}(\partial\Gamma') \cap \Delta \subseteq \Sigma$. Note that $E \cup \Theta = \mathrm{lk}(\partial\Gamma') \in \mathcal{L}^\phi$ since ϕ is link-preserving. Now $\Sigma \cap (E \cup \Theta) \in \mathcal{L}^\phi$ and thus

$$\partial\Gamma' * (\Sigma \cap (E \cup \Theta)) = \mathrm{st}(\Sigma \cap (E \cup \Theta)) \in \mathcal{L}^\phi$$

where the equality follows from the observation that $\Theta \subseteq \Sigma \cap (E \cup \Theta)$. But $\partial\Gamma' * (\Sigma \cap (E \cup \Theta)) \in \mathcal{S}$, and therefore

$$\Delta \subseteq \partial\Gamma' * (\Sigma \cap (E \cup \Theta))$$

This in turn implies that

$$\Delta \subseteq \partial\Gamma' \cup \Sigma$$

and so $\Delta \cap \mathrm{lk}(\partial\Gamma') \subseteq \Sigma$. We assumed that $\Theta \subseteq \Sigma$, and so observing that $\Theta \cap \mathrm{lk}(\Gamma') = \Theta$ we get

$$\mathrm{lk}(\partial\Gamma') \cap (\Theta \cup \Delta) \subseteq \Sigma$$

that is Σ satisfies $(*)$.

Lastly, suppose that Σ satisfies $(*)$. Then

$$\overline{\Sigma} = \Sigma \cup \partial\Gamma' = \Sigma \cup (\Delta \smallsetminus \Delta \cap \mathrm{lk}(\partial\Gamma')) = \Sigma \cup \Delta$$

as required. □

Since $\Sigma \not\subseteq \Delta \cup \Theta$ implies $\Sigma \not\subseteq \overline{\Theta}$, we see that any $\Sigma \in \mathcal{L}^\phi$ not covered by Step 2 satisfies $(*)$ and so $\overline{\Sigma} \in \mathcal{S}$. Hence given such a Σ we have

$$\Sigma \subseteq \mathrm{st}_{\overline{\Sigma}}(\Sigma) \subseteq \overline{\Sigma}$$

We will deal with each graph in this chain in turn.

Step 3: Constructing X_Σ for $\Sigma \in \mathcal{S} \smallsetminus \{\Delta \cup \Theta\}$. In this case

$$\Sigma_1 = \Sigma \cap \Gamma' \in \mathcal{S}_{\Gamma'}$$

and so (by construction) we have $\widetilde{Z}_{\Sigma_1}$ in $\widetilde{X}_\Gamma$ containing $\widetilde{Z}_\Delta$ (which in particular implies that it is fixed). We also have the image under our gluing map $\widetilde{\iota}_{\Delta\cup\Theta,\Gamma}$ of $\widetilde{X}''_{\Delta\cup\Theta} = \widetilde{X}_{\Delta\cup\Theta}$; let us call it $\widetilde{R}_{\Delta\cup\Theta}$. This subcomplex contains $\widetilde{Z}_\Delta$ by construction. We obtain $\widetilde{X}_\Sigma$ from the two complexes by gluing the two instances of $\widetilde{Z}_\Delta$. Its projection carries the desired marking by construction. The action of H is also the desired one; taking any $h \in H$, and looking at a geometric representative obtained by choosing a basepoint and a path in the subcomplex X_Σ, we get an automorphism of A_Γ which preserves A_Σ. This automorphism is a representative of $\phi(h)$, and so the restriction to A_Σ is the desired one. But this is equal to the geometric representative of the action of h on X_Σ obtained using the same basepoint and path.

Our construction also gives us maps $\widetilde{\iota}_{\Sigma_1,\Sigma}$ and $\widetilde{\iota}_{\Delta\cup\Theta,\Sigma}$. These maps are as required, since $\Delta \cup \Theta$ and Σ_1 both have trivial links in Σ; the former statement is clear, and the latter follows from the observation that

$$\mathrm{lk}_\Sigma(\Sigma_1) \neq \emptyset$$

implies that $\Sigma_1 \subseteq \partial\Gamma'$ and hence $\Sigma_1 = \Delta = \partial\Gamma'$, which in turn gives $\Sigma = \Delta \cup \Theta$, contradicting our assumption.

We also get a map $\widetilde{\iota}_{\Sigma,\Gamma}$, since we define $\widetilde{X}_\Sigma$ as a subcomplex of $\widetilde{X}_\Gamma$. It is as required since $\mathrm{lk}(\Sigma) = \emptyset$ for all $\Sigma \in \mathcal{S}$.

Step 4: Constructing X_Σ for Σ with $(*)$ and such that $\mathrm{lk}_{\overline{\Sigma}}(\Sigma) = \emptyset$. By the Composition Property there exists a standard copy of $\widetilde{X}_{\partial\Gamma'}$ in $\widetilde{X}_{\Delta\cup\Theta}$ which lies within $\widetilde{R}_\Delta$; let us denote it by $\widetilde{R}_{\partial\Gamma'}$. The gluing $\widetilde{R}_\Delta = \widetilde{Z}_\Delta$ gives us the corresponding standard copy $\widetilde{Z}_{\partial\Gamma'}$ in $\widetilde{Z}_\Delta$.

Note that the assumption implies that $\mathrm{lk}_{\overline{\Sigma}_1}(\Sigma_1) = \emptyset$. Let us take the unique standard copy of $\widetilde{X}_{\Sigma_1}$ in $\widetilde{X}_{\overline{\Sigma}_1}$; we will denote it by $\widetilde{Z}_{\Sigma_1}$. By the Matching Property, it intersects $\widetilde{Z}_\Delta$ in a standard copy of $\widetilde{X}_{\Sigma\cap\Delta}$. Using the Matching Property again, this time in $\widetilde{Z}_\Delta$, we see that this copy of $\widetilde{X}_{\Sigma\cap\Delta}$ intersects $\widetilde{Z}_{\partial\Gamma'}$ in a copy of $\widetilde{X}_{\Sigma\cap\partial\Gamma'}$; we will denote thus copy by $\widetilde{Z}_{\Sigma\cap\partial\Gamma'}$, and the corresponding one in $\widetilde{R}_{\partial\Gamma'}$ by $\widetilde{R}_{\Sigma\cap\partial\Gamma'}$.

Recall that $\Sigma_2 = \Sigma \cap (\Delta \cup \Theta)$. Since $\mathcal{X}''$ is a product of $\mathcal{X}''_{\partial\Gamma'}$ and $\mathcal{X}''_{\mathrm{lk}_{\Delta\cup\Theta}(\partial\Gamma')}$, there exists a standard copy of $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_{\Delta\cup\Theta}$ which contains $\widetilde{R}_{\Sigma\cap\partial\Gamma'}$; we will denote it by $\widetilde{R}_{\Sigma_2}$. We define $\widetilde{X}_\Sigma$ to be the subcomplex of $\widetilde{X}_{\overline{\Sigma}}$ obtained by gluing $\widetilde{Z}_{\Sigma_1}$ to $\widetilde{R}_{\Sigma_2}$. Note that these two copies overlap in a copy of $\widetilde{X}_{\Sigma\cap\Delta}$, which is the unique such copy containing $\widetilde{R}_{\Sigma\cap\partial\Gamma'}$.

Note that again our gluing procedure determines maps $\widetilde{\iota}_{\Sigma_1,\Sigma}$ and $\widetilde{\iota}_{\Sigma_2,\Sigma}$ of the required type.

From this construction we also obtain a map $\widetilde{\iota}_{\Sigma,\overline{\Sigma}}$, since we define $\widetilde{X}_\Sigma$ as a subcomplex of $\widetilde{X}_{\overline{\Sigma}}$.

Step 5: Constructing the remaining complexes. As remarked above we are left with graphs $\Sigma \in \mathcal{L}^\phi$ satisfying $(*)$ and such that $\Sigma \subset \mathrm{st}_{\overline{\Sigma}}(\Sigma)$ is a proper subgraph. Let $\Sigma' = \mathrm{st}_{\overline{\Sigma}}(\Sigma)$, and note that Σ' is covered by the previous step. We need to exhibit a product structure on $\widetilde{X}_{\Sigma'}$, one factor of which will be the desired complex for Σ, the other for $\Lambda = \mathrm{lk}_{\overline{\Sigma}}(\Sigma)$.

The complex $\widetilde{X}_{\Sigma'}$ is obtained from complexes $\widetilde{X}_{\Sigma'_1}$ and $\widetilde{X}_{\Sigma'_2}$ by gluing them along a copy of $\widetilde{X}_{\Sigma' \cap \Delta}$. Since $\Lambda \subseteq \partial\Gamma'$, each of these three complexes is a product of $\widetilde{X}_\Lambda$ and some other complex by the Product Axiom. Moreover, the embeddings $\widetilde{X}_{\Sigma' \cap \Delta} \to \widetilde{X}_{\Sigma'_i}$ with $i \in \{1, 2\}$ respect the product structure, that is the image of any standard copy of $\widetilde{X}_\Lambda$ in $\widetilde{X}_{\Sigma' \cap \Delta}$ is still a standard copy in $\widetilde{X}_{\Sigma'_i}$ by the Composition Property. Hence the glued-up complex $\widetilde{X}_{\Sigma'}$ is a product of $\widetilde{X}_\Lambda$ and a complex obtained by gluing some standard copies of $\widetilde{X}_{\Sigma_i}$ in $\widetilde{X}_{\Sigma'_i}$ along $\widetilde{X}_{\Sigma \cap \Delta}$; we call this latter complex $\widetilde{X}_\Sigma$. For notational convenience we pick some such standard copies of $\widetilde{X}_{\Sigma_1}$ and $\widetilde{X}_{\Sigma_2}$, and denote them by $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{R}_{\Sigma_2}$ respectively.

This last gluing gives us maps $\widetilde{\iota}_{\Sigma_i, \Sigma}$ for both values of i, which are as required.

The construction also gives us a map

$$\widetilde{\iota}_{\Sigma, \Sigma'} \colon \widetilde{X}_\Sigma \times \widetilde{X}_\Lambda \to \widetilde{X}_{\Sigma'}$$

which is again as wanted.

10.2. **Constructing the maps.** Let $\Sigma, \Sigma' \in \mathcal{L}^\phi$ be such that $\Sigma \subseteq \Sigma'$. We need to construct a map $\widetilde{\iota}_{\Sigma, \Sigma'}$. We will do it in several steps.

(1) $\Sigma' = \Sigma'_i$ **for some** $i \in \{1, 2\}$

In this case the cube complexes X_Σ and $X_{\Sigma'}$ are obtained directly from another cubical system (in Step 2), and we take $\widetilde{\iota}_{\Sigma, \Sigma'}$ to be the map coming from that system.

We will now assume that the hypothesis of this step is not satisfied, which immediately tells us that Σ' satisfies $(*)$ and that $\Sigma' \neq \Delta \cup \Theta$.

(2) $\Sigma = \Sigma_i$ **and** $\Sigma \neq \Sigma_j$ **with** $\{i, j\} = \{1, 2\}$

In this case we have $\mathrm{st}(\Sigma) = \mathrm{st}(\Sigma)_i$, since if $\mathrm{lk}(\Sigma) \neq \mathrm{lk}(\Sigma)_i$ then (knowing that $\Sigma = \Sigma_i$) we must have $\Sigma \subseteq \partial\Gamma'$, and thus $\Sigma = \Sigma_j$ which contradicts the assumption. Therefore $\mathrm{lk}_{\Sigma'}(\Sigma) = \mathrm{lk}_{\Sigma'}(\Sigma)_i$ as well. We define

$$\widetilde{\iota}_{\Sigma, \Sigma'} = \widetilde{\iota}_{\Sigma'_i, \Sigma'} \circ \widetilde{\iota}_{\Sigma, \Sigma'_i}$$

where the last map was defined in the previous step, and the map $\widetilde{\iota}_{\Sigma'_i, \Sigma'}$ was constructed together with the complex $\widetilde{X}_{\Sigma'}$ in Step 3, 4 or 5 of Subsection 10.1.

(3) Σ **satisfies** $(*)$ **and** $\Sigma \neq \Delta \cup \Theta$

Observe that $\mathrm{st}_{\overline{\Sigma}}(\Sigma) = \mathrm{st}_{\overline{\Sigma'}}(\Sigma)$ since $\mathrm{lk}(\Sigma) \subseteq \partial\Gamma'$. Hence Step 5 above gives us the map $\widetilde{\iota}_{\Sigma, \mathrm{st}_{\overline{\Sigma'}}(\Sigma)}$.

Let $\Omega = \mathrm{st}_{\overline{\Sigma'}}(\Sigma) \cup \mathrm{st}_{\overline{\Sigma'}}(\Sigma')$

Claim. $\Omega \in \mathcal{L}^\phi$.

Proof. We use part (3) of Lemma 4.2. Thus we only need to observe that

$$\mathrm{lk}\big(\mathrm{st}_{\overline{\Sigma'}}(\Sigma) \cap \mathrm{st}_{\overline{\Sigma'}}(\Sigma')\big) \subseteq \mathrm{lk}(\Sigma) \subseteq \mathrm{st}(\Sigma) = \mathrm{st}_{\overline{\Sigma'}}(\Sigma)$$

since $\mathrm{lk}(\Sigma) \subseteq \partial\Gamma'$ as Σ satisfies $(*)$. □

We have

$$\mathrm{lk}(\mathrm{st}_{\overline{\Sigma}}(\Sigma)) \subseteq \mathrm{lk}(\Sigma) \smallsetminus \mathrm{st}_{\overline{\Sigma}}(\Sigma) = \emptyset$$

since $\mathrm{lk}(\Sigma) \subseteq \partial\Gamma'$ and so $\mathrm{lk}(\Sigma) = \mathrm{lk}_{\overline{\Sigma}}(\Sigma)$. Analogously we have $\mathrm{lk}(\mathrm{st}_{\overline{\Sigma'}}(\Sigma')) = \emptyset$. It immediately follows that $\mathrm{lk}(\Omega) = \emptyset$ as well. Now the complex $\widetilde{X}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)}$ is formed by gluing $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)_1}$ and $\widetilde{R}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)_2}$; we have similar statements for $\widetilde{X}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma')}$ and $\widetilde{X}_\Omega$. By construction (see Step 4), $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)_1}$ and $\widetilde{Z}_{\Omega_1}$ are both unique standard copies of the corresponding complexes in $\widetilde{Z}_{\overline{\Sigma}_1}$. The Composition Property in $\mathcal{X}'$ implies that there exists a standard copy of $\widetilde{X}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)_1}$ contained in $\widetilde{Z}_{\Omega_1}$, and thus

$$\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma}}(\Sigma)_1} \subseteq \widetilde{Z}_{\Omega_1}$$

Similarly

$$\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma')_1} \subseteq \widetilde{Z}_{\Omega_1}$$

Now the Matching Property (in $\mathcal{X}'$) tells us that $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma')_1}$ and $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma')_1}$ intersect non-trivially. Since both have a product structure, we find standard copies $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{Z}_{\Sigma'_1}$ of $\widetilde{X}_{\Sigma_1}$ and $\widetilde{X}_{\Sigma'_1}$ respectively (in $\widetilde{Z}_{\Omega_1}$) which also intersect non-trivially.

Let us look more closely at the product structure of $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma)_1}$. It is isomorphic to a product of $\widetilde{X}_{\Sigma_1}$ and $\widetilde{X}_{\mathrm{lk}_{\overline{\Sigma'}}(\Sigma)_1}$. The latter complex contains a standard copy of $\widetilde{X}_{\mathrm{lk}_{\Sigma'}(\Sigma)_1}$. Hence there is a standard copy $\widetilde{Z}_{\mathrm{st}_{\Sigma'}(\Sigma)_1}$ of $\widetilde{X}_{\mathrm{st}_{\Sigma'}(\Sigma)_1}$ in $\widetilde{Z}_{\mathrm{st}_{\overline{\Sigma'}}(\Sigma)_1}$ containing the chosen standard copy $\widetilde{Z}_{\Sigma_1}$ (by the Intersection Axiom). Thus $\widetilde{Z}_{\mathrm{st}_{\Sigma'}(\Sigma)_1}$ and $\widetilde{Z}_{\Sigma'_1}$ intersect non-trivially, and so the intersection Axiom tells us that

$$\widetilde{Z}_{\mathrm{st}_{\Sigma'}(\Sigma)_1} \subseteq \widetilde{Z}_{\Sigma'_1}$$

After the gluing this yields a map $\widetilde{\iota}_{\mathrm{st}_{\Sigma'}(\Sigma),\Sigma'}$, and since $\widetilde{Z}_{\mathrm{st}_{\Sigma}(\Sigma')_1}$ had a product structure, so does $\widetilde{X}_{\mathrm{st}_{\Sigma'}(\Sigma)}$ (arguing as in Step 5 above). This gives us the desired map $\widetilde{\iota}_{\Sigma,\Sigma'}$.

(4) $\Sigma \subseteq \partial\Gamma'$

In this (last) case we observe that $\mathrm{lk}_{\Sigma'}(\Sigma)$ satisfies $(*)$, and so we have already constructed the map

$$\widetilde{\iota}_{\mathrm{lk}_{\Sigma'}(\Sigma),\Sigma'} \colon \widetilde{X}_{\mathrm{lk}_{\Sigma'}(\Sigma)} \times \widetilde{X}_\Sigma \to \widetilde{X}_{\Sigma'}$$

We define $\widetilde{\iota}_{\Sigma,\Sigma'}$ by reordering the factors in the domain.

10.3. **Verifying the Axioms.** The first two axioms depend only on two subgraphs $\Sigma, \Sigma' \in \mathcal{L}^\phi$ with $\Sigma \subseteq \Sigma'$. This is the same assumption as in the maps part of our proof, and hence the verification of the two axioms will follow the structure as the construction of maps – we will consider four cases, and the assumption in each will be identical to the assumptions of the corresponding case above.

Product Axiom. Suppose that $\Sigma' = \mathrm{st}_{\Sigma'}(\Sigma)$.

(1) In this case the Product Axiom follows from the Product Axiom in $\mathcal{X}'$ or $\mathcal{X}''$. Otherwise we assume that Σ' satisfies $(*)$ and $\Sigma' \neq \Delta \cup \Theta$.
(2) If $\Sigma = \Sigma_i$ and $\Sigma \neq \Sigma_j$ with $\{i,j\} = \{1,2\}$, then $\Sigma' = \mathrm{st}_{\Sigma'}(\Sigma)$ also satisfies $\Sigma' = \Sigma'_i$, and so we are in the previous case.
(3) The standard copy $\widetilde{Z}_{\mathrm{st}_{\Sigma'}(\Sigma)_1}$ used in case (3) above is equal to the image of $\widetilde{\iota}_{\Sigma_1,\mathrm{st}_{\Sigma'}(\Sigma)_1}$, and hence the corresponding statement is still true after the gluing.
(4) In this case we defined the map $\widetilde{\iota}_{\Sigma,\Sigma'}$ using $\widetilde{\iota}_{\mathrm{lk}_{\Sigma'}(\Sigma),\Sigma'}$, and the graph $\mathrm{lk}_{\Sigma'}(\Sigma)$ is covered by the previous cases.

Orthogonal Axiom. Let $\Lambda = \mathrm{lk}_{\Sigma'}(\Sigma)$.

(1) If $\Sigma' \subseteq \Gamma'$ or $\Sigma' \subseteq \Delta \cup \Theta$ then the axiom is satisfied, since it is satisfied in $\mathcal{X}'$ and $\mathcal{X}''$. Otherwise we assume that Σ' satisfies $(*)$ and $\Sigma' \neq \Delta \cup \Theta$.
(2) If $\Sigma = \Sigma_i$ and $\Sigma \neq \Sigma_j$ with $\{i,j\} = \{1,2\}$, then also $\Lambda = \Lambda_i$. Suppose that $\Lambda \neq \Lambda_j$. In this case both maps $\widetilde{\iota}_{\Sigma,\Sigma'}$ and $\widetilde{\iota}_{\Lambda,\Sigma'}$ factorise through $\widetilde{\iota}_{\mathrm{st}_{\Sigma'}(\Sigma),\Sigma'}$, and thus it is enough to verify the axiom within $\widetilde{X}_{\mathrm{st}_{\Sigma'}(\Sigma)}$. But $\mathrm{st}_{\Sigma'}(\Sigma) = \mathrm{st}_{\Sigma'}(\Sigma)_i$ and so we are done by the previous case.
    Now suppose that $\Lambda \subseteq \partial\Gamma'$. In this case the axiom follows trivially from the construction of the map $\widetilde{\iota}_{\Lambda,\Sigma'}$.
(3) In this case we have $\Lambda \subseteq \partial\Gamma'$ and so we are done as above.
(4) When $\Sigma \subseteq \partial\Gamma'$ we are again done by construction.

Intersection Axiom. Let us now verify that $\mathcal{X}$ has the Intersection Axiom. Take $\Sigma, \Sigma', \Omega \in \mathcal{L}^\phi$ such that $\Sigma \subseteq \Omega$ and $\Sigma' \subseteq \Omega$, and let $\widetilde{Y}_\Sigma$ and $\widetilde{Y}_{\Sigma'}$ be standard copies of, respectively, $\widetilde{X}_\Sigma$ and $\widetilde{X}_{\Sigma'}$ in $\widetilde{X}_\Omega$ with non-empty intersection. We need to show that the intersection is the image of a standard copy of $\Sigma \cap \Sigma'$ in each.

As in the first two cases, the details depend on the inclusions $\Sigma, \Sigma' \subseteq \Omega$. The cases will thus be labeled by pairs of integers (n,m), the first determining in which step the map $\widetilde{\iota}_{\Sigma,\Omega}$ was constructed, and the second playing the same role for $\widetilde{\iota}_{\Sigma',\Omega}$. By symmetry we only need to consider $n \leqslant m$.

(1,1) In this case $\Omega = \Omega_i$, and so the axiom follows from the Intersection Axiom in $\mathcal{X}'$. In what follows we can assume that $\Omega \neq \Delta \cup \Theta$ satisfies $(*)$. Hence $\widetilde{X}_\Omega$ is obtained by gluing $\widetilde{Z}_{\Omega_1}$ and $\widetilde{R}_{\Omega_2}$ along a subcomplex of $\widetilde{Z}_\Delta$ which is a standard copy of $\widetilde{X}_{\Delta\cap\Omega}$; let us denote it by $\widetilde{Z}_{\Delta\cap\Omega}$.

(2,2) This splits into two cases. If $\Sigma = \Sigma_i$ and $\Sigma' = \Sigma'_i$ then both maps $\widetilde{\iota}$ factor through $\widetilde{\iota}_{\Omega_i,\Omega}$, and so the problem is reduced to checking the axiom for the triple $\Sigma, \Sigma', \Omega_i$, for which it holds.

In the other case we have, without loss of generality, $\Sigma = \Sigma_1$ and $\Sigma' = \Sigma'_2$. By construction, the given standard copies $\widetilde{Y}_\Sigma$ and $\widetilde{Y}_{\Sigma'}$ must lie within the standard copies $\widetilde{Z}_{\Omega_1}$ and $\widetilde{R}_{\Omega_2}$ respectively, and hence intersect within $\widetilde{Z}_{\Delta\cap\Omega}$.

We use the Intersection Axiom of $\mathcal{X}'$ for $\widetilde{Z}_{\Delta\cap\Omega}$ and $\widetilde{Y}_\Sigma$ inside $\widetilde{Z}_{\Omega_1}$ and see that the two copies intersect in a copy of $\widetilde{X}_{\Delta\cap\Sigma}$, which is also the image of a standard copy of $\widetilde{X}_{\Delta\cap\Sigma}$ in $\widetilde{Z}_{\Delta\cap\Omega}$.

We repeat the argument for Σ' and obtain a standard copy of $\widetilde{X}_{\Delta\cap\Sigma'}$ in $\widetilde{Z}_{\Delta\cap\Omega}$. Now this copy intersects the one of $\widetilde{X}_{\Delta\cap\Sigma}$, and hence, applying the Intersection Axiom again, they intersect in a copy of $\widetilde{X}_{\Delta\cap\Sigma\cap\Sigma'}$ in $\widetilde{Z}_{\Delta\cap\Omega}$. But $\Delta\cap\Sigma\cap\Sigma' = \Sigma\cap\Sigma'$, and so we have found the desired standard copy in $\widetilde{Z}_{\Delta\cap\Omega}$. Now the Composition Property (Lemma 7.7) implies that this is also a standard copy in $\widetilde{X}_\Sigma$, $\widetilde{X}_{\Sigma'}$ and $\widetilde{X}_{\Omega_i}$ for any $i \in \{1, 2\}$, and thus this is also a standard copy in $\widetilde{X}_\Omega$ by construction.

(2,3) The non-trivial intersection of any standard copy of $\widetilde{X}_\Sigma$ and any standard copy of $\widetilde{X}_{\Sigma'}$ in $\widetilde{X}_\Omega$ is in fact contained in the standard copy of $\widetilde{X}_{\Omega_i}$, since any copy of $\widetilde{X}_\Sigma$ is contained therein. Therefore the intersection is also contained in a standard copy of $\widetilde{X}_{\Sigma'_i}$ by construction of $\widetilde{\iota}_{\Sigma',\Omega}$. We apply the Intersection Axiom in $\widetilde{X}_{\Omega_i}$, and observe that the standard copy of $\widetilde{X}_{\Sigma\cap\Sigma'}$ in $\widetilde{X}_{\Omega_i}$ obtained this way is also a standard copy in $\widetilde{X}_\Omega$ by construction.

(2,4) In this case $\widetilde{Y}_\Sigma$ must in fact be contained in $\widetilde{Q}$, where $\widetilde{Q}$ is either $\widetilde{Z}_{\Omega_1}$ or $\widetilde{R}_{\Omega_2}$.

We have $\widetilde{Y}_\Sigma \cap \widetilde{Y}_{\Sigma'} \subseteq \widetilde{Q}$, and hence we only need to prove that $\widetilde{Y}_{\Sigma'}$ is a standard copy in $\widetilde{Q}$.

Let $\Lambda = \mathrm{lk}_\Omega(\Sigma')$. Note that $\mathrm{lk}(\partial\Gamma') \cap (\Delta \cup \Theta) \subseteq \Lambda$. By definition of $\widetilde{\iota}_{\Sigma',\Omega}$, we have

$$\widetilde{Y}_{\Sigma'} = \widetilde{\iota}_{\Lambda,\Omega}(\{\widetilde{x}\} \times \widetilde{X}_{\Sigma'})$$

for some point $\widetilde{x} \in \widetilde{X}_\Lambda$. Since $\widetilde{Y}_{\Sigma'}$ contains a point in $\widetilde{Q}$, there exists $\widetilde{y} \in \widetilde{X}_{\Sigma'}$ such that $\widetilde{\iota}_{\Lambda,\Omega}(\widetilde{x}, \widetilde{y}) \in \widetilde{Q}$.

If $\Lambda \not\subseteq \Delta \cup \Theta$ then this is only possible if $\widetilde{x}$ lies in $\widetilde{Z}_{\Lambda_i}$ or $\widetilde{R}_{\Lambda_2}$ (depending on what $\widetilde{Q}$ is), by the construction of $\widetilde{X}_\Lambda$ and $\widetilde{\iota}_{\Lambda,\Omega}$. But then, again by the construction of $\widetilde{X}_\Lambda$, we have $\widetilde{Y}_{\Sigma'} \subseteq \widetilde{Q}$ being a standard copy as claimed.

We still need to check what happens when $\Lambda \subseteq \Delta \cup \Theta$. Suppose that $\widetilde{Q} = \widetilde{R}_{\Omega_2}$. In this case $\widetilde{Y}_{\Sigma'}$ is a standard copy in $\widetilde{Q}$ by the Composition Property of $\mathcal{X}''$. Lastly, let us suppose that $\widetilde{Q} = \widetilde{Z}_{\Omega_1}$. Since $\mathrm{im}(\widetilde{\iota}_{\Lambda,\Omega}) \subseteq \widetilde{R}_{\Omega_2}$ by construction, the Intersection Axiom in $\mathcal{X}''$ tells us that $\widetilde{Y}_{\Sigma'}$ is a standard copy in $\widetilde{Z}_{\Omega\cap\Delta}$. But then it is also a standard copy in $\widetilde{Q} = \widetilde{Z}_{\Omega_1}$ by the Composition Property in $\mathcal{X}'$.

(3,3) In this case $\widetilde{Y}_\Sigma$ and $\widetilde{Y}_{\Sigma'}$ are obtained from $\widetilde{Z}_{\Sigma_1}$, $\widetilde{R}_{\Sigma_2}$, and $\widetilde{Z}_{\Sigma_1'}$, $\widetilde{R}_{\Sigma_2'}$ respectively.

Suppose that $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{Z}_{\Sigma_1'}$ do not intersect. Then $\widetilde{R}_{\Sigma_2}$ and $\widetilde{R}_{\Sigma_2'}$ do intersect, and the Intersection Axiom in $\mathcal{X}''$ tells us that they intersect in a standard copy of $\widetilde{X}_{\Sigma_2\cap\Sigma_2'}$. The graph $\Sigma_2 \cap \Sigma_2'$ satisfies $(*)$, and so $(\Sigma_2\cap\Sigma_2')\cup\Delta = \Delta\cup\Theta$. This implies that the standard copy of $\widetilde{X}_{\Sigma_2\cap\Sigma_2'}$ intersects $\widetilde{Z}_\Delta$ non-trivially (by the Matching Property in $\mathcal{X}''$). But this intersection lies in $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{Z}_{\Sigma_1'}$, and hence they did intersect.

Now the Intersection Axiom in $\mathcal{X}'$ tells us that $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{Z}_{\Sigma_1'}$ intersect in a standard copy of $\widetilde{X}_{\Sigma_1\cap\Sigma_1'}$; let us call it $\widetilde{Z}_{\Sigma_1\cap\Sigma_1'}$.

Note that the standard copy $\widetilde{Z}_{\overline{\Sigma}_1}$ which contains $\widetilde{Z}_{\Sigma_1}$ by construction; similarly $\widetilde{Z}_{\overline{\Sigma'}_1}$ contains $\widetilde{Z}_{\Sigma'_1}$. Now $\widetilde{Z}_{\overline{\Sigma}_1}$ and $\widetilde{Z}_{\overline{\Sigma'}_1}$ intersect in the copy $\widetilde{Z}_{\overline{\Sigma}_1\cap\overline{\Sigma'}_1}$, which contains $\widetilde{Z}_{\Sigma_1\cap\Sigma_1'}$, since $\widetilde{Z}_{\Sigma_1\cap\Sigma_1'}$ lies in both $\widetilde{Z}_{\overline{\Sigma}_1}$ and $\widetilde{Z}_{\overline{\Sigma'}_1}$. The copy $\widetilde{Z}_{\overline{\Sigma}_1\cap\overline{\Sigma'}_1}$ intersects $\widetilde{Z}_\Delta$, and so the Matching Property implies that $\widetilde{Z}_{\Sigma_1\cap\Sigma_1'}$ intersects $\widetilde{Z}_\Delta$ as well. The Intersection Axiom implies that this intersection is a copy of $\widetilde{X}_{\Sigma_1\cap\Sigma_1'\cap\Delta}$. This in turn implies that $\widetilde{R}_{\Sigma_2}$ and $\widetilde{R}_{\Sigma_2'}$ intersect, and we have already shown above that in this case they intersect in a standard copy of $\widetilde{X}_{\Sigma_2\cap\Sigma_2'}$. The union of this copy with $\widetilde{Z}_{\Sigma_1\cap\Sigma_1'}$ is by construction a standard copy of $\widetilde{X}_{\Sigma\cap\Sigma'}$ in $\widetilde{X}_\Omega$, and again by construction it is the image of a standard copy of $\widetilde{X}_{\Sigma\cap\Sigma'}$ in $\widetilde{X}_\Sigma$ and $\widetilde{X}_{\Sigma'}$.

(3,4) The standard copy $\widetilde{Y}_\Sigma$ is obtained from $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{R}_{\Sigma_2}$. If $\widetilde{Y}_{\Sigma'}$ intersects $\widetilde{Z}_{\Sigma_1}$, then we apply case (2,4) to the triple $\Sigma_1, \Sigma', \Omega$, and see that $\widetilde{Y}_{\Sigma'}$ intersects $\widetilde{Z}_{\Sigma_1}$ in a copy of $\widetilde{X}_{\Sigma_1\cap\Sigma'} = \widetilde{X}_{\Sigma\cap\Sigma'}$.

If $\widetilde{Y}_{\Sigma'}$ intersects $\widetilde{R}_{\Sigma_2}$, then we apply case (2,4) to the triple $\Sigma_2, \Sigma', \Omega$, and see that $\widetilde{Y}_{\Sigma'}$ intersects $\widetilde{R}_{\Sigma_2}$ in a copy of $\widetilde{X}_{\Sigma_1\cap\Sigma'} = \widetilde{X}_{\Sigma\cap\Sigma'}$.

If $\widetilde{Y}_{\Sigma'}$ intersects both $\widetilde{Z}_{\Sigma_1}$ and $\widetilde{R}_{\Sigma_2}$, then the two copies we obtained intersect. But they are copies of the same complex, and hence they coincide.

(4,4) In case (2,4) we have shown that if $\widetilde{Y}_{\Sigma'}$ intersects $\widetilde{Z}_{\Omega_1}$, then it lies within as a standard copy; the analogous statement holds for $\widetilde{R}_{\Omega_2}$. Now the standard copies $\widetilde{Y}_\Sigma$ and $\widetilde{Y}_{\Sigma'}$ intersect, and hence they both lie in $\widetilde{Q}$ as standard copies, where $\widetilde{Q}$ is $\widetilde{Z}_{\Omega_1}$ or $\widetilde{R}_{\Omega_2}$. But now we just need to apply the Intersection Axiom in $\mathcal{X}'$ or $\mathcal{X}''$.

System Intersection Axiom. Take a subsystem $\mathcal{P} \subseteq \mathcal{L}^\phi$ closed under taking unions. If all elements of $\mathcal{P}$ lie in Γ' or in $\Delta \cup \Theta$, then we are done (from the System Intersection Axiom of $\mathcal{X}'$ or $\mathcal{X}''$). So let us suppose this is not the case, that is suppose that there exists $\Sigma \in \mathcal{P}$ satisfying $(*)$ and $\Sigma \neq \Delta \cup \Theta$. Hence $\bigcup \mathcal{P} \neq \Delta \cup \Theta$ satisfies $(*)$.

Define

$$\begin{aligned} \mathcal{P}' &= \{\Sigma \in \mathcal{P} \mid \Theta \subseteq \Sigma\} \\ \overline{\mathcal{P}}' &= \{\overline{\Sigma} = \Sigma \cup \partial\Gamma' \mid \Sigma \in \mathcal{P}'\} \end{aligned}$$

and $\overline{\mathcal{P}} = \mathcal{P} \cup \overline{\mathcal{P}}'$. Observe that $\overline{\mathcal{P}}' \subseteq \mathcal{L}^\phi$, since all graphs in $\mathcal{P}'$ satisfy $(*)$, and so for any $\Sigma' \in \mathcal{P}'$ we have $\overline{\Sigma'} \in \mathcal{S} \subseteq \mathcal{L}^\phi$.

Claim. $\overline{\mathcal{P}}$ is closed under taking unions.

Proof. Take $\Sigma, \Sigma' \in \overline{\mathcal{P}}$. If both lie in $\mathcal{P}$ then we are done. Let us first suppose that $\Sigma \in \mathcal{P}$ and $\Sigma' \in \overline{\mathcal{P}}'$. Then $\Sigma' = \overline{\Sigma''}$ for some $\Sigma'' \in \mathcal{P}'$. Now

$$\Sigma \cup \Sigma' = \Sigma \cup (\Sigma'' \cup \partial\Gamma') = (\Sigma \cup \Sigma'') \cup \partial\Gamma' \in \overline{\mathcal{P}}'$$

since $\Theta \subseteq \Sigma \cup \Sigma'' \in \mathcal{P}$. If both $\Sigma, \Sigma' \in \overline{\mathcal{P}}'$ then an analogous argument shows that $\Sigma \cup \Sigma' \in \overline{\mathcal{P}}'$. □

Now observe that $\overline{P}'$ is a subsystem of $\overline{\mathcal{P}}$, which is closed under taking unions. Hence the same is true for systems $(\overline{P})_{\Gamma'}$ and $(\overline{P}')_{\Gamma'}$.

We are now going to construct standard copies for all elements in $\mathcal{P}$, such that they all intersect non-trivially.

Observe that $\overline{P}' \subseteq \mathcal{S}$, and so for each element $\overline{\Sigma} \in (\overline{P}')_{\Gamma'}$ we are given a standard copy $\widetilde{Z}_{\overline{\Sigma}}$ in $\widetilde{X}_{\Gamma'}$ which contains $\widetilde{Z}_\Delta$.

Let $\Sigma \in \overline{P}$. Then $\Sigma \cup \bigcap \overline{P}' \in \overline{P}'$, and hence $\Sigma_1 \cup \bigcap (\overline{P}')_{\Gamma'} \in (\overline{P}')_{\Gamma'}$, and so we are able to apply Lemma 7.5 to the collection of standard copies we just discussed, and extend it by adding copies $\widetilde{Z}_{\Sigma_1}$ of $\widetilde{X}_{\Sigma_1}$ with $\Sigma \in \overline{P}$, such that all these copies intersect non-trivially. Moreover, for every $\Sigma \in \mathcal{P}'$, the copy $\widetilde{Z}_{\Sigma_1}$ will intersect $\widetilde{Z}_\Delta$ (thanks to the Matching Property in $\widetilde{Z}_{\overline{\Sigma}_1}$).

By this point we have constructed standard copies $\widetilde{Z}_{\Sigma_1}$ for each $\Sigma \in \mathcal{P}$ which all intersect non-trivially. We will now extend these copies to copies of $\widetilde{X}_\Sigma$.

Let $\Sigma \in \mathcal{P}'$. If $\Sigma \not\subseteq \Delta \cup \Theta$, then we can extend $\widetilde{Z}_{\Sigma_1}$ to a standard copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_\Gamma$ by construction.

When $\Sigma \subseteq \Delta \cup \Theta$ (but still $\Sigma \in \mathcal{P}'$) we need to show that there exists a standard copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_\Gamma$ which contains $\widetilde{Z}_{\Sigma_1}$. By construction it is enough to find such a copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\Delta\cup\Theta}$, bearing in mind that $\widetilde{Z}_{\Sigma_1}$ lies in $\widetilde{Z}_\Delta = \widetilde{R}_\Delta$.

By $(*)$ we know that $\Delta \cap \mathrm{lk}(\partial\Gamma') \subseteq \Sigma$.

The map $\widetilde{\iota}_{\mathrm{lk}_{\Delta\cup\Theta}(\partial\Gamma'),\Delta\cup\Theta}$ is onto by the Product Axiom, and so there exists a standard copy of $\widetilde{X}_{\mathrm{lk}_{\Delta\cup\Theta}(\partial\Gamma')}$ which intersects $\widetilde{Z}_{\Sigma_1}$. But we have $\mathrm{lk}_{\Delta\cup\Theta}(\partial\Gamma') \subseteq \Sigma$, and thus there exists a copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\Delta\cup\Theta}$ (since $\widetilde{X}_{\Delta\cup\Theta}$ is a product) which contains the given copy of $\widetilde{X}_{\mathrm{lk}_{\Delta\cup\Theta}(\partial\Gamma')}$, and thus is as required by the Intersection Axiom.

We have finished extending the copies for all $\Sigma \in \mathcal{P}'$. For all $\Sigma \in \mathcal{P}$ with $\Sigma = \Sigma_1$ we do not even need to extend.

It is still possible that there exists $\Sigma \in \mathcal{P} \smallsetminus \mathcal{P}'$ such that $\Sigma \not\subseteq \Gamma'$. Such a Σ must satisfy $\Sigma \subseteq \overline{\Theta}$ (by $(*)$) and $\Sigma \cap \Theta \notin \{\emptyset, \Theta\}$. But then for all $\Sigma' \in \mathcal{P}$ we have $\Theta \subseteq \Sigma'$ or $\Sigma' \subseteq \overline{\Theta}$, as otherwise $\Sigma \cup \Sigma'$ would violate $(*)$. So in this situation $\mathcal{P} \smallsetminus \mathcal{P}' \subseteq \mathcal{L}^\phi_{\Delta\cup\Theta}$.

Consider the subsystem $\mathcal{P}'_{\Delta\cup\Theta}$ of $\mathcal{P}_{\Delta\cup\Theta}$. It is closed under taking unions, since $\mathcal{P}'$ is, and for each $\Sigma \in \mathcal{P}$ we have $\Sigma_2 \cup \bigcap \mathcal{P}'_{\Delta\cup\Theta} \in \mathcal{P}'_{\Delta\cup\Theta}$. Now our standard copies of $\widetilde{X}_\Sigma$ for $\Sigma \in \mathcal{P}'$ give us (by the Intersection Axiom) standard copies of $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_{\Delta\cup\Theta}$ which intersect in a standard copy of $\widetilde{X}_{(\bigcap \mathcal{P}')_2}$, and hence non-trivially. Lemma 7.5 gives us a collection of standard copies of $\widetilde{X}_{\Sigma_2}$ in $\widetilde{X}_{\Delta\cup\Theta}$ for all $\Sigma \in \mathcal{P}$, which contains the previously discussed collection, and such that all of these standard copies intersect non-trivially. For each $\Sigma \in \mathcal{P} \smallsetminus \mathcal{P}'$ we have $\Sigma = \Sigma_2$, and the standard copy of $\widetilde{X}_\Sigma$ in $\widetilde{X}_{\Delta\cup\Theta}$ becomes a standard copy in $\widetilde{X}_\Gamma$ by construction. □

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